for the like price, take these thirteen *kákinís*, merchant, and give me quickly two parts of rice with one of kidney-beans ; for we must make a hasty meal and depart, since my companion will proceed onwards.

Statement: $\frac{2}{1}$ $\frac{1}{1}$. Mixed sum $\frac{13}{64}$.
$\frac{1}{1}$ $\frac{1}{1}$
$\frac{7}{2}$ $\frac{8}{1}$

The prices, $\frac{1}{1}$, $\frac{1}{1}$, multiplied by the portions $\frac{2}{1}$, $\frac{1}{1}$, and divided by the goods $\frac{7}{2}$, $\frac{8}{1}$, make $\frac{4}{7}$, $\frac{1}{8}$, the sum of which is $\frac{39}{56}$. By this divide the same fractions $\frac{4}{7}$, $\frac{1}{8}$, taken into the mixed sum $\frac{13}{64}$; and the portions $\frac{2}{1}$, $\frac{1}{1}$, taken into that mixed sum $\frac{13}{64}$. There result the prices of the rice and kidney-beans, $\frac{1}{6}$ and $\frac{7}{192}$ of a *dramma* ; or 10 *kákinís* and $13\frac{1}{3}$ shells for the rice, and 2 *kákinís* and $6\frac{2}{3}$ shells for the kidney-beans ; and the quantities are $\frac{7}{12}$ and $\frac{7}{24}$ of a *mána* of rice and kidney-beans respectively.

[Let x denote the *mánas* of kidney-beans.
Then $2x$ will denote the *mánas* of rice.
Now the price paid $= \frac{13}{64}$ of a *dramma ;*
$\therefore 2x \times \frac{2}{7} + x \times \frac{1}{8} = \frac{13}{64}$;
Whence $x\ (\frac{4}{7} + \frac{1}{8}) = \frac{13}{64}$,

$$\therefore x = \frac{1 \times \frac{13}{64}}{\frac{4}{7} + \frac{1}{8}} = \frac{7}{24},$$

$$\text{and } 2x = \frac{2 \times \frac{13}{64}}{\frac{4}{7} + \frac{1}{8}} = \frac{7}{12};$$

and the prices are, $\frac{7}{24} \times \frac{1}{8} = \frac{7}{192}$
and $\frac{7}{24} \times \frac{4}{7} = \frac{1}{6}$.

Hence the reason for the rule is evident. General formulæ corresponding to the rule in § 96 may be easily established. It is, however, not worth while to do so.

The example in § 98 may be worked out in a similar manner.]

98. Example. If a *pala* of best camphor may be had for two *nishkas*, and a *pala* of sandal-wood[1] for the eighth part of a *dramma*, and half a *pala* of alœ-wood[2] also for the eighth of a *dramma*, good merchant, give me the value of one *nishka* in the proportions of one, sixteen and eight; for I wish to prepare a perfume.

Statement: $\begin{array}{ccc} 32 & \frac{1}{8} & \frac{1}{8}. \\ 1 & 1 & \frac{1}{2} \\ 1 & 16 & 8 \end{array}$ Mixed sum 16.

Answer. Prices: *drammas*, $14\frac{2}{9}$, $\frac{8}{9}$, $\frac{8}{9}$.
Quantities: *palas*, $\frac{4}{9}$, $7\frac{1}{9}$, $3\frac{5}{9}$.

SECTION IV.

99. Rule. Problem concerning a present of gems.[3] From the gems subtract the gift multiplied by the persons; and any arbitrary number being divided by the remainders, the quotients are numbers expressive of the prices. Or the remainders being multiplied together, the product, divided by the several reserved remainders, gives the values in whole numbers.

[The reason for the rule will appear from the solution of the example in § 100, to which the rule specially refers.]

100. Example. Four jewellers, possessing respectively eight rubies, ten sapphires, a hundred pearls, and five diamonds, presented, each from his own stock, one apiece to the rest in token of regard and gratification at meeting; and they thus became owners of stock of precisely equal value. Tell me severally, friend, the prices of their gems.

---

[1] *Chandana*; santalum album.

[2] *Aguru*; aquillaria agallochum.

[3] The problem is an indeterminate one. The solution gives relative values only.

Statement : rub. 8 ; sapph. 10 ; pearls 100 ; diam. 5 ; gift 1 ; persons 4.

Here, the product of the gift 1 by the persons 4, *viz.*, 4, being severally subtracted, there remain rubies 4, sapphires 6, pearls 96, diamond 1. Any number arbitrarily assumed being divided by these remainders, the quotients are the relative values. Taking it at random, they may be fractional values ; or by judicious selection, whole numbers Thus, put 96 ; and the prices thence deduced are 24, 16, 1, 96 ; and the equal stock 233.

Or the remainders being multiplied together, and the product severally divided by those remainders, the prices are 576, 384, 24, 2304 ; and the equal amount of stock (after interchange of presents) is 5592.

[Let the relative values of a ruby, sapphire, pearl, and diamond be respectively x, y, z, w. Then we shall evidently get from the conditions of the problem the following equations :—

$$\begin{aligned} &5x+y+z+w \\ &=7y+x+z+w \\ &=97z+x+y+w \\ &=2w+x+y+z \end{aligned}$$

$\therefore 4x=6y=96z=w=k$ suppose ; then $x=\frac{k}{4}$, $y=\frac{k}{6}$, $z=\frac{k}{96}$, $w=k$. Putting $k=$ L. C. M. of 4, 6, 96, *i.e.*, 96, we get the least integral values of x, y, z, w, *viz.*, 24, 16, 1, 96 ; and putting k $=$product of 4, 6, 96, we get for x, y, z, w, the values 576, 384, 24, 2304.

The reason for the rule in § 99 will be evident from the above algebraical solution. The coefficients of x, y, &c. in the final equations will be

$=$the no. of the respective gems—the no. of that gem given to each person $\times$ the no. of persons altogether.

For instance, if there were 9 rubies, and each presented 2 from his pocket to each of the rest, the coefficient of x in the left hand side of the first equation would have been 3; and that of x in the right hand side would have been 2; thus the coefficient of x in the simplified equations would have been 1, *i.e.*, $9 - 4 \times 2$, which agrees with the rule in § 99.

Súryadása cites the *Víja-ganita* for the solution of the problem. Ranganátha gives an arithmetical explanation which, however, is meaningless add obscure.]

SECTION V.

ALLIGATION.[1]

101. Rule.[2] The sum of the products of the touch[3] and (weight of several parcels)[4] of gold being divided by the aggregate of the gold, the touch of the mass is found: or (after refining) being divided by the fine gold, the touch is ascertained; or divided by the touch, the quantity of purified gold is determined.

[This is simply the ordinary rule for alligation medial. We may consider the prices per *másha* of the several kinds of gold as *proportional to* the fineness. The reason for the rule is obvious.]

102—103. Example. Parcels of gold weighing severally ten, four, two and four *máshas*, and of the fineness of thirteen, twelve, eleven and ten respectively, being melted together, tell me quickly, merchant, who art conversant with the computation of gold, what the fineness of the mass is. If the twenty *máshas* above

---

[1] *Suvarna-ganita*, computation of gold, that is, of its weight and fineness; alligation medial.

[2] To find the fineness produced by mixture of parcels of gold; and, after refining, to find the weight, if the fineness be known; and the fineness, if the weight of refined gold be given.—Gan.

[3] *Varna*, colour of gold on the touchstone; fineness of gold determined by that touch. See § 77. "The degrees of fineness increase as the weight is reduced by refining."—Gan.

[4] Gang.

described be reduced to sixteen by refining, tell me instantly the touch of the purified mass. Or, if its purity when refined be sixteen, prithee, what is the number to which the twenty *máshas* are reduced ?

Statement : touch 13 12 11 10 ;
weight 10 4 2 4.

Answer[1]: after melting, fineness 12 ; weight 20.

After refining, the weight being sixteen *máshas*, the touch is 15. The touch being sixteen, the weight is 15.

104. Rule.[2] From the acquired fineness of the mixture, taken into the aggregate quantity of gold, subtract the sum of the products of the weight and fineness (of the parcels, the touch of which is known), and divide the remainder by the quantity of gold of unknown fineness ; the quotient is the degree of its touch.[3]

105. Example. Eight *máshas* of ten, and two of eleven by the touch, and six of unknown fineness, being mixed together, the mass of gold, my friend, became of the fineness of twelve ; tell the degree of unknown fineness.

Statement : 10 11 Fineness of the mixture 12.
8 2 6 .

Answer : degree of the unknown fineness 15.

[Let x denote the unknown fineness. Then, $(8+2+6)\times 12 = 8\times 10+2\times 11+6\times x$,

whence $x=\dfrac{(8+2+6)\times 12-(8\times 10+2\times 11)}{6}$. The reason for the rule in § 104 is obvious.]

---

[1] Products 130, 48, 22, 40. Their sum 240, divided by 20, gives 12 ; divided by 16, gives 15.

[2] To discover the finness of a parcel of unknown degree of purity mixed with others of which the touch is known.—Gan.

[3] *The rule being the converse of the preceding, the principle of it is obvious.*—Rang.

106. Rule.[1] The acquired fineness of the mixture being multiplied by the sum of the gold (in the known parcels), subtract therefrom the aggregate products of the weight and fineness (of the parcels): divide the remainder by the difference between the fineness of the gold of unknown weight and that of the mixture, the quotient is the weight of gold that was unknown.

107. Example. Three *máshas* of gold of the touch of ten, and one of the fineness of fourteen, being mixed with some gold of the fineness of sixteen, the degree of purity of the mixture, my friend, is twelve. How many *máshas* are there of the fineness of sixteen?

Statement: 10 14 16. Fineness of the mixture 12.
3 1

Answer: *másha* 1.

[Let x denote the number of *máshas* required.
Then, $(3+1+x)\times 12 = 3\times 10 + 1\times 14 + x\times 16$;

$$\therefore x = \frac{(3+1)\times 12 - (3\times 10 + 1\times 14)}{16-12},$$

whence the rule in § 106.]

108. Rule.[2] Subtract the effected fineness from that of the gold of a higher degree of touch, and that of the one of lower touch from the effected fineness; the differences, multiplied by an arbitrarily assumed number, will be the weights of gold of the lower and higher degrees of purity respectively.

109. Example. Two ingots of gold, of the touch of sixteen and ten respectively, being mixed together,

---

[1] To find the weight of a parcel of known fineness, but unknown weight, mixed with other parcels of known weight and fineness.—Gan.

[2] To find the weight of two parcels of given fineness and unknown weight.—Gan. and Súr. The problem is an indeterminate one, as is intimated by the author.

the gold became of the fineness of twelve. Tell me, friend, the weight of gold in both lumps.

Statement: 16, 10. Fineness resulting 12.

Putting one, and proceeding as directed, the weights of gold are found, *máshas* 2 and 4. Assuming two, they are 4 and 8. Taking half, they come out 1 and 2. Thus, manifold answers are obtained by varying the assumption.

[Let x and y be the weights required.

Then, $x \times 16 + y \times 10 = (x + y) \times 12$;

$\therefore (16-12)\,x = (12-10) \times y$;

$\therefore \dfrac{x}{y} = \dfrac{12-10}{16-12}$;

$\therefore x = (12 - 10)\,k,\; y = (16-12)\,k$,

where k is any positive quantity.

The general solution in positive integers evidently is, $x = k$, $y = 2k$, where k is any positive integer.

The reason for the rule in § 108 is obvious.]

Section VI.

Permutations and Combinations.

110—112. Rule[1]: three stanzas.

Let the figures from one upwards, differing by one, put in the inverse order, be divided by the same (arithmeticals) in direct order; and let the subsequent be multiplied by the preceding, and the next following by the foregoing (result). The several results are the changes, ones, twos, threes, &c.[2] This is termed a

[1] To find the possible permutations of long and short syllables in prosody; combinations of ingredients in pharmacy; variations of notes, &c., in music; as well as changes in other instances.—Gan.

[2] According to Ganesa, there is no demonstration of the rule, besides acceptation and experience. [This, however, is not correct.—Ed.]

general rule.[1] It serves in prosody, for those versed therein, to find the variations of metre; in the arts (as in architecture) to compute the changes upon apertures (of a building); and (in music) the scheme of musical permutations[2]; in medicine, the combinations of different savours. For fear of prolixity, this is not (fully) set forth.

[The reason for the rule will appear from the solution of the example which follows.]

113. A single example in prosody. In the permutations of the *Gáyatrí* metre,[3] say quickly, friend, how many the possible changes of the verse are; and tell severally, how many the permutations are with one, (two, three,) &c., long syllables.

Here the verse of the *Gáyatrí* stanza comprises six syllables. Wherefore, the figures from one to six are set down, and the statement of them, in direct and inverse order is $\frac{6\ 5\ 4\ 3\ 2\ 1}{1\ 2\ 3\ 4\ 5\ 6}$. Proceeding as directed, the results are :—changes with one long syllable, 6; with two, 15; with three, 20; with four, 15; with five, 6; with six, 1; with all short, 1. The sum of these is the whole number of permutations of the verse, 64.

---

[1] Commentators appear to interpret this as a name of the rule here taught; *sádhárana*, or *sádhárana-chhandoganita*, general rule of prosodian permutation, subject to modification in particular instances, as in music, where a special method (*asádhárana*) must be applied.—Gang. and Súr.

[2] *Khanda-meru*, a certain scheme.—Gan. It is more fully explained by other commentators; but the translator is not sufficiently conversant with the theory of music to understand the term distinctly.

[3] The *Gáyatrí* metre in sacred prosody is a triplet comprising twenty-four syllables, as in the famous prayer containing the Brahmanical creed, called *Gáyatrí* [*Rigveda*, *Mandala* 3, *súkta* 62, *rik* 10.—Ed.] See As. Res., vol. X, p. 463. But in the prosody of profane poetry, the same number of syllables is distributed in a tetrastic; and the verse consequently contains six syllables. (As. Res., vol. X, p. 469.)

In like manner, setting down the numbers of the whole tetrastic, in the mode directed, and finding the changes with one, two, &c., and summing them, the permutations of the entire stanza are found, *viz.*, 16777216.

In the same way may be found the permutations of all varieties of metre, from *Ukthá* (which consists of monosyllabic verses) to *Utkriti* (the verses of which contain twenty-six syllables).[1]

[In the *Gáyatrí* metre, the number of syllables is 6. In finding the number of changes with *one* long and the rest *five* short syllables, we have to find the permutations of 6 things taken all at a time, when one of them is of one kind, and the rest, of another kind. Hence the number of changes $= \frac{\underline{|6}}{\underline{|1}\;\underline{|5}}$, which is precisely the number of combinations of 6 things taken 1 at a time. Similarly, in finding the number of changes with *two* long, and therefore the rest *four* short, we get the number $= \frac{\underline{|6}}{\underline{|2}\;\underline{|4}}$, and so on; thus finally the total number of changes = sum of combinations of 6 things taken 1, 2, 3, 4, 5, 6, at a time + 1 (with all short syllables) $= (2^6 - 1) + 1 = 64$.

Now $\frac{\underline{|6}}{\underline{|1}\;\underline{|5}} = \frac{6}{1}$; $\frac{\underline{|6}}{\underline{|2}\;\underline{|4}} = \frac{6 \times 5}{1 \times 2}$; &c.

Hence the reason for the rule is clear.

If the aggregate number of changes only is wanted, this can be found at once from the proposition, *viz.*, the total number of combinations of n things taken 1, 2, 3......n at a time $= 2^n - 1$ (see Todhunter's Algebra, art. 515). This proposition is given in a concrete shape in § 130 – 131.

[1] As. Res., vol. X, pp. 468—473.

Similarly, taking the whole tetrastic, *i.e.*, 24 syllables, the total number of changes = $(2^{24}-1)+1$ (with all short syllables) = 16777216.]

114. Example. In a pleasant, spacious and elegant edifice, with eight doors,[1] constructed by a skilful architect, as a palace for the lord of the land, tell me the permutations of apertures taken one, two, three, &c.[2] Say, mathematician, how many are the combinations in one composition, with ingredients of six different tastes, sweet, pungent, astringent, sour, salt and bitter,[3] taking them by ones, twos, threes, &c.

Statement, first example:

8 7 6 5 4 3 2 1
1 2 3 4 5 6 7 8

Answer: the number of ways in which the doors may be opened by ones, twos, threes, &c., is 8, 28, 56, 70, 56, 28, 8, 1, respectively. And the changes on the apertures of the octagonal palace[4] amount to 255.

Statement, second example:

6 5 4 3 2 1
1 2 3 4 5 6

Answer: the number of various preparations with ingredients of divers tastes is 6, 15, 20, 15, 6, 1.[5]

[In the first example, the total number of variations $= 2^8 - 1 = 255$. The case of all the windows being shut is not taken

---

[1] *Múshá*, aperture for the admission of air; a door or window; (same with *gavákshá*.—Gan.) A portico or terrace, (*bhúmi-visesha*.—Gang. and Súr.)

[2] The variations of one window or portico open (or terrace unroofed) and the rest closed; two open, and the rest shut; and so forth.

[3] *Amara-kosha*, *swarga-varga*, 147.

[4] An octagonal building, with eight doors or windows or porticos or terraces facing the eight cardinal points of the horizon, is meant.—Gan.

[5] Total number of possible combinations is 63.—Gang.

into account ; otherwise the total number of variations would be 256.

The second example from its very nature is a case of combinations and not of permutations, *i.e.*, we have to find the number of combinations of 6 things taken 1, 2......6 at a time. The rule in §110—112, however, equally applies, as has been explained above. The total number of combinations in this case $= 2^6 - 1 = 63$.]

CHAPTER V.
PROGRESSIONS.[1]

SECTION I.
ARITHMETICAL PROGRESSION.

115. Rule.[2] Half the period[3] multiplied by the period added to unity, is the sum of the arithmeticals one, &c., and is named their addition.[4] This, being multiplied by the period added to two, and being divided by three, is the aggregate of the additions.[5]

---

[1] *Sredhí*, a term employed by the older authors for any set of distinct substances or other things put together.—Gan. It signifies sequence or progression. *Sredhí-vyavahára*, ascertainment or determination of progressions.

[2] To find the sums of the arithmeticals. Gan.

[3] *Pada*, the place.—Gan. Any one of the figures or digits, being that of which the sum is required.—Súr. The last of the numbers to be summed.—*Mano.* See below, note to §119.

[4] *Sankalita*, the first sum or addition of arithmeticals. *Sankalitaikya*, aggregate of additions, summed sums or second sum.

[5] The first figure is unity. The sum of that and the period being halved, is the middle figure. As the figures decrease behind it, so they increase before it: wherefore the middle figure, multiplied by the period, is the sum of the figures one, &c., continued to the period. The only proof of the rule for the aggregate of sums is acceptation.—Gan. [This last remark is not correct. The proof of the formula for the sum of the first n arithmeticals given by Ganesa does not apply where n is even, but requires to be modified. In that case, the sum of any two terms equidistant from the first and last $=n+1$; whence the sum of n terms evidently $=\frac{n}{2}(n+1)$.—Ed.] It is a maxim, that a number multiplied by the next following arithmetical, and halved, gives the sum of the preceding; wherefore, &c.—Súr. Kamalákara is quoted by Ranganátha for a demonstration grounded on placing the numbers of the series in the reversed order under the direct one and adding the two series—[the same proof as that given in modern works on Algebra.—Ed.]

[$1+2+3 \ldots\ldots$ to n terms $= \frac{n(n+1)}{2}$. By "the aggregate of the additions," the author evidently means the sum of n terms of the series whose nth term is $\frac{1}{2}n(n+1)$, in other words, the sum of n *triangular numbers*. This sum is $\frac{1}{6}n(n+1)(n+2) = \frac{n(n+1)}{2} \cdot \frac{n+2}{3}$. See Todhunter's Algebra, Art. 666. The reason for the rule is obvious.]

116. Example. Tell me quickly, mathematician, the sums of the several (progressions of) numbers one, &c., continued to nine; and the summed sums of those numbers.

Statement: arithmeticals: 1 2 3 4 5 6 7 8 9.

Answer: sums: 1 3 6 10 15 21 28 36 45.

Summed sums: 1 4 10 20 35 56 84 120 165.

117. Rule.[1] Twice the period added to one and divided by three, being multiplied by the sum (of the arithmeticals), is the sum of the squares. The sum of the cubes of the numbers one, &c., is pronounced by the ancients equal to the square of the addition.

$$[1^2+2^2+\ldots\ldots+n^2 = \frac{n(n+1)(2n+1)}{6}$$
$$= \frac{n(n+1)}{2} \cdot \frac{2n+1}{3}.$$

$1^3+2^3+\ldots\ldots+n^3 = \left\{\frac{n(n+1)}{2}\right\}^2$. See Todhunter's Algebra, Arts. 460, 461.]

118. Example. Tell promptly the sum of the squares, and the sum of the cubes, of those numbers, if thy mind be conversant with the way of summation.

Statement: 1 2 3 4 5 6 7 8 9.

Answer: sum of squares, 285. Sum of cubes, 2025.

---

[1] To find the sums of squares and of cubes.—Gan. and Súr.

119. Rule.[1] The increase multiplied by the period less one, and added to the first quantity, is the amount of the last.[2] That, added to the first, and halved, is the amount of the mean; which multiplied by the period is the amount of the whole, and is denominated (*ganita*) the computed sum.

[Consider the series a, $a+b$, $a+2b$......The nth term $=a+(n-1)b$. If n be *odd*, middle term $=\frac{n+1}{2}$ th term $=\frac{2a+(n-1)b}{2}$. If n be *even*, there will be two middle terms, *viz.*, $\frac{n}{2}$ th and $\frac{n+2}{2}$ th terms, and the mean amount = average of these two terms $=\frac{n}{2}$ th term $+\frac{1}{2}$ of common difference $=a+\left(\frac{n}{2}-1\right)b+\frac{1}{2}b=\frac{2a+(n-1)b}{2}$. And the sum of n terms $=\frac{n}{2}\left\{2a+(n-1)b\right\}$.

Thus the rule holds good whether n be odd or even. The author does not notice that the first part of the rule in § 115 is only a particular case of the present rule.]

120. Example. A person, having given four *drammas* to priests on the first day, proceeded, my friend, to distribute daily alms at a rate increasing by five a

[1] Where the increase is arbitrary.—Gang. In such cases, to find the last term, mean amount, and sum of the progression.—Súr. From first term, common difference and period, to find the whole amount, &c.—Gan.

[2] *Adi* and *mukha*, *vadana*, *vaktra*, and other synonyms of face—the initial quantity of the progression, the first term: (that, from which as an origin the sequence commences.—Súr.)

Chaya, *prachaya* or *uttara*—the more (*adhika*—Súr.) or augment (*vriddhi*—Gang.) by which each term increases, the common difference. *Antya*, the last term. *Madhya*, the middle term. *Pada* or *gachchha*, the period, the number of terms: (so many days as the sequence reaches.—Súr.) *Sarva-dhana*, *sredhi-phala* or *ganita*—the amount of the whole, the sum of the progression. 'It is called *ganita*, because it is found by computation (*gananá*).—Gan.

day. Say quickly how many were given by him in half a month.

Statement: initial quantity 4; com. diff. 5; period 15.

Here, first term 4. Middle term 39. Last term 74. Sum 585.

121. Another example.[1] The initial term being seven, the increase five, and the period eight, tell me what the magnitudes of the middle and last terms are, and what the total sum is.

Statement: first term 7; com. diff. 5; period 8.

Answer: mean amount $\frac{49}{2}$. Last term 42. Sum 196.

Here, the period consisting of an even number of days, there is no middle day; wherefore half the sum of the days preceding and following the mean place, must be taken for the mean amount: and the rule is thus proved.

[See note to § 119.]

122. Rule[2]: half a stanza. The sum of the progression being divided by the period, and half the common difference multiplied by one less than the number of terms, being subtracted, the remainder is the initial quantity.[3]

[$s=\frac{n}{2}\left\{2a+(n-1)b\right\}=n\left\{a+\frac{n-1}{2}b\right\}; \therefore \frac{s}{n}-\frac{n-1}{2}b=a$, whence the rule.]

123. Example. We know the sum of the progression, one hundred and five; the number of terms, seven; the increase, three; tell us, dear boy, the initial quantity.

---

[1] To exhibit an instance of an even number of terms, where there can consequently be no middle term (but a mean amount).—Gan.

[2] The difference, period and sum being given, to find the first term.—Gan. and Súr.

[3] The rule is the converse of the preceding.—Gan. and Súr.

Statement : com. diff. 3 ; period 7 ; sum 105.

Answer : first term, 6.

Rule[1]; half a stanza.[2] The sum being divided by the period, and the first term subtracted from the quotient, the remainder, divided by half of one less than the number of terms, will be the common difference.[3]

[$s=n\left\{a+\frac{n-1}{2}b\right\}$; $\therefore \left\{\frac{s}{n}-a\right\} \div \frac{n-1}{2}=b$, whence the rule.]

124. Example. On an expedition to seize his enemy's elephants, a king marched two *yojanas* the first day. Say, intelligent calculator, with what increasing rate of daily march he proceeded, he reaching his foe's city, a distance of eighty *yojanas*, in a week.

Statement : first term 2 ; period 7 ; sum 80.

Answer : com. diff. $\frac{22}{7}$.

125. Rule.[4] From the sum of the progression multiplied by twice the common increase, and added to the square of the difference between the first term and half that increase, the square root being extracted, this root less the first term and added to the (above-mentioned) portion of the increase, being divided by the increase, is pronounced[5] to be the period.

[The translation is rather obscure. A clearer rendering would be as follows :—" The sum of the progression multiplied by twice the common increase, being added to the square of the

---

[1] The first term, period and sum being known, to find the common difference which is unknown.—Gan.

[2] Second half of one, the first half of which contained the preceding rule, § 122.

[3] This rule also is converse of the foregoing.—Gan.

[4] The first term, common difference and sum being known, to find the period which is unknown.—Gan.

[5] By Brahmagupta and the rest.—Gan.

The rules are substantially the same ; the square being completed for the solution of the quadratic equation in the manner taught by Srídhara (cited in *Víja-ganita*, § 131) and by Brahmagupta.

difference between the first term and half that increase, the square root of the result is extracted ; this root less, &c."

We have $s=\frac{n}{2}\left\{2a+(n-1)b\right\}$;

whence $n=\frac{b-2a\pm\sqrt{\{(2a-b)^2+8sb\}}}{2b}$

(Todhunter's Algebra, Art. 454)

$$=\frac{1}{b}\left\{\frac{b}{2}-a\pm\sqrt{\left\{(a-\frac{b}{2})^2+2sb\right\}}\right\}$$, whence the rule,

the *upper sign* only being taken by the author. He does not discuss the meaning of the *two* values of n.]

126. Example. A person gave three *drammas* on the first day, and continued to distribute alms increasing by two (a day) ; and he thus bestowed on the priests three hundred and sixty *drammas* : say quickly in how many days.

Statement : first term 3 ; com. diff. 2 ; sum 360.

Answer : period 18.

Section II.

Geometrical Progression.

127. Rule[1]: a couplet and a half. The period being an uneven number, subtract one, and note 'multiplicator'; being an even one, halve it, and note 'square,' until the period be exhausted. Then the produce arising from multiplication and squaring (of the common multiplier) in the inverse order from the last,[2] being lessened by one, the remainder divided by the common

[1] To find the sum of a progression, the increase being a multiplier.—Gan. In other words, to find the sum of an increasing geometrical progression.

[2] The last note is of course 'multiplicator.' For in exhausting the number of the period (when odd) you arrive at last at unity, an uneven number. The proposed multiplier (the common multiplicator of the progression) is therefore put in the last place ; and the operations of squaring and multiplying by it are continued in the inverse order of the line of the notes.—Gan.

multiplier less one, and multiplied by the initial quantity, will be the sum of a progression increasing by a common multiplier.[1]

[Let a denote the first term and r the common ratio. Then, $a+ar+\dots\dots\dots+ar^{n-1}=\frac{a(r^n-1)}{r-1}$. The first part of the rule is clumsily and obscurely stated. It is difficult to make out what the author means. He wants us to find r^n. Now if n is even and therefore of the form $2m$, we can find r^n by first squaring r, then squaring the square, and so on m times. If n is odd and therefore of the form $2m+1$, we can find r^n by first finding r^{2m} as before, and then multiplying the result by r. This is probably what the author means by the words multiplying and squaring. See the explanation of Ganesa in the footnotes.]

128. Example. A person gave a mendicant a couple of cowry shells first, and promised a twofold increase of the alms daily. How many *nishkas* did he give in a month ?

Statement : first term, 2 ; increasing multiplier, 2 ; period, 30.

Answer : 2147483646 cowries ; or 104857 *nishkas*, 9 *drammas*, 9 *panas*, 2 *kákinís*, and 6 shells.

129. Example. The initial quantity being two, my friend ; the daily augmentation, a threefold increase ; and the period, seven ; say what the sum in this case is.

---

[1] The effect of squaring and multiplying, as directed, is the same as the continued multiplication of the multiplier for as many times as the number of the period. For dividing by the multiplier the product of the multiplication continued to the uneven number, equals the product of multiplication continued to one less than the number ; and the extraction of the square root of a product of multiplication continued to the even number, equals continued multiplication to half that number. Conversely, squaring and multiplying equals multiplication for double and for one more time.—Gan.

Statement : first term, 2 ; increasing multiplier, 3 ; period, 7.

Answer : sum, 2186.

130—131. Rule :[1] a couplet and a half. The number of syllables in a verse being taken for the period, and the increase twofold, the produce of multiplication and squaring (as above directed, § 127) will be the number (of variations) of like verses.[2] Its square, and square's square, less their respective roots, will be (the variations) of alternately similar and of dissimilar verses (in tetrastics).[3]

[The rule refers specially to the example in § 132. It is a statement in a concrete shape of the following proposition :— The total number of combinations of n things taken, 1, 2, ... n at a time $= 2^n - 1$. (See note to § 113).]

132. Example. Tell me directly the number (of

---

[1] Incidentally introduced in this place, showing a computation serviceable in prosody.—Súr. and *Mano.* To calculate the variations of verse, which are also found by the sum of permutations (§118).—Gan.

[2] Sanskrit prosody distinguishes metre in which the four verses of the stanza are alike, or the alternate ones only so, or all four dissimilar. Asiat. Res., Vol. X, syn. tab., v, vi and vii.

[3] The number of possible varieties of verse found by the rule of permutation (§113) is the same as the continued multiplication of two : this number being taken, because the varieties of syllables are so many, long and short. Accordingly this is assumed for the common multiplier. The product of its continued multiplication is to be found by this method of squaring and multipiying (§127) ; assuming for the period a number equal to that of syllables in the verse. The varieties of similar verses are the same as those of one verse containing twice as many syllables ; and the changes in the four verses are the same as those of one verse comprising four times as many syllables, excepting, however, that these permutations embracing all the possible varieties, comprehend those of like and half-alike metre. Wherefore the number first found is squared, and this again squared, corresponding to twice or four times the number of places ; and the roots of these squares are subtracted [for obtaining the varieties of alternately like and dissimilar verses respectively,—Ed.]—Gan.

varieties) of like, alternately like, and dissimilar verses respectively, in the metre named *anushtubh*.[1]

Statement : increasing multiplier, 2 ; period, 8.

Answer : variations of like verses, 256 ; of alternately alike verses, 65280 ; of dissimilar verses, 4294901760.

[The total number of syllables in the four *charanas* of the *anushtubh* metre being 32 (8 to each *charana*), the possible varieties of arrangements of long and short syllables in the metre are $2^{32} = 4294967296$. (See note to § 113). These evidently include cases of (*a*) all like, and (*b*) alternately like *charanas*. To find the number of these cases, we find the number of varieties of the syllables in *two charanas* ; which number is 2^{16} or 65536. It is clear that if we place each one of these varieties under itself, we shall get all the cases included in (*a*) and (*b*). Hence the total number of cases in (*a*) and (*b*) is 65536. Of these the number of cases in (*a*) clearly is the number of varieties that may occur in one *charana* $= 2^8 = 256$. Consequently the number of cases in (*b*) is $65536 - 256$ or 65280. Subtracting 65536 from 4294967296 we get 4294901760, which is considered as the number of *dissimilar* verses or *charanas*. It is to be observed, however, that these last include cases in which the first two *charanas* are like, as also the last two ; or cases in which the first two are like, but the last two unlike ; and so forth. These cases are not separately considered.]

[1] Asiat. Res., Vol. X, p. 438 ; syn. tab., p. 469.

CHAPTER VI.

PLANE FIGURE.[1]

133. Rule. A side[2] is assumed. The other side in

---

[1] *Kshetra-vyavahára*, determination of plane figure. *Kshetra*, as expounded by Ganesa, signifies plane surface bounded by lines, straight or curved; as triangle, &c. *Vyavahára* is the ascertainment of its dimensions, as diagonal, perpendicular, area, &c. Ganesa says plane figure is fourfold; triangle, quadrangle, circle and bow. Triangle (*tryasra, trikona* or *tribhuja*) is a figure containing (*tri*) three (*asra* or *kona*) angles, and consisting of as many (*bhuja*) sides. Quadrangle or tetragon (*chaturasra, chatushkona, chaturbhuja*), is a figure comprising (*chatur*) four (*asra*, &c.) angles or sides. The circle and bow, he observes, need no definition. Triangle is either (*játya*) right-angled, as that which is first treated of in the text; or it is (*tribhuja*) trilateral (and oblique) like the fruit of the *sringáta* (Trapa natans). This again is distinguished according as the (*lamba*) perpendicular falls within or without the figure: *viz.*, *antarlamba*, acute-angled; *bahirlamba*, obtuse-angled. Quadrangle also is in the first place twofold: with equal, or with unequal, diagonals. [This is not a proper classification.—Ed.] The first of these, or equidiagonal tetragon (*sama-karna*), comprises four distinctions: 1st, *sama-chaturbhuja*, equilateral, a square; 2d. *vishama-chaturbhuja*, a trapezium; 3d, *áyata-dírgha-chaturasra*, an oblique parallelogram; [this is not correct; for a parallelogram with equal diagonals must be either a rectangle or a square, so that this 3d. cannot be a distinct species.—Ed.]; 4th. *áyata-samalamba*, oblong with equal perpendiculars, *i.e.*, a rectangle. The second sort of quadrangle, or the tetragon with unequal diagonals (*vishamakarna*), embraces six sorts: 1st. *sama-chaturbhuja*, equilateral, a rhombus; 2d. *sama-tribhuja*, having three equal sides; 3d. *sama-dwi-dwi-bhuja*, consisting of two pairs of equal sides, a rhomboid; 4th *sama-dwi-bhuja*, having two equal sides; 5th. *vishama-chaturbhuja*, composed of four unequal sides, a trapezium; 6th. *sama-lamba*, having equal perpendiculars, a trapezoid. The several sorts of figures, observes the commentator, are fourteen; the circle and bow being but of one kind each. He adds, that pentagons (*panchásra*), &c. comprise triangles (and are reducible to them).

[2] *Báhu, dosh, bhuja*, and other synonyms of arm are used for the leg of a triangle, or side of a quadrangle or polygon; so called, as resembling the human arm.—Gan. and Súr.

the rival direction is called the upright,[1] whether in a triangle or tetragon, by persons conversant with the subject.

134. The square root of the sum of the squares of those legs is the diagonal.[2] The square root, extracted from the difference of the squares of the diagonal and side, is the upright; and that, extracted from the difference of the squares of the diagonal and upright, is the side.[3]

[Euclid I. 47.]

135.[4] Twice the product of two quantities, added to the square of their difference, will be the sum of their squares. The product of their sum and difference will be the difference of their squares: as must be everywhere understood by the intelligent calculator.[5]

---

[1] Either leg being selected to retain this appellation, the others are distinguished by different names. That which proceeds in the opposite direction, meaning at right angles, is called *koti*, *uchchhráya*, *uchchhriti*, or any other term signifying upright or elevated. Both are alike sides of the triangle or of the tetragon, differing only in assumed situation and name.—Gan. and Súr.

[2] A thread or oblique line from the two extremities of the legs, joining them, is the *karna*, also termed *sruti*, *sravana*, on any other word signifying ear. It is the diagonal of a tetragon.—Súr., Rang., &c. Or, in the case of a triangle, it is the diagonal of the parallelogram, whereof the triangle is the half: and is the hypotenuse of a right-angled triangle.

[3] The rule concerns (*játya*) right-angled triangles. The proof is given both algebraically and geometrically by Ganesa (*upapatti avyakta-kriyayá*, proof by algebra; *kshetragatopapatti*, geometrical demonstration); and the algebraical proof is also given by Súryadása. Ranganátha cites one of those demonstrations from his brother Kamalákara, and the other from his father Nrisinha, in the *Vártika*, or critical remarks on the (*Vásaná*) annotations of the *Siromani*; and censures the *Sringára-tilaka* for denying any proof of the rule besides experience. Bháskara has himself given a demonstration of the rule in his *Víja-ganita*, § 146.

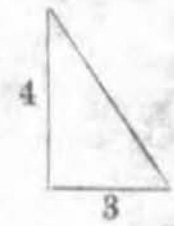
[4] A stanza of six *charanas* of *anushtubh* metre.

[5] Ganesa here also gives both an algebraic and a geometrical proof of the latter rule; and an algebraical one only of the first. See *Víja-ganita* under § 148, whence the latter demonstration is borrowed; and § 147, where the first of the rules is given and demonstrated.

$[2ab + (a-b)^2 = a^2 + b^2.$
$(a+b)\ (a-b) = a^2 - b^2.$

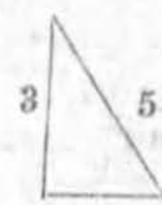
Geometrical proofs of these formulæ are furnished by Euc. II. 5 and 9. The object of introducing them here is to facilitate the calculations required in § 134.]

136. Example. Where the upright is four and the side three, what is the hypotenuse? Tell me also the upright from the hypotenuse and side; and the side from the upright and hypotenuse.



Statement: side 3; upright 4.

Sum of their squares 25. Or, the product of the sides, doubled, 24; square of the difference, 1; added together, 25. The square root of this is the hypotenuse 5.

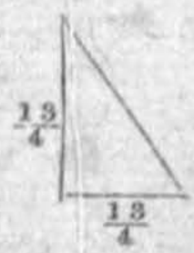


Difference of the squares 5 and 3 is 16. Or the sum 8 multiplied by the difference 2, makes 16. Its square root is the upright 4.

Difference of squares, found as before, 9. Its square root is the side 3.

137. Example. Where the side measures three and a quarter, and the upright, as much; tell me quickly, mathematician, what the length of the hypotenuse is.

Statement: side $\frac{13}{4}$; upright $\frac{13}{4}$. Sum of the squares $\frac{338}{16}$ or $\frac{169}{8}$. Since this has no (assignable) root, the hypotenuse is a surd. A method of finding its approximate root follows:—



138. Rule. From the product of numerator and

denominator,[1] multiplied by any large square number assumed, extract the square root: that, divided by the denominator taken into the root of the multiplier, will be an approximation.

The square of the above hypotenuse, $\frac{169}{8}$ (is proposed). The product of its numerator and denominator is 1352. Multiplied by a myriad (the square of a hundred) the product is 13520000. Its root is 3677 nearly.[2] This divided by the denominator taken into the square root of the multiplier, *viz.*, 800, gives the approximate root $4\frac{477}{800}$. It is the hypotenuse. So in every similar instance.

[$\sqrt{\frac{a}{b}} = \frac{\sqrt{a \times b}}{b} = \frac{\sqrt{a \times b \times c^2}}{bc}$.

The object of multiplying the product of numerator and denominator by a large square number (being some power of ten, as the above process shows), and then taking the square root approximately is practically to get the square root to a certain number of decimal places. Bháskara, however, does not use the decimal notation which was probably not known in his time, and expresses the result as a fraction. In the above example, the result obtained will be found to be correct to two decimal places.]

139. Rule.[3] A side is put. From that multiplied by twice some assumed number, and divided by one less than the square of the assumed number, an upright is obtained. This, being set apart, is multiplied by the arbitrary number, and the side as put is subtracted;

[1] If the surd be not a fraction, unity may be put for the denominator, and the rule holds good.—Gan.

[2] The remainder being unnoticed.

[3] Either the side or upright being given, to find the other two sides.—Súr. To find the upright and hypotenuse from the side; or the side and hypotenuse from the upright.—Gan. The problem is an indeterminate one, as is intimated by the author.

the remainder will be the hypotenuse. Such a triangle is termed right-angled.[1]

[Let a denote the given side, and n the assumed number. Then proceeding by the rule, we get $\frac{2an}{n^2-1}$ for upright, and $\frac{2an}{n^2-1}\times n - a = a\,\frac{n^2+1}{n^2-1}$ for hypotenuse. To verify this we have $a^2 + \left(\frac{2an}{n^2-1}\right)^2 = \frac{a^2}{(n^2-1)^2}\left\{(n^2-1)^2 + 4n^2\right\} = \frac{a^2\,(n^2+1)^2}{(n^2-1)^2}$ $= \left(a\,\frac{n^2+1}{n^2-1}\right)^2$. The proof of this rule given by Súryadása shows how these expressions for the upright and hypotenuse are arrived at, although it is rather difficult to follow it. The quantities $2n$, n^2-1, n^2+1 may be taken to represent the upright, side and hypotenuse of a right-angled triangle, because $(2n)^2 + (n^2-1)^2 = (n^2+1)^2$. Now consider another right-angled triangle similar to the above, the side being a. Then, since the sides of the two triangles are proportional, the upright of the second triangle will obviously be $\frac{2an}{n^2-1}$. Again, as $n \times$ upright of the first triangle = its side + its hypotenuse, so $n \times$ upright of the second triangle = its side + its hypotenuse. Thus, $n \times \frac{2an}{n^2-1} = a +$ hypotenuse, whence hypotenuse $= n \times \frac{2an}{n^2-1} - a$. Thus we see how the expressions are got.]

140. Or a side is put. Its square, divided by an arbitrary number, is set down in two places: and the arbitrary number being added and subtracted, and the sum and difference halved, the results are the hypotenuse and upright.[2] Or, in like manner, the side and

---

[[1] Colebrooke uses the word *rectangular*. But the more usual word in modern geometry is *right-angled*.—Ed.]

[2] Assume any number for the difference between the upright and hypotenuse. The difference of their squares (which is equal to the square of the given side) being divided by that assumed difference, the quotient is the sum of the upright and hypotenuse. For the difference of the squares is

hypotenuse may be deduced from the upright. Both results are rational quantities.

[Let a denote the given side, and n the assumed number. Then by the rule we have $\frac{1}{2}\left(\frac{a^2}{n}+n\right)$ for hypotenuse, and $\frac{1}{2}\left(\frac{a^2}{n}-n\right)$ for upright. To verify this we have $a^2+\frac{1}{4}\frac{(a^2-n^2)^2}{n^2}$ $=\frac{4a^2n^2+(a^2-n^2)^2}{4n^2}=\frac{(a^2+n^2)^2}{4n^2}=\left\{\frac{1}{2}\left(\frac{a^2}{n}+n\right)\right\}^2$. Ganesa gives an elegant demonstration of this rule, based on the fact that the assumed number n is the difference between the hypotenuse and upright, as is obviously the case. See foot-note.]

141. Example. The side being in both cases twelve, tell quickly by both methods, several uprights and hypotenuses, which shall be rational numbers.

Statement: side 12; assumption 2. The side, multiplied by twice that, *viz.*, 4, is 48. Divide by the square of the arbitrary number less one, *viz.*, 3, the quotient is the upright 16. This upright multiplied by the assumed number is 32, from which subtract the given side; the remainder is the hypotenuse 20.

Assume 3. The upright is 9, and the hypotenuse 15. Or, putting 5, the upright is 5, and the hypotenuse 13.

By the second method: the side, as put, 12. Its square 144. Divide by 2, the arbitrary number being 2, the quotient is 72. Add and subtract the arbitrary number, and halve the sum and difference. The hypotenuse and upright are found, *viz.*, hypotenuse 37, upright 35.

---

equal to the product of the sum and difference of the roots (§ 135). The upright and hypotenuse are therefore found by the rule of concurrence (§ 55).—Gan.

Assume 4. The upright is 16, and the hypotenuse 20. Assuming 6, the upright is 9, and the hypotenuse 15.[1]

142. Rule.[2] Twice the hypotenuse taken into an arbitrary number, being divided by the square of the arbitrary number added to one, the quotient is the upright. This taken apart is to be multiplied by the number put: the difference between the product and the hypotenuse is the side.

[Let a denote the hypotenuse, and n the assumed number. Then, by the rule, the upright is $\frac{2an}{n^2+1}$, and the side, $\frac{2an^2}{n^2+1}-a$ $=a\frac{n^2-1}{n^2+1}$. For $\left(\frac{2an}{n^2+1}\right)^2+\left(a\frac{n^2-1}{n^2+1}\right)^2=a^2$. The proof of this rule given by Súryadása is exactly similar to that of the rule in §139. See note to §139.]

143. Example. The hypotenuse being measured by eighty-five, say promptly, learned man, what uprights and sides will be rational.

Statement: hypotenuse 85. This doubled is 170, and multiplied by an arbitrary number two is 340. This, divided by the square of the arbitrary number added to one, *viz.*, 5, is the upright 68. This upright multiplied by the arbitrary number makes 136; and subtracting the hypotenuse, the side comes out 51. Or putting four, the upright will be 40, and the side 75.

144. Rule. Or else the hypotenuse is doubled and divided by the square of an assumed number added to one. The hypotenuse less that quotient is the upright.

---

[1] In like manner, if the upright be given 16, its square 256 divided by the arbitrary number 2 is 128. The arbitrary number, subtracted and added, makes 126 and 130; which halved give the side 63, and the hypotenuse 65.—Gang. and Súr.

[2] From the hypotenuse given, to find the side and upright in rational numbers.—Gan. The problem is an indeterminate one.

The same quotient multiplied by the assumed number is the side.[1]

The same hypotenuse 85. Putting two, the upright and side are 51 and 68. Or, with four, they are 75 and 40.

Here the distinction between side and upright is in name only, and not essential.

[Taking a and n as in § 142, we get $a-\frac{2a}{n^2+1}=a\frac{n^2-1}{n^2+1}$ for upright, and $\frac{2an}{n^2+1}$ for side. Verification as in § 142. To see how these expressions are arrived at, take $2n$, n^2-1 and n^2+1 for the side, upright and hypotenuse of a right-angled triangle, and proceed as in § 139, note.]

145. Rule.[2] Let twice the product of two assumed numbers be the upright; and the difference of their squares, the side: the sum of their squares will be the hypotenuse, and a rational number.

[Let a and b be the assumed numbers.

Then $2ab$ is the upright, and a^2-b^2 the side. The hypotenuse is $\sqrt{(2ab)^2+(a^2-b^2)^2}=a^2+b^2$.

Thus the three sides are all rational. Ganesa gives a proof of this rule after the manner of the *Vija-ganita*; but it is very obscure and cannot be easily followed.]

146. Example. Tell quickly, friend, three numbers, none being given, with which as upright, side and hypotenuse, a right-angled triangle may be (constructed.)

---

[1] This and the preceding rule are founded on the same principle, differing only in the order of the operation and names of the sides: the same numbers come out for the side and upright in one mode, which were found for the upright and side by the other.

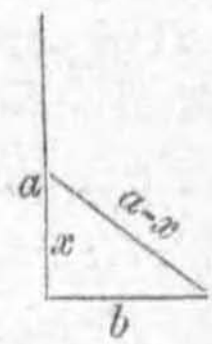
[2] Having taught the mode of finding a third side from any two of hypotenuse, upright and side; and in like manner from one, the other two; the author now shows a method of finding all three rational (none being given).—Gan. The problem is an indeterminate one.

Let two numbers be put, 1 and 2. From these, the side, upright and hypotenuse are found, 4, 3, 5. Or, putting 2 and 3, the side, upright and hypotenuse deduced from them are, 12, 5, 13. Or let the assumed numbers be 2 and 4 : from which will result 16, 12, 20. In like manner, manifold (answers are obtained).

147. Rule.[1] The square of the ground intercepted between the root and tip is divided by the (length of the) bambu, and the quotient severally added to, and subtracted from, the bambu : the moieties (of the sum and difference) will be the two portions of it representing hypotenuse and upright.[2]

[The rule bears reference to the example which follows.

Let a denote the height of the bambu, b, the distance between root and tip, and x the height at which the bambu is broken.



Then, $b^2=(a-x)^2-x^2$

$$\therefore \frac{b^2}{(a-x)+x}=(a-x)-x$$

$$i.e.\ \frac{b^2}{a}=(a-x)-x$$

also $a=(a-x)+x$, identically.

Hence $\frac{1}{2}\left(a+\frac{b^2}{a}\right)=a-x$ or hypotenuse, and $\frac{1}{2}\left(a-\frac{b^2}{a}\right)=x$ or upright, whence the rule. The above proof is the same as that given by Ganesa. See foot-note.]

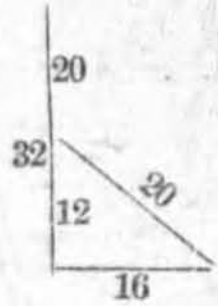
---

[1] The sum of hypotenuse and upright being known, as also the side, to discriminate the hypotenuse and upright.—Gan.

[2] The height from the root to the fracture is the upright. The remaining portion of the bambu is the hypotenuse. The whole bambu, therefore, is the sum of hypotenuse and upright. The ground intercepted between the root and tip is the side: it is equal to the square root of the difference between the squares of the hypotenuse and upright. Hence the square of the side, divided by the sum of hypotenuse and upright, is their difference (§ 135). With these (sum and difference) the upright and hypotenuse are found by the rule of concurrence (§ 55).—Gan.

148. Example. If a bambu, measuring thirty-two cubits and standing upon level ground, be broken in one place by the force of the wind, and the tip of it meet the ground at sixteen cubits : say, mathematician, at how many cubits from the root it is broken.

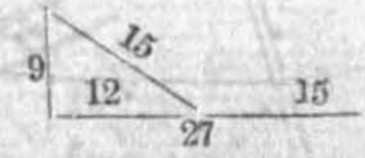
Statement. Bambu 32. Interval between the root and tip of the bambu, 16. It is the side of the triangle. Proceeding as directed, the upper and lower portions of the bambu are found to be 20 and 12.



149. Rule.[1] The square (of the height) of the pillar is divided by the distance of the snake from his hole; the quotient is to be subtracted from that distance. The meeting of the snake and peacock is from the snake's hole half the remainder, in cubits.

150. Example. A snake's hole is at the foot of a pillar, nine cubits high, and a peacock is perched on its summit. Seeing a snake at the distance of thrice the pillar gliding towards his hole, he pounces obliquely upon him. Say quickly at how many cubits from the snake's hole they meet, both proceeding an equal distance.

Statement. Pillar 9. It is the upright. Distance of the snake from his hole, 27. It is the sum of hypotenuse and side. Proceeding



---

[1] The sum of the side and hypotenuse being known, as also the upright, to discriminate the hypotenuse and side.—Gan. The rule bears reference to the example which follows. The principle is the same as that of the preceding rule.

as directed, the distance between the hole and the place of meeting is found to be 12 cubits.[1]

[The principle of the rule in § 149 is, as Colebrooke observes, and as is also evident from the example in §150, the same as that of the rule in § 147. The peacock is supposed not to change his direction, and to pounce in such a direction that the distance traversed by him being the hypotenuse of a right-angled triangle, is equal to the distance traversed by the snake. Practically, however, such a thing does not happen; but the bird of prey changes its direction at every instant, and describes a curved path known as the curve of pursuit. See Tait and Steele's Dynamics of a Particle, Art. 33.

Let a denote the distance of the snake from the hole, b the height of the pillar, and x the distance required.

Then, $b^2=(a-x)^2-x^2$, whence as in § 147, $x=\frac{1}{2}\left(a-\frac{b^2}{a}\right)$. Hence the rule.]

151. Rule.[2] The quotient of the square of the side divided by the difference between the hypotenuse and upright is twice set down; and the difference is subtracted from the quotient (in one place) and added to it (in the other). The moieties (of the remainder and sum) are in their order the upright and hypotenuse.[3]

This[4] is to be generally applied by the intelligent mathematician.

---

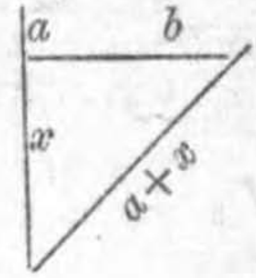
[1] Subtracted from the sum of hypotenuse and side, this leaves 15 for the hypotenuse. The snake had proceeded the same distance of 15 cubits towards his hole, as the peacock in pouncing upon him. Their progress is therefore equal.—Súr.

[2] The difference between the hypotenuse and upright being known, as also the side, to find the upright and hypotenuse.—Gan.

[3] The demonstration, distinctly set forth under a preceding rule, is applicable to this.—Gan.

[4] Beginning from the instance of the broken bambu (§ 147) and including what follows.—Gan.

[The demonstration of the rule in §147 applies to this rule as Ganesa observes.

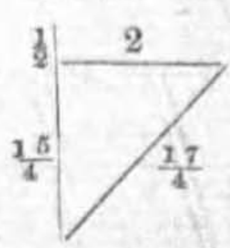


Let a denote the difference between hypotenuse and upright, b the side, and x the upright.

Then, $b^2=(a+x)^2-x^2$, whence as in § 147, $x=\frac{1}{2}\left(\frac{b^2}{a}-a\right)$, and $a+x=\frac{1}{2}\left(\frac{b^2}{a}+a\right)$. Hence the rule.]

152. Friend, the space between the lotus (as it stood) and the spot where it submerged, is the side. The lotus as seen (above water) is the difference between the hypotenuse and upright. The stalk is the upright, for the depth of water is measured by it. Say, what the depth of the water is.[1]

153. Example.[2] In a certain lake swarming with ruddy geese[3] and cranes, the tip of a bud of lotus was seen a span above the surface of the water. Forced by the wind it gradually advanced, and was submerged at the distance of two cubits. Compute quickly, mathematician, the depth of the water.



Statement. Diff. of hypotenuse and upright, $\frac{1}{2}$ cubit. Side 2 cubits. Proceeding as directed, the upright is found $\frac{15}{4}$. It is the depth of the water. Adding to it the height of the bud, the hypotenuse comes out $\frac{17}{4}$.

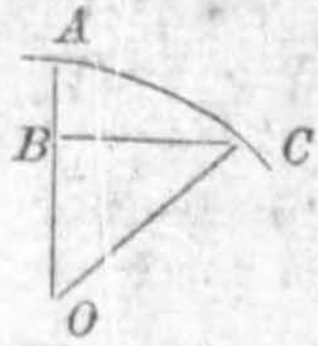
---

[1] The sides constituting the figure in the example which follows, are here set forth, to assist the apprehension of the student.—Súr. and Gan.

[2] [This example is inserted in Barnard Smith's Arithmetic, Appendix, p. 300.—Ed.]

[3] Anas Casarca.

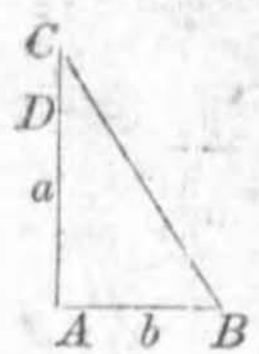
[Let O be the root of the lotus, A its tip, and C the point on the surface of the water where it is submerged. Then, while it advances by the force of the wind, O remains fixed, and the lotus describes an arc of a circle, of which O is the centre, and OA the radius. Hence $OC = OA$. Then, as the author himself explains in § 152, the solution will follow from the method of § 151, where we have only to substitute $\frac{1}{2}$ for a, and 2 for b.]



154. Rule.[1] The height of the tree multiplied by its distance from the pond, is divided by twice the height of the tree added to the space between the tree and the pond : the quotient will be the measure of the leap.

[The rule refers to the example which follows.

Let D be the top of the tree, and B the position of the pond. The first ape is supposed to descend from D to A, and then to go from A to B; while the second ape is supposed to jump vertically upwards from D to C, and then to leap directly from C to B. Now let $AD = a$, $AB = b$, and $CD = x$, which is required. Then by the question, we have $x + \sqrt{(a+x)^2 + b^2} = a + b$;



$$\therefore (a+x)^2 + b^2 = (a+b)^2 - 2(a+b)x + x^2,$$

whence $x = \frac{ab}{2a+b}$. Hence the rule.]

155. Example. From a tree a hundred cubits high, an ape descended and went to a pond two hundred cubits distant : while another ape, vaulting to some height off the tree, proceeded with velocity diagonally

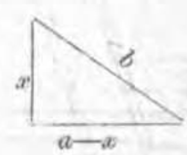
[1] The sum of the hypotenuse and upper portion of the upright being given, and the lower portion being known, as also the side : to discriminate the upper portion of the upright from the hypotenuse.—Gan. As in several preceding instances, a reference to the example is requisite to the understanding of the rule.

to the same spot. If the space travelled by them be equal, tell me quickly, learned man, the height of the leap, if thou have diligently studied calculation.

Statement. Tree 100 cubits. Distance of it from the pond 200. Proceeding as directed, the height of the leap comes out 50.

156. Rule.[1] From twice the square of the hypotenuse subtract the sum of the upright and side multiplied by itself, and extract the square root of the remainder. Set down the sum twice, and let the root be subtracted in one place and added in the other. The moieties will be measures of the side and upright.[2]

[Let a denote the sum of side and upright, and b the hypotenuse. Also let x denote the upright. Then we have evidently, $(a-x)^2+x^2=b^2$;



$\therefore\ 2x^2-2ax+a^2-b^2=0,$

whence $x=\dfrac{a\pm\sqrt{2b^2-a^2}}{2}$.

If we take the upper sign, then $a-x=\dfrac{a-\sqrt{2b^2-a^2}}{2}$, and

---

[1] Hypotenuse being known, as also the sum of the side and upright, or their difference; to discriminate those sides.—Gan.

[2] In like manner, the difference of the side and upright being given, the same rule is applicable.—Gan. [A slight variation will be necessary; see note to § 158.—Ed.] The principle of the rule is this: the square of the hypotenuse is the sum of the squares of the sides. But the sum of the squares with twice the product of the sides added to it is the square of the sum; and with the same subtracted is the square of the difference. Hence cancelling equal quantities affirmative and negative, twice the square of the hypotenuse will be the sum of the squares of the sum and difference. Therefore, subtracting from twice the square of hypotenuse the square of the sum, the remainder is the square of the difference; or conversely, subtracting the square of the difference, the residue is the square of the sum. The square root is the sum or difference. With these, the sides are found by the rule of concurrence.—Gan. and Súr.

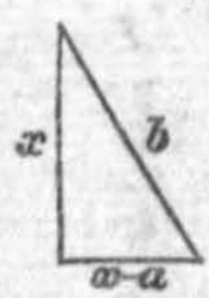
if we take the lower sign, then $a-x=\frac{a+\sqrt{2b^2-a^2}}{2}$. It is evident that we may take either sign. The reason for the rule is obvious. The method of solving an adfected quadratic equation by completing the square has been mentioned before. See §§ 62--63. Interesting geometrical interpretations of the expressions for x and $a-x$ are given by Ganesa and Súryadása. See foot-note.]

157. Example. Where the hypotenuse is seven above ten ; and the sum of the side and upright, three above twenty ; tell them to me, my friend.

Statement : Hypotenuse 17 ; sum of side and upright 23. Proceeding as directed, the side and upright are found 8 and 15.

158. Example.[1] Where the difference of the side and upright is seven, and hypotenuse is thirteen, say quickly, eminent mathematician, what the side and upright are.

Statement : hypotenuse 13 ; difference of side and upright 7. Proceeding as directed, the side and upright come out 5 and 12.



[Let x denote the upright, and a the diff. between upright and side, the upright being supposed > side. Then $x-a$ will denote the side ; and we evidently have $x^2+(x-a)^2=b^2$, whence as in §156, $x=\frac{a\pm\sqrt{2b^2-a^2}}{2}$. But here we must take the upper sign alone, since $x-a$ is necessarily positive. Thus we have $x=\frac{\sqrt{2b^2-a^2}+a}{2}$, and $x-a=\frac{\sqrt{2b^2-a^2}-a}{2}$.

[1] This example of a case where the difference of the sides is given, is omitted by Súryadása, but noticed by Ganesa. Copies of the text vary : some containing, and others omitting, the instance.

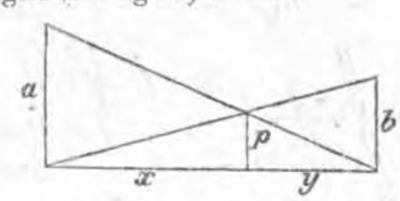
From these values it is clear that the rule in § 156 must be slightly varied in order to be applicable to the present case.]

159. Rule.[1] The product of two erect bambus being divided by their sum, the quotient is the perpendicular[2] from the junction (intersection) of threads passing reciprocally from the root (of one) to the tip (of the other). The two bambus, multiplied by an assumed base, and divided by their sum, are the portions of the base on the respective sides of the perpendicular.

[From similar triangles (see figure) we have

$$\frac{p}{a} = \frac{y}{x+y},$$

$$\frac{p}{b} = \frac{x}{x+y};$$



$$\therefore\ p\left(\frac{1}{a}+\frac{1}{b}\right)=1, \text{ and } \therefore\ p=\frac{ab}{a+b}.$$

Thus p is independent of x and y, provided a and b be given.

Again, let $x+y=k$, any assumed number.

Then $x=\frac{pk}{b}=\frac{ak}{a+b}$,

and $y=\frac{pk}{a}=\frac{bk}{a+b}$.

This rule shows that the property of similar triangles was known. See Bháskara's remark at the end of § 160, and the proof given by Ganesa, cited in the foot-note, which is slightly different and more cumbrous.]

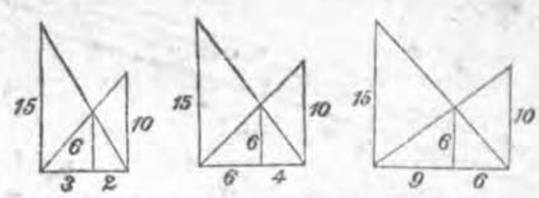
160. Example. Tell the perpendicular drawn from the intersection of strings stretched mutually from the roots to the summits of two bambus fifteen and ten cubits high, standing upon ground of unknown extent.

[1] Having taught fully the method of finding the sides in a right-angled triangle, the author next propounds a special problem.—Gan. To find the perpendicular, the base being unknown.—Súr.

[2] *Lamba, avalamba, valamba, adholamba*, the perpendicular.

Statement : bambus 15, 10. The perpendicular is found 6.

Next to find the segments of the base. Let the ground be assumed 5 ; the segments come out 3 and 2. Or putting 10, they are 6 and 4. Or taking 15, they are 9 and 6. See the figures. In every instance the perpendicular is the same,[1] *viz.*, 6.



The proof is in every case by the rule of three : if with a side equal to the base, the bambu be the upright, then with the segment of the base, what will be the upright ?[2]

161. Aphorism.[3] That figure, though rectilinear, of which sides are proposed by some presumptuous

---

[1] However the base may vary by assuming a greater or less quantity for it, the perpendicular will always be the same.—Gan.

[2] On each side of the perpendicular, are segments of the base relative to the greater and smaller bambus, and larger or less analogously to them. Hence this proportion : "If with the sum of the bambus, this sum of the segments equal to the entire base be obtained, then, with the smaller bambu, what is had ?" [This proportion cannot be at once obtained easily, but may be got by dividing corresponding members of the first two equations in the note to §159, whence we have $\frac{a}{b}=\frac{x}{y}$, and therefore $\frac{a+b}{x+y}=\frac{b}{y}$.—Ed.] The answer gives the segment which is relative to the least bambu. Again : "If with a side equal to the whole base, the higher bambu be the upright, then with a side equal to the segment found as above, what is had ?" The answer gives the perpendicular let fall from the intersection of the threads. Here a multiplicator and a divisor equal to the entire base are both cancelled as equal and contrary : and there remain the product of the two bambus for numerator and their sum for denominator. Hence the rule.—Gan.

[3] The aphorism explains the nature of impossible figures proposed by dunces.—Súr. It serves as a definition of plane figure (*kshetra*).—Gan. In a triangle or other plane rectilinear figure, one side is always less than the sum of the rest. If equal, the perpendicular is nought, and there is no complete figure. If greater, the sides do not meet.—Súr. Containing no area, it is no figure.—*Kaumudí* cited by Ranganátha.

person, wherein one side[1] exceeds or equals the sum of the other sides, may be known to be no figure.

[This follows from Euclid I. 20. Any side of a triangle or of any polygon must be less than the sum of the remaining sides. Hence the numbers 2, 3, 6, 12 cannot represent the sides of a quadrilateral.]

162. Example. Where sides are proposed two, three, six and twelve in a quadrilateral, or three, six and nine in a triangle, by some presumptuous dunce, know it to be no figure.

Statement. The figures are both incongruous. Let straight rods exactly of the lengths of the proposed sides be placed on the ground, the incongruity will be apparent.[2]

163—164. Rule[3] in two couplets. In a triangle, the sum of two sides being multiplied by their difference, is divided by the base[4]; the quotient is subtracted from, and added to, the base which is twice set down: and being halved, the results are segments corresponding to those sides.[5]

---

[1] The principal or greatest side.—Gan. *Kaum.* Rang.

[2] The rods will not meet.—Súr.

[3] In any triangle to find the perpendicular, segments and area. This is introductory to a fuller consideration of areas.—Gan. and Súr.

[4] *Bhúmi, bhú, ku, mahí* or any other term signifying earth; the ground or base of a triangle or other plane figure. Any one of the sides is taken for the base, and the rest are termed simply sides. Ganesa restricts the term to the greatest side. See note § 168.

Lamba, &c., the perpendicular. See note § 159. *Abádhá, abadhá, avabádhá*, segment of the base made by the perpendicular. These are terms introduced by earlier writers. These segments are internal in an acute-angled triangle, but external in an obtuse-angled one. *Phala, ganita, kshetra-phala, sama-koshtha-miti*: the measure of like compartments, or number of equal squares of the same denomination (as cubit, fathom, finger, &c.) in which the dimension of the side is given; the area or superficial content.—Gan. and Súr.

[5] The relative or corresponding segments. The smaller segment answers to the less side, and the larger to the greater side.—Gan.

The square root of the difference of the squares of the side and its own segment of the base becomes the perpendicular. Half the base multiplied by the perpendicular[1] is in a triangle the exact[2] area.[3]

[$x+y$ is given, as also a and b.

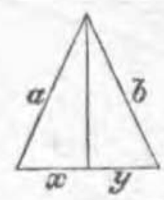
We have $a^2-b^2=x^2-y^2$,

$$\therefore \frac{a^2-b^2}{x+y}=x-y.$$

Then $(x+y)+(x-y)=2x$,

and $(x+y)-(x-y)=2y$;

and half of these results give x and y.



Also perpendicular $=\sqrt{a^2-x^2}$,

and area $=\frac{1}{2}\times$ base $\times$ altitude.

---

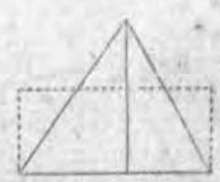
[1] Or half the perpendicular taken into the base.—Gan.

[2] *Sphuta-phala*, distinct or precise area; opposed to *asphuta*—or *sthûla-phala*, indistinct or gross area. See § 167.

[3] Demonstration. In both the right-angled triangles formed in the proposed triangle, one on each side of the perpendicular, this line is the upright; the side is hypotenuse, and the corresponding segment is side. Hence, subtracting the square of the perpendicular from the square of the side, the remainder is the square of the segment. So, subtracting the square of the other side, there remains the square of the segment answering to it. Their difference is the difference of the squares of the segments, and is equal to the difference of the squares of the sides, since an equal quantity has been taken from each; for any two quantities less an equal quantity have the same difference. It is equal to the product of the sum and difference of the simple quantities. Therefore, the sum of the sides multiplied by their difference is the difference of the squares of the segments. But the base is the sum of the segments. The difference of the squares, divided by that, is the difference of the segments. From which by the rule of concurrence (§ 55) the segments are found.

The square root of the difference between the squares of the side and segment (taken as hypotenuse and side) is the upright or perpendicular.

Dividing the triangle by a line across the middle (of the perpendicular), and placing the two parts of the upper portion disjoined by the perpendicular on the two sides of the lower portion (as in the annexed figure), an oblong is formed in which the half of the perpendicular is one side, and the base is the other. Wherefore half the perpendicular multiplied by the base is the area or number of equal compartments. Or, half the base multiplied by the



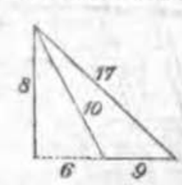
The above is practically the demonstration given by Ganesa, although it is rather long as it is expressed in words.]

165. Example. In a triangular figure in which the base is fourteen and its sides thirteen and fifteen, tell quickly the length of the perpendicular, the segments, and the dimension by like compartments termed area.

Statement: base 14; sides 13 and 15. Proceeding as directed, the segments are found, 5 and 9; the perpendicular, 12; the area 84.

166. Example. In a triangle, wherein the sides measure ten and seventeen, and the base nine, tell me promptly, expert mathematician, the segments, perpendicular, and area.

Statement: base 9; sides 10 and 17. By the rule in §163, the quotient found is 21. This cannot be subtracted from the base; wherefore the base is subtracted from it. Half the remainder is the segment, 6, and is negative, that is to say, in the contrary direction.[1] (See figure.) Thus the two segments are found 6 and 15.



---

perpendicular is just so much. In an obtuse-angled triangle also, the base multiplied by half the perpendicular is the area.—Gan.

[The proof given by Sûryadâsa is practically the same as the above, and so we omit it here.—Ed.]

[1] When the perpendicular falls without the base, as overpassing the angle in consequence of the side exceeding the base, the quotient found by the rule in § 163 cannot be taken from the base; for both origins of sides are situated in the same quarter from the fall of the perpendicular. Therefore subtracting the base from the quotient, half the residue is the segment and situated on the contrary side, being negative. Wherefore, as both segments stand on the same side, the smaller is comprehended in the greater, and, in respect of it, is negative. Thus all is congruous and unexceptionable.—Gan.

From which, both ways too, the perpendicular comes out 8. The area is 36.

[As the commentators observe, the segments of the base made by the perpendicular in an obtuse-angled triangle are external, and their algebraical sum is their arithmetical difference.]

167. Rule.[1] Half the sum of all the sides is set down in four places; and the sides are severally subtracted. The remainders being multiplied together, the square root of the product is the area inexact in the quadrilateral, but pronounced exact in the triangle.[2]

[Let s denote the sermi-perimeter of a triangle. Then area $=\sqrt{s\,(s-a)\,(s-b)\,(s-c)}$. See Todhunter's Trigonometry, Art. 247. This rule does not apply to the case of a quadrilateral. In fact, a quadrilateral in general is not determined by the

When the sum of the segments is to be taken, as they have contrary signs affirmative and negative, the difference of the quantities is that sum.—Súr. See *Víj-gan.* § 5.

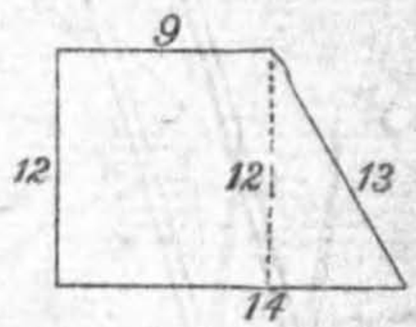
[1] For finding the gross area of a quadrilateral, and, by extension of the rule, the exact area of a traugle.—Gan. For finding the area by a method delivered by Sridhara—Rang.

[2] If the three remainders be added together, their sum is equal to half the sum of all the sides. The product of the continued multiplication of the three remainders being taken into the sum of those remainders, the product so obtained is equal to the product of the square of the perpendicular taken into the square of half the base. [It is not explained how this is the case. The last mentioned product $=\frac{1}{4}a^2\left\{b^2-\frac{1}{4}\left(a+\frac{b^2-c^2}{a}\right)^2\right\}$, (taking a as base, and supposing $b > c$, and applying §163–164) $=\frac{4a^2b^2-(a^2+b^2-c^2)^2}{16}=\frac{1}{16}(a+b+c)\,(a+b-c)\,(a-b+c)\,(b+c-a)=s\,(s-a)\,(s-b)(s-c)=(s-a)(s-b)(s-c)\,(s-a+s-b+s-c.)$—Ed.] It is a square quantity; for a square multiplied by a square gives a square. The square root being extracted, the product of the perpendicular by half the base is the result; and that is the area of the triangle. Therefore the true area is thus found. In a quadrilateral, the product of the multiplication does not give a square quantity, but an irrational one. Its approximate root is the area of the figure; not, however, the true one: for, when divided by the perpendicular, it should give half the sum of the base and summit.—Súr. [The last remark does not hold good unless the quadrilateral be a trapezium.—Ed.]

four sides alone without an angle, whereas a triangle is determined by its three sides. Hence it is incorrect to say that the expression derived from the above rule represents the gross area of a quadrilateral. See note to §§169—170. The proof of the rule given by Súryadása is not at all clear. See foot-note.]

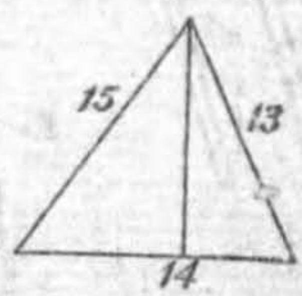
168. Example. In a quadrilateral figure, of which the base[1] is fourteen, the summit[2] nine, the flanks thirteen and twelve, and the perpendicular twelve, tell the area as it was taught by the ancients.

Statement: base 14; summit 9; sides 13 and 12; perpendicular 12. By the method directed, the result obtained is the surd 19800, of which the approximate root is somewhat less than 141. That, however, is not in this figure the true area. But, found by the method which will be set forth (§ 175), the true area is 138.



Statement of the triangle before instanced (§ 165).

By the (present) method the area comes out the same, *viz.*, 84.



169—170. Aphorism comprised in a stanza and a half. Since the diagonals of the quadrilateral are indeterminate, how should the area be in this case determinate? The diagonals found as assumed by the ancients[3] do not answer in another case. With the

---

[1] The greatest of the four sides is called the base.—Gan. This definition is, however, too restricted. See §§178, 185.

[2] *Mukha*, *vadana*, or any other word denoting mouth; the side opposite to the base, the summit.

[3] By Srídhara and the rest.—Gan.

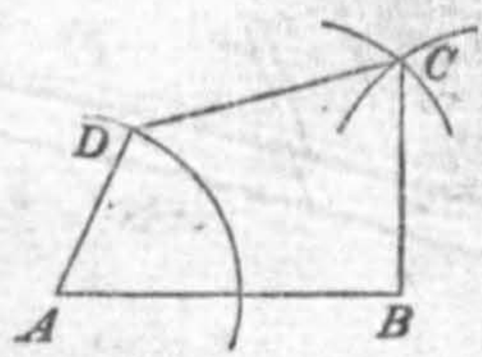
same sides, there are other diagonals; and the area of the figure is accordingly manifold.

For, in a quadrilateral, opposite angles being made to approach, contract their diagonal as they advance inwards: while the other angles receding outwards lengthen their diagonal. Therefore it is said that with the same sides there are other diagonals.

171. How can a person, neither specifying one of the perpendiculars, nor either of the diagonals, ask the rest?[1] Or how can he demand a determinate area, while they are indefinite?

172. Such a questioner is a blundering devil.[2] Still more so is he, who answers the question. For he considers not the indefinite nature of the lines[3] in a quadrilateral figure.

[The four sides alone without an angle do not determine the quadrilateral in general, and the area is consequently indeterminate. For, take AB as one side; and with centre A and radius equal to another side describe a circle; take *any point* D in the circumference, and join AD. With D, B as centres and radii equal to the other two sides, describe circles cutting each other in C. Join CB, CD. Then $ABCD$ is the quadrilateral which is indeterminate, since the angle BAD being not given, the point D may be taken anywhere in the circumference of the first described circle. Hence with the same sides, the diagonals may vary. But if the perpendicular from D to the line AB be given, or if the diagonal BD be given, it is easy to see that the point D becomes a fixed point in the circumference of the first de-



[1] The perpendiculars, diagonals, &c.—Gan.
[2] *Piśácha*, a demon or vampire; so termed because he blunders.—Súr.
[3] Of the diagonal and perpendicular lines.—Súr.

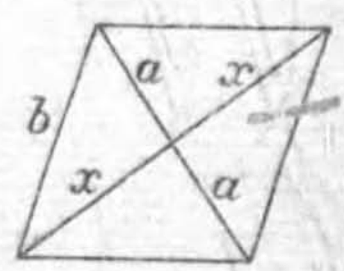
scribed circle, and so the quadrilateral is determinate. In the case of a trapezium, it is easily seen that the four sides being given, the distance between two paralled sides is known, and so the figure is determinate, The rule in §167, however, is equally inapplicable to this case.]

173—175. Rule[1] in two and a half stanzas. Let one diagonal of an equilateral tetragon be put as given. Then subtract its square from four times the square of the side. The square root of the remainder is the measure of the second diagonal.

The product of unequal diagonals multiplied together, being divided by two, will be the precise area in an equilateral tetragon. In a regular one with equal diagonals, as also in an oblong,[2] the product of the side and upright will be so.

In any other quadrilateral with equal perpendiculars, the moiety of the sum of the base and summit, multiplied by the perpendicular (is the area).

[In an equilateral tetragon, the diagonals bisect each other at right angles. Hence (see figure), $x = \sqrt{b^2 - a^2}$; $\therefore$ $2x$ = unknown diagonal = $\sqrt{4b^2 - 4a^2}$.



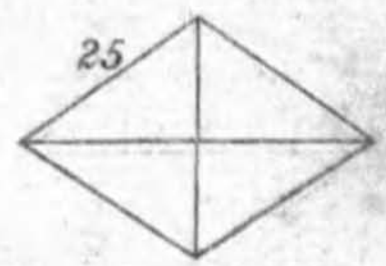
The area of the above figure is evidently equal to half the product of the diagonals. The other propositions stated above are well known elementary geometrical results. By "a quadrilateral with equal perpendiculars," the author means a trapezium.

[1] In an equilateral tetragon, one diagonal being given, to find the second diagonal and the area; also in an equiperpendicular tetragon (trapezium) to find the area.—Gan. Equilateral tetragons are two-fold: with equal and with unequal diagonals. The first rule regards the equilateral tetragon with unequal diagonals (the rhombus).—Súr.

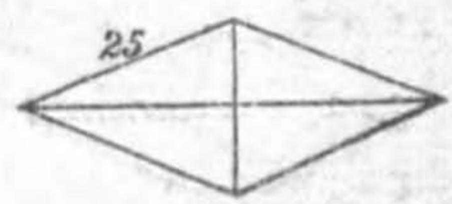
[2] *Áyata*, a long quadrilateral which has pairs of equal sides.—Gan.

176. Example. Mathematician, tell both diagonals and the area of an equilateral quadrangular figure whose side is the square of five: and the area of it, the diagonals being equal: also (the area) of an oblong, the breadth of which is six and length eight.

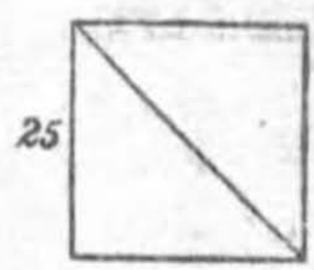
Statement of the first figure (rhombus). Here, assuming one diagonal 30, the other is found 40; and the area is 600.



Or put one diagonal 14; the other is found 48; and the area is 336. See figure.

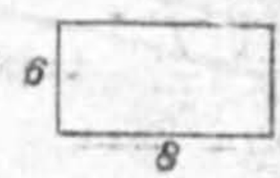


Statement of the second figure (square).



Here, taking the square root of the sum of the squares (§ 134), the diagonal comes out as the surd $\sqrt{1250}$, alike both ways. The area is 625.

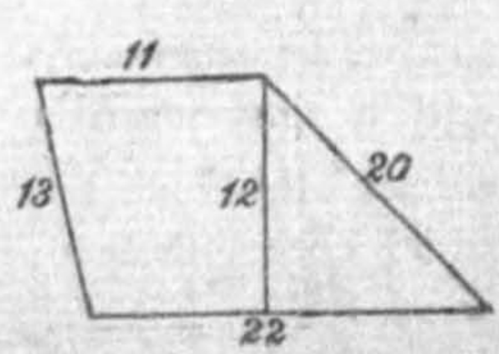
Statement of the third figure (oblong). Area 48.



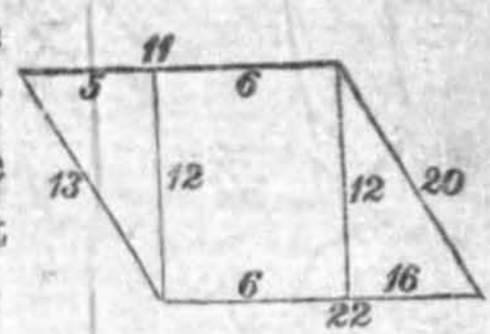
177. Example. Where the summit is eleven, the base twice as much as the summit, the flanks thirteen and twenty, and the perpendicular twelve; say what the area will be.

Statement:

The gross area (§ 167) is 250. The true area (§175) is 198.



Or making three portions of the figure, and severally finding their areas, we get 30, 72, 96 (see figure); and summing up we get for the total area 198 as before.



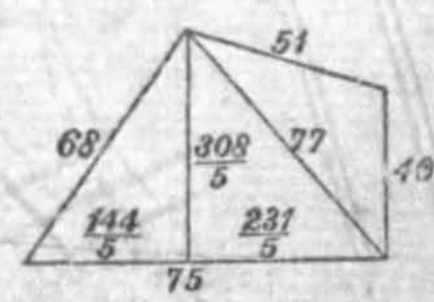
[The four sides of the trapezium being given, the perpendicular is necessarily known.]

178. Another example. Declare the diagonal, perpendicular and dimensions of the area, in a figure of which the summit is fifty-one, the base seventy-five, the left side sixty-eight, and the other side twice twenty.

179. Aphorism showing the connection of area, perpendicular and diagonal. If the perpendicular be known, the diagonal is so; if the diagonal be known, the perpendicular is so. If they be definite, the area is determinate. For, if the diagonal be indefinite, so is the perpendicular. Such is the meaning.

179 *continued.* Rule for finding the perpendicular.[1] In the triangle within the quadrilateral, the perpendicular is found as before taught (§163—164); the diagonal and side being sides, and the base, a base.

Here, to find the perpendicular, a diagonal proceeding from the extremity of the left side to the origin of the right one is assumed to be 77; see figure. By this a triangle is constituted within the quadrilateral. In it that diagonal is one side, 77; the left side is another, 68; the base continues such, 75. Then, proceeding by the rule (§§163—164),



[1] The diagonal being either given or assumed.—Gan.

the segments are found, $\frac{144}{5}$ and $\frac{231}{5}$; and the perpendicular, $\frac{308}{5}$.

[The problem given is indeterminate, unless a diagonal, or an angle, or a perpendicular distance be given. So one diagonal is supposed to be 77. The process then adopted is the same as that shown in the note to §§163—164.]

180. Rule to find the diagonal, when the perpendicular is known.

The square root of the difference of the squares of the perpendicular and its adjoining side is pronounced the segment. The square of the base less that segment being added to the square of the perpendicular, the square root of the sum is the diagonal.

In the above quadrilateral, the perpendicular from the extremity of the left side is put $\frac{308}{5}$. Hence the segment is found $\frac{144}{5}$; and by the rule (§180) the diagonal comes out 77.

[The reason for the rule is manifest.]

181—182. Rule to find the second diagonal: two stanzas.

In this figure, first a diagonal is assumed.[1] In the two triangles situated one on each side of the diagonal, this diagonal is made the base of each; and the other sides are given: the perpendiculars and segments[2] must be found. Then the square of the difference of two segments on the same side[3] being added to the square of the sum of the perpendiculars, the square root of

---

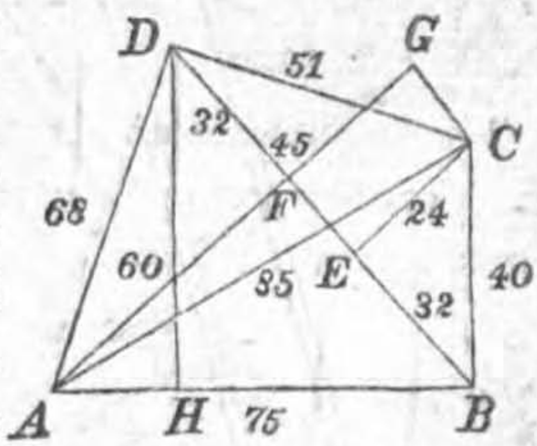
[1] Either arbitrarily (see §183) or as given by the conditions of the question.—Gan.

[2] The two perpendiculars and the four segments.—Gan.

[3] Square of the interval of two segments measured from the same extremity.

the sum of those squares will be the second diagonal in all tetragons.[1]

In the same quadrilateral, the length of the diagonal passing from the extremity of the left side to the origin of the right one, is put 77. Within the figure cut by that diagonal line, two triangles are formed, one on each side of the diagonal. Taking the diagonal for the base of each, and the two other sides as given, the two perpendiculars and the several segments must be found by the method before taught. Thus the perpendiculars are found, 24 and 60. Segments of the base made by the former, 45 and 32 ; those made by the latter, 32 and 45. Difference of the segments on the same side (that is, so much of the base as is intercepted between the perpendiculars) is 13. Its square 169. Sum of the perpendiculars 84. Its square 7056. Sum of the squares 7225. Square root of the sum 85. It is the length of the second diagonal. So in every like instance.



---

[1] In the figure which is divided by the diagonal line, two triangles are contained, one on each side of that line; and their perpendiculars, which fall one on each side of the diagonal, are thence found. The difference between two segments on the same side will be the interval between the perpendiculars. It is taken as the upright of a triangle. Producing (see above figure) one perpendicular by the addition of the other, (*i. e.*, drawing CG perpendicular to AF produced), the sum (AG) is made the side of the triangle. The second diagonal (AC) is hypotenuse. A triangle (AGC) is thus formed. From this is deduced, that the square root of the sum of the squares of the upright (which $= CG = EF = BF - BE$) and side (which $= AG = AF + CE$) will be the second diagonal: and the rule is demonstrated.—Gan.

In an equilateral tetragon, there is no interval between the perpendiculars; and the second diagonal is the sum of the perpendiculars.—*Ibid.*

[The reason for the rule will be clear from the explanation given by Ganesa, cited in the foot-note. In the particular example chosen, it happens (see above figure) that $DF=EB$, but this need not always be the case. It also happens from the values of AB, BC, CA, that the angle ABC is a right angle.]

183. Rule restricting the arbitrary assumption of a diagonal: a stanza and a half. The sum of the shortest pair of sides containing the diagonal being taken as a base, and the remaining two as the legs (of a triangle), the perpendicular is to be found: and, in like manner, with the other diagonal. The diagonal cannot by any means be longer than the corresponding base, nor shorter than the perpendicular answering to the other. Adverting to these limits, an intelligent person may assume a diagonal.

For a quadrilateral, contracting as the opposite angles approach, becomes a triangle; wherein the sum of the least pair of sides about one angle is the base, and the other two are taken as the legs. The perpendicular is found in the manner before taught. Hence the shrinking diagonal cannot by any means be less than the perpendicular; nor the other be greater than the base. It is so both ways. This, even though it were not mentioned, would be readily perceived by the intelligent student.

[What the author intends to say is (see figure to §§ 181-182) that the diagonal BD cannot be longer than $DC+BC$, but always shorter; nor can it be shorter than the perpendicular DH, but always longer. (Euclid, I. 20 and 19). The first sentence of the above section is meaningless; and so also is the proof given, *viz.*, "For a quadrilateral, contracting, &c."]

184. Rule to find the area: half a stanza. The

sum of the areas of the two triangles on either side of the diagonal is assuredly[1] the area in this figure.

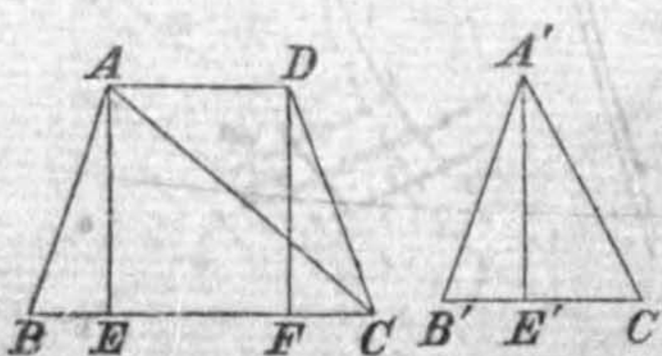
In the figure last specified, the areas of the two triangles are 924 and 2310. The sum of these is 3234, the area of the tetragon.

[The area of the triangle *BCD* (see figs. to §§ 181-182) is 924, and that of the triangle *ABD* is 2310.]

185—186. Rule: two stanzas. Making the difference between the base and summit of a (trapezium or) quadrilateral that has equal perpendiculars, the base (of a triangle), and the sides (its) legs, the segments of it and the length of the perpendicular are to be found as for a triangle. From the base of the trapezium subtract the segment, and add the square of the remainder to the square of the perpendicular; the square root of the sum will be the diagonal.

In a (trapezium or) quadrilateral that has equal perpendiculars, the sum of the base and least flank is greater than the aggregate of the summit and other flank.

[The rule gives the method of finding the diagonals of a trapezium the sides of which are given. It is demonstrated by Ganesa in the following manner :—



Let the two triangles *ABE*, *DFC* be united into one triangle *A′B′C′*, their altitudes being equal. Then the altitude *A′E′* of the new triangle *A′B′C′* is the altitude of the trapezium, and the segments *B′E′*, *E′C′* will be equal to *BE*, *FC*. Hence $AC^2 = AE^2 + EC^2 = A'E'^2 + (BC - B'E')^2$, which leads to the rule.

---

[1] It is the true and correct area, contrasted with the gross or inexact area of former writers.—Gan. and Súr.

Also, $A'B'+B'C'>A'C'$,

$\therefore AB+BE+FC+EF>AD+DC$, ($\because AD=EF$)

i. e., $AB+BC \quad >AD+DC$.

AB need not be the 'least flank.']

187—189. Example. The sides measuring fifty-two and one less than forty, the summit equal to twenty-five, and the base sixty, this was given as an example by former writers for a figure having unequal perpendiculars; and definite measures of the diagonals were stated, fifty-six and sixty-three. Assign to it other diagonals, and those particularly which appertain to it as a figure with equal perpendiculars.

Statement.

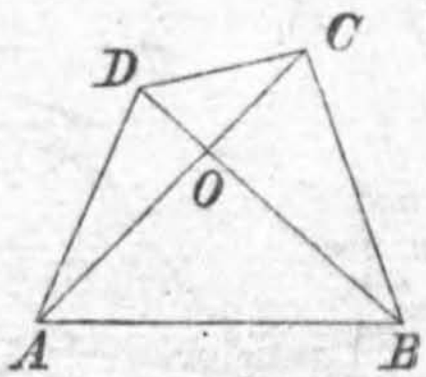


Fig. 1.

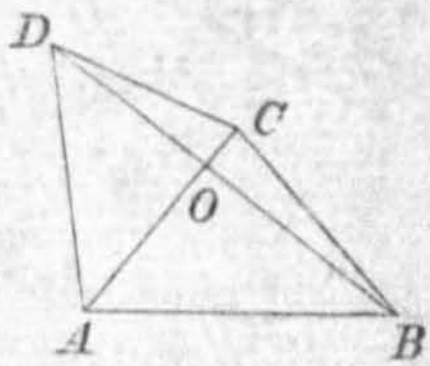


Fig. 2.

Here assuming one diagonal 63, the other is found as before, 56. Or, putting 32 instead of 56 for a diagonal (Fig. 2), the other, found by the process before shown, comes out in two portions, both surds, $\sqrt{621}$ and $\sqrt{2700}$. The sum of the roots (extracted by approximation) is the second diagonal $76\frac{22}{25}$.

[In Fig. 2, if we drop perpendiculars from B, D on AC, and find the segments of AC by § 163, we shall find that AC, BD intersect at right angles, and that $AO=30$.

Hence $BD=BO+DO=\sqrt{60^2-30^2}+\sqrt{39^2-30^2}$
$=\sqrt{2700}+\sqrt{621}=30\sqrt{3}+3\sqrt{69}=30\times 1{\cdot}732\ldots+3\times 8{\cdot}307\ldots$
$=51{\cdot}96\ldots+24{\cdot}921\ldots=76{\cdot}88\ldots=76\frac{22}{25}$ nearly. The root is approximately extracted in the manner indicated in § 138.]

Again, if the above quadrilateral (Fig. 1) be one with equal perpendiculars, *i.e.*, a trapezium, (Fig. 3), consider the triangle $A'B'D'$ (Fig. 4), put to find the perpendicular DE, and the segments AE, FB, according to the rule in §§ 185-186. Here the segments are found $\frac{3}{5}$ and $\frac{172}{5}$; and the perpendicular, the surd $\sqrt{\frac{38016}{25}}$, of which the root found by approximation is $38\frac{622}{625}$. It is the equal perpendicular of the trapezium.

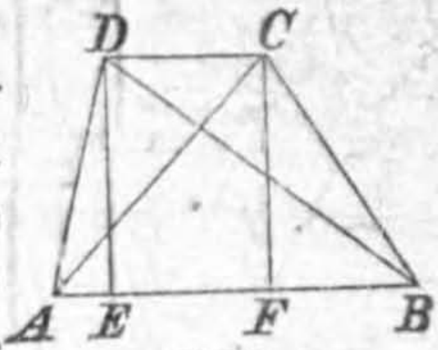


Fig. 3.

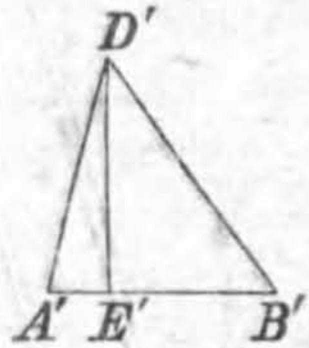


Fig. 4.

Next to find the sum of the squares of the perpendicular and difference between base and segment, we have, base of trapezium, 60; least segment $\frac{3}{5}$; difference $\frac{297}{5}$; square of the difference $\frac{88209}{25}$; square of the perpendicular $\frac{38016}{25}$; sum $\frac{126225}{25}$, or dividing by the denominator, 5049. It is the square of one diagonal (BD). Similarly, the square of the other diagonal (AC) is 2176. Extracting the roots of these squares by approximation, the two diagonals come out $71\frac{1}{25}$ and $46\frac{16}{25}$.

In the above trapezium, the short side 39 added to the base 60 makes 99, which is greater than the aggregate of the summit and other flank, 77. Such is the limitation.

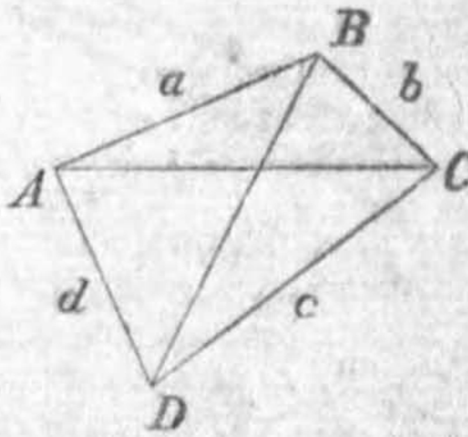
Thus, with the same sides, may be various diagonals in the tetragon. Yet, though indeterminate, diagonals have been sought as determinate, by Brahmagupta and others. Their rule is as follows :—

190. Rule.[1] The sums of the products of the sides about both the diagonals being divided by each other, multiply the quotients by the sum of the products of opposite sides; the square roots of the results are the diagonals in a quadrilateral.

The objection to this mode of finding the diagonals s its operoseness, as I shall show by proposing a shortr method.

[The rule applies only to a quadrilateral which can be inbed in a circle. This, however, is not mentioned in the '.

t $ABCD$ be such a quadrilateral, let $AB=a$, $BC=b$, $CD=c$, $=d$.



ıen $AC^2=\dfrac{(ac+bd)\,(ad+bc)}{ab+cd}$,

and $BD^2=\dfrac{(ac+bd)\,(ab+cd)}{ad+bc}$.

(See Todhunter's Trigonometry, Art. 254.) Thus the reason r the rule is obvious.

In the quadrilateral in §§ 187-189, Fig. 1, the sides are so ken that the diagonals intersect at right angles; and we sily find $OB=48$, $OD=15$. Thus $\cos(ODA)=\frac{15}{39}=\frac{5}{13}$, ıd $\cos(OCB)=\frac{20}{52}=\frac{5}{13}$. Hence angle ODA=angle OCB, ıd therefore a circle passes round the quadrilateral. Conquently, the rule in § 190 will apply to this quadrilateral. is probable that the rule was derived *a posteriori* from this ırticular instance, and not *a priori* from the fact of the quadateral being inscribable in a circle. The same quadrilateral is ven as an example of the rule by Chaturveda Prithúdaka wámí, in his commentary on Brahmagupta's treatise.]

191—192. Rule: two stanzas. The uprights and

---

[1] A couplet cited from Brahmagupta, XII, 28.

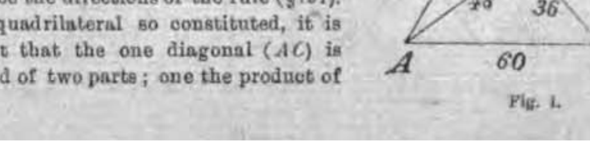
sides of two assumed right-angled triangles,[1] being multiplied by the reciprocal hypotenuses, become sides (of a quadrilateral) : and in this manner is constituted a quadrilateral, in which the diagonals are deducible from the two triangles.[2] The product of the uprights added to the product of the sides is one diagonal ; the sum of the products of uprights and sides reciprocally multiplied, is the other.[3] When this short methoc exists, why an operose one was practised by forme writers, we know not.

---

[1] Assumed conformably with the rule in §145. An objection, to wl Ganesa adverts and which he endeavours to obviate, is that this sh method requires sagacity in the selection of assumed triangles ; wh the longer method is adapted to all capacities.

[2] This method of constructing a quadrilateral is taken from Brahmag XII, 38.

[3] A quadrilateral is divided into four triangles by its intersecting di nals ; and conversely, by the junction of four triangles, a quadrilater constituted. For that purpose, four triangles are assumed in this man Two triangles are first put in the mode directed (§145), the sides of wh are all rational. Such sides, multiplied by any assumed number, will c stitute other right-angled triangles, of which also the sides will be ration By the twofold multiplication of hypotenuse, upright and side of one assum triangle by the upright and side of the other, four (right-angled) triangl are formed, such that turning and adapting them and placing the multipl of the hypotenuses for sides, a quadrilateral is composed, (as shown below



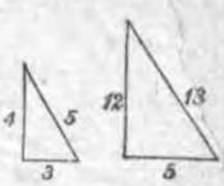
Here the uprights and sides of the arbitrary triangles (the first two on the left side), reciprocally multiplied by the hypotenuses, become sides of the quadrilateral ; and hence the directions of the rule (§191).

In a quadrilateral so constituted, it is apparent that the one diagonal (AC) is composed of two parts ; one the product of

Fig. 1.

[This rule is of no importance whatever. It applies only to quadrilaterals constructed in the artificial manner indicated by the author, and fully explained by Ganesa ; see foot-note. It is curious that the author, while censuring Brahmagupta's important rule (§190) as operose, entirely forgets that the rule propounded by himself is comparatively unimportant and of very limited application.]

Assuming two right-angled triangles, multiply the upright and side of one by the hypotenuse of the other : the greatest of the products is taken for the base ; the least for the summit ; and the other two for the flanks. (See Fig. 1 in the foot-note.)



---

the uprights, the other the product of the sides of the arbitrary triangles. The other diagonal (*BD*) consists of two parts, *viz.*, the products of the reciprocal multiplication of uprights and sides. These two portions are the perpendiculars, for there is no interval between the points of intersection. This holds, provided the shortest side be the summit; the longest, the base; and the rest, the flanks. But if the component triangles be otherwise adapted, the summit and a flank change places, as in the adjoining figure. Here the two portions of the first diagonal as above found (*viz.*, 48 and 15) do not face, but are separated by an interval, which is equal to the difference between the two portions (36 and 20) of the other diagonal, *viz.*, 16. It is the difference of two segments on he same side, found by a preceding rule (§§181-82), and is taken for the upright of a triangle

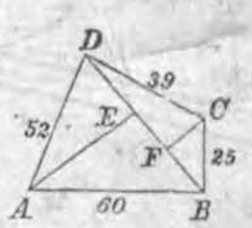


Fig. 2.

already explained (§§181-182, note); the sum of the two portions of the agonal equal to the two perpendiculars is made the side. The square t of the sum of the squares of such upright and side is equal to the duct of the hypotenuses (13 and 5): wherefore the author adds, "if the mit and flank change places, the *first* diagonal will be the product of hypotenuses." (The MSS. have *first*, but Bháskara's text exhibits *second* ead.)

This last remark is not clearly explained by the commentator. From the lated values of AE and FB (see Fig. 2), we find that $\sin ABE = \frac{AE}{AB} = \frac{4}{5}$; and $\cos \hat{CBF} = \frac{FB}{BC} = \frac{20}{25} = \frac{4}{5}$. Hence the angle ABC is a right angle.

Here with much labour (by the former method) the diagonals are found 63 and 56.

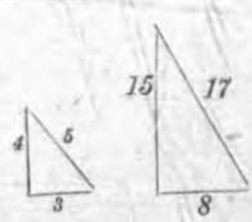
With the same pair of right-angled triangles, the products of uprights and sides reciprocally multiplied are 36 and 20; the sum of which is one diagonal, 56. The products of uprights multiplied together, and sides taken into each other, are 48 and 15; their sum is the other diagonal, 63. Thus they are found with ease.

But if the summit and flank change places, and the figure be stated accordingly, the second diagonal will be the product of the hypotenuses of the two right-angled triangles, *viz.*, 65. (See Fig. 2 in the foot-note.)

And ∵ $AB = 5 \times 12$, and $BC = 5 \times 5$,

∴ $AC = 5 \times 13$, and the truth of the remark is obvious. It need hardly be added that the remark applies only to quadrilaterals constructed in the artificial manner indicated by the author.—ED.]

In like manner, for the tetragon before instanced (§178), to find the diagonals, a pair of rectangular triangles is put. Proceeding as directed, the diagonals come out 77 and 84. (See Fig. 3.)



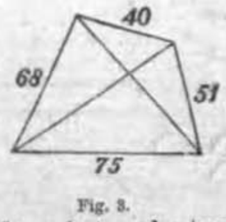


Fig. 3.

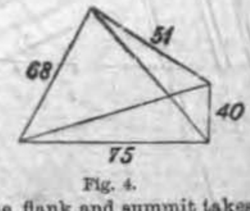
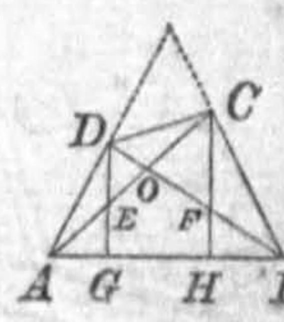


Fig. 4.

In the figure instanced, a transposition of the flank and summit takes p (see Fig. 4 which corresponds to the tetragon instanced in §178); where the product of the hypotenuses (5 and 17) of the two right-angled trian will be the second diagonal; and they thus come out 77 and 85.—GAN.

193—194. Example.[1] In a figure in which the base is three hundred, the summit a hundred and twenty-five, the flanks two hundred and sixty and one hundred and ninety-five, one diagonal two hundred and eighty and the other three hundred and fifteen, and the perpendiculars a hundred and eighty-nine and two hundred and twenty-four, what are the portions of the perpendiculars and diagonals below the intersections of them ? and the perpendicular let fall from the intersection of the diagonals, with the segments answering to it ? and the perpendicular of the needle formed by the prolongation of the flanks until they meet, as well as the segments corresponding to it ? and the measure of both the needle's sides ? All this declare, mathematician, if thou be thoroughly skilled in this (science of[2]) plane figure.

Statement. Length of the base 300. Summit 125. Flanks 260 and 195. Diagonals 280 and 315. Perpendiculars 189 and 224.



195—196. Rule: two stanzas. The interval between the perpendicular and its correspondent flank is termed the *sandhi*[3] or link of that perpendicular. The

---

[1] Having thus, from §173 to this place, shown the method of finding the area, &c., in the fourteen sorts of quadrilaterals, the author now exhibits another quadrilateral, proposing questions concerning segments produced by intersection.—Gan. For the instruction of the pupil, he exhibits the figure called (*súchí*) a needle.—*Mano.*

The problem is taken from Brahmagupta with a slight variation ; and this example differs from his only in the scale, his numbers being here increased five-fold. See Brahmagupta, XII, 32.

[2] *Manoranjana.*

[3] *Sandhi*, union, alliance ; connecting link.

base less the link or segment is called the *pítha*[1] or complement of the same. The link or segment contiguous to that portion (of perpendicular or diagonal) which is sought, is twice set down. Multiplied by the other perpendicular in one instance, and by the diagonal in the other, and divided (in both instances) by the complement belonging to the other (perpendicular), the quotients will be the lower portions of the perpendicular and diagonal below the intersection.

Statement. Perpendicular 189. Flank contiguous to it 195. Segment intercepted between them (found by §134), 48. It is the link. The second segment is 252, and is called the complement.

In like manner, the second perpendicular is 224. The flank contiguous to it, 260. Interval between them, being the segment called link, 132. Complement 168.

Now to find the lower portion of the first perpendicular 189. Its link 48 separately multiplied by the other perpendicular 224 and by the diagonal 280, and divided by the other complement 168, gives quotients 64, the lower portion of the perpendicular, and 80, the lower portion of the diagonal.

So for the second perpendicular 224, its link 132, severally multiplied by the other perpendicular 189 and by the diagonal 315, and divided by the other complement 252, gives 99 for the lower portion of the perpendicular, and 165 for that of the diagonal.

[The object of the rule is to find EG, EA, FH, FB. (See figure, §§193-194).

From similar triangles we have,

$$\frac{EG}{GA}=\frac{CH}{HA}, \text{ whence } EG=\frac{GA\times CH}{HA};$$

---

[1] *Pítha*, lit. stool. Here the complement of the segment.

and $\frac{AE}{AC}=\frac{GA}{HA}$, whence $AE=\frac{GA\times AC}{HA}$.

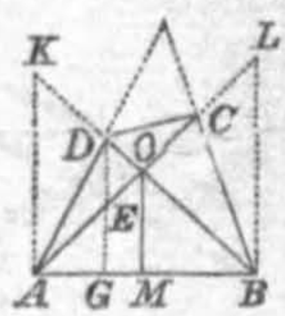
Similarly for *FH* and *FB*.

Hence the reason for the rule is clear.]

197. Rule to find the perpendicular below the intersection of the diagonals.

The perpendiculars multiplied by the base and divided by the respective complements, are the erect poles : from which the perpendicular let fall from the intersection of the diagonals, as also the segments of the base, are to be found as before.[1]

Statement. Proceeding as directed, the erect poles are found 225 and 400. Whence, by a former rule (§159), the perpendicular below the intersection of the diagonals is deduced, 144 ; and the segments of the base 108 and 192.



[The method employed is first to find *AK*, *BL*, the erect poles or perpendiculars on *AB*, and then to apply §159 to find *OM*. Now from similar triangles we have $\frac{AK}{AB}=\frac{GD}{GB}$, whence $AK=\frac{GD\times AB}{GB}$, with a similar value for *BL*. Hence the rule.]

198—200. Rule to find the perpendicular of the needle,[2] its legs and the segments of its base : three stanzas. The proper link multiplied by the other perpendicular and divided by its own, is termed the mean;[3] and the sum of this and the opposite link is called the divisor.[4] Those two quantities, namely, the mean and

[1] By the rule in §159.

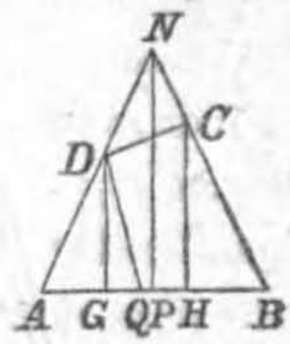
[2] *Súchí*, needle ; the triangle formed by the flanks of the quadrilateral until they meet.

[3] *Sama*, mean ; a fourth proportional to the two perpendiculars and the link or segment.

[4] *Hara*, divisor ; the sum of the *mean* and the other link or segment.

the opposite link, being multiplied by the base and divided by that divisor, will be the respective segments of the needle's base. The other perpendicular, multiplied by the base and divided by the divisor, will be the perpendicular of the needle. The flanks, multiplied by the perpendicular of the needle and divided by their respective perpendiculars, will be the legs of the needle. Thus may the subdivision of a plane figure be conducted by the intelligent, by means of the Rule of Three.

Here the perpendicular being 224, its link is 132. This multiplied by the other perpendicular, *viz.*, 189, and divided by its own, *viz.*, 224, gives the mean as it is named, $\frac{891}{8}$. The sum of this and the other link 48 is the divisor as it is called, $\frac{1275}{8}$. The mean and the other link severally taken into the base, being divided by this divisor, give the segments of the needle's base, $\frac{1536}{17}$ and $\frac{3564}{17}$. The other perpendicular 189, multiplied by the base and divided by the same divisor, yields the perpendicular of the needle, $\frac{6048}{17}$. The sides 195 and 260, multiplied by the needle's perpendicular and divided by their own perpendiculars respectively, *viz.*, 189 and 224, give the legs of the needle, which are the sides of the quadrilateral produced, *viz.*, $\frac{6240}{17}$ and $\frac{7020}{17}$.



Thus in all instances under this head, taking the divisor for the argument, and making the multiplicand or multiplicator, as the case may be, the fruit or requisition, the Rule of Three is to be inferred by the intelligent mathematician.

[The object of the rule is to find *PA*, *PB*, *NP*, *NA*, *NB* (see above figure). It is demonstrated by Ganesa in the following