manner :—Through D draw DQ parallel to NB, meeting AB in Q. Then the triangles AQD and ABN are similar, and we evidently get $\frac{AB}{AQ} = \frac{AN}{AD} = \frac{NP}{DG} = \frac{PB}{GQ} = \frac{PA}{GA}$. . . . (1). Now $\frac{GQ}{DG} = \frac{HB}{CH}$ from similar triangles ; thus $GQ = \frac{HB \times DG}{CH}$ = what the author calls *mean*, and consequently AQ or *mean* $+ AG$ = what the author calls *divisor*. Thus finally, $PB = \frac{GQ \times AB}{AQ}$ from (1) $= \frac{mean \times \text{base}}{divisor}$; and $PA = \frac{GA \times AB}{AQ}$ from (1) $= \frac{\text{opposite link} \times \text{base}}{divisor}$. Again, from (1) $\frac{NP}{DG} = \frac{AB}{AQ}$, whence $NP = \frac{\text{other perpendicular} \times \text{base}}{divisor}$. Lastly, $\frac{NA}{DA} = \frac{NP}{DG}$, whence $NA = \frac{DA \times NP}{DG}$, and similarly for NB.]

201. Rule. When the diameter of a circle[2] is multiplied by three thousand nine hundred and twenty-seven and divided by twelve hundred and fifty, the quotient is the near[3] circumference : or multiplied by twenty-two and divided by seven, it is the gross circumference adapted to practice[4].

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[1] To deduce the circumference of a circle from its diameter, and the diameter from the circumference.—Gan.

[2] *Vritta*, *vartula*, a circle. *Vyása*, *vishkambha*, *vistriti*, *vistára*, the breadth or diameter of a circle. *Paridhi*, *parinâha*, *vritti*, *nemi* (and other synonyms of the felloe of a wheel), the circumference of a circle.

[3] *Súkshma*, delicate or fine, nearly precise ; contrasted with *sthûla*, gross, or somewhat less exact, but sufficient for common purposes.—Gang. and Súr.

Brahmagupta puts the ratio of the circumference to the diameter as three to one for the gross value, and takes the square root of ten times the square of the diameter for the neat value of the circumference. See Brahma. XII, 40. [This is more rough even than $\frac{22}{7}$; for $\sqrt{10} = 3{\cdot}1622...$, and $\frac{22}{7} = 3.\dot{1}4285\dot{7}$.—Ed.]

[4] As the diameter increases or diminishes, so does the circumference increase or diminish : therefore to find the one from the other, make proportion, as the diameter of a known circle is to the known circumference, so is the given diameter to the circumference sought : and conversely, as the

[Ganesa shows (see foot-note) that if the measure of the diameter of a circle be 1250, that of the side of a regular polygon of 384 sides inscribed in the circle will be very nearly 392 (more accurately it will be $= \sqrt{98683} \times 12{\cdot}5 = 3926{\cdot}625 \ldots\ldots$ This shows the degree of approximation of the fraction $\frac{3927}{1250}$ to the value of π. Converting the fraction into a decimal we get 3·1416, the true value of π being 3·14159 . . . .]

202. Example. Where the measure of the diameter is seven, friend, tell the measure of the circumference: and where the circumference is twenty-two find the diameter.

Statement : diameter 7.

Answer : circumference $21\frac{1239}{1250}$, or gross circumference 22.

Statement : circumference 22.

Reversing multiplier and divisor, the diameter comes out $7\frac{11}{3927}$; or gross diameter 7.

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circumference is to the diameter, so is the given circumference to the diameter sought.

Further, the semi-diameter is equal to the side of a regular hexagon within the circle, as will be shown. From this the side of a regular dodecagon may be found in this manner:—the semi-diameter being hypotenuse, and half the side of the hexagon, the side, the square root of the difference of their squares is the upright: subtracting which from the semi-diameter, the remainder is the arrow (or height of the arc). Again, this arrow being the upright and half the side of the hexagon, a side, the square root of the sum of their squares is the side of the dodecagon. From this in like manner, may be found the side of a polygon of 24 sides: and so on doubling the number of sides in the polygon, until the side be near to the arc. The sum of such sides will be the circumference of the circle nearly. Thus, the diameter being 100, the side of the dodecagon is the surd $\sqrt{673}$ and that of a polygon of 384 sides is nearly equal to the arc. By computation it comes out the surd $\sqrt{98683}$. Now the proportion, if to the square of the diameter put at 100, *viz.*, 10000, this be the square of the circumference, *viz.*, 98683, then to the square of the assumed diameter 1250, *viz.*, 1562500, what will be the circumference? Answer: the square root 3927 without remainder.—Gan.

203. Rule. In a circle, a quarter of the diameter multiplied by the circumference is the area. That multiplied by four[1] is the net all around the ball.[2] This content of the surface of the sphere, multiplied by the diameter and divided by six, is the precise solid, termed cubic, content within the sphere.[3]

[This rule gives the well-known expressions for the area of a circle, and the surface and volume of a sphere. Ganesa gives

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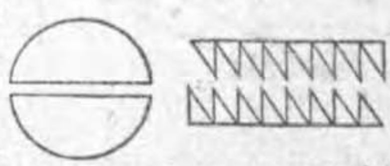
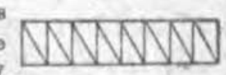
[1] Or rejecting equal multiplier and divisor, the circumference multiplied by the diameter is the surface.—Gan.

[2] *Prishtha-phala*, superficial content: compared to the net formed by the string with which cloth is tied to make a playing ball. *Ghana-phala*, solid content: compared to a cube, and denominated from it, cubic.

[3] Dividing the circle into two equal parts, cut the content of each into any number of equal angular spaces, and expand it so that the circumference may become a straight line as in the adjoining figure. Then let the



two portions approach so that the angular spaces of the one may enter into the similar intermediate vacant spaces of the other, as in the figure, thus constituting an oblong, of which the semi-diameter is one side, and half the circumference the other. The product of their multiplication is the area. Half by



half is a quarter. Therefore a quarter of the diameter multiplied by the circumference is the area.—Gan.

See in the *Goládhyáya* of the *Siddhánta-siromani*, a demonstration of the rule, that the surface of a sphere is four times the area of a great circle, or equal to the circumference multiplied by the diameter.—*Ibid.* [See the *Goládhyáya*, Wilkinson's translation, III, 52.—Ed.]

To demonstrate the rule for the solid content of a sphere, suppose the sphere divided into as many little pyramids, or long needles with an acute tip and square base, as is the number by which the surface is measured, the height of each pyramid being equal to the radius of the sphere. The base of each pyramid is a unit of the scale by which the dimensions of the surface are reckoned; and the altitude being a semi-diameter, one-third of their product is the content: for a needle-shaped excavation is one-third of an excavation in the form of a rectangular parallelopiped of the same base and height, as will be shown (§221). Therefore (unit taken into) a sixth part of the diameter is the content of one such pyramidical portion: and that multiplied by the surface gives the solid content of the sphere.—*Ibid.*

interesting but rough demonstrations in the case of the area of a circle and the volume of a sphere, and refers to the *Goládhyáya* for the case of the surface of a sphere. See foot-note.]

204. Example. Intelligent friend, if thou know well the spotless *Lílávatí*, say what the area of a circle is, the diameter of which is measured by seven : and the surface of a globe, or area like a net upon a ball, the diameter being seven: and the solid content within the same sphere.

Statement: diameter 7.

Answer: area of the circle, $38\frac{2423}{5000}$. Superficial content of the sphere, $153\frac{1173}{1250}$. Solid content of the sphere, $179\frac{1487}{2500}$.

205—206. Rule: a stanza and a half. The square of the diameter being multiplied by three thousand nine hundred and twenty-seven, and divided by five thousand, the quotient is the nearly precise area ; or multiplied by eleven and divided by fourteen, it is the gross area adapted to common practice. Half the cube of the diameter, with its twenty-first part added to it, is the solid content of the sphere.

The area of the circle, nearly precise, comes out as before $38\frac{2423}{5000}$, or gross area $38\frac{1}{2}$. Gross solid content $179\frac{2}{3}$.

[The area of a circle $=\frac{1}{4}\pi d^2$, d being the diameter

$$=\frac{3927}{4\times 1250}d^2=\frac{3927}{5000}d^2;$$

$$\text{or more roughly }=\frac{22}{4\times 7}d^2 \quad =\frac{11}{14}d^2.$$

Again, the volume of a sphere $= \frac{4}{3}\pi\frac{d^3}{8}$, d being the diameter

$$= \frac{22}{7\times6}d^3 = \frac{11}{21}d^3 \text{ roughly}$$

$$= \frac{1}{2}d^3\left(1+\frac{1}{21}\right).$$

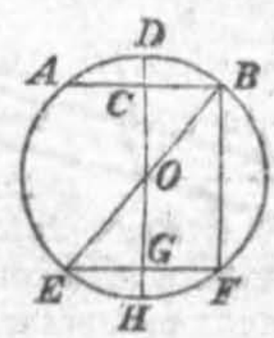
Thus the reasons for the rules are obvious.]

206—207. Rule[1] : a stanza and a half. The sum and difference of the chord and diameter being multiplied together, and the square root of the product being subtracted from the diameter, half the remainder is the arrow.[2] The diameter less the arrow being multiplied by the arrow, twice the square root of the product is the chord. The square of half the chord being divided by the arrow, the quotient added to the arrow is pronounced to be the diameter of the circle.

[By the word *arrow* is meant the height of the arc.

Súryadása gives the following proof of the formula for the arrow.

Let AB be the chord, CD, the arrow, and O the centre. Join BO, and produce it to meet the circumference in E. Draw the chord EGF perpendicular to the diameter DH. Join BF.



Then $CD = \frac{1}{2}(DH - CG) = \frac{1}{2}(DH - BF)$
$= \frac{1}{2}(DH - \sqrt{BE^2 - EF^2}) = \frac{1}{2}(DH - \sqrt{DH^2 - AB^2})$,
whence the rule.

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[1] In a circle cut by a right line, to find the chord, arrow, &c.; that is, either the chord, the arrow, or the diameter being unknown, and the other two given, to find the one from the others.—Gan. and Súr.

[2] A portion of the circumference is a bow (*dhanush*, *chápa*). The right line between its extremities, like the string of a bow is its chord (*jívá*, *jyá*, *guna*, *maurví*). The line between them is the arrow (*sara*, *ishu*), as resembling one set on a bow.—Gan. and Súr.

Again, $CH.\ CD = AC^2$ (Euclid III. 35)

$$\therefore AB = 2AC = 2\sqrt{CH.\ CD}.$$

$$\text{Also } \frac{AC^2}{CD} = CH;$$

$$\therefore DH = \frac{AC^2}{CD} + CD.]$$

208. Example. In a circle of which the diameter is ten, the chord being measured by six, say friend what the arrow is : and from the arrow tell the chord : and from chord and arrow, the diameter.

Statement : diameter 10. When the chord is 6, the length of the arrow comes out 1.

Or, the arrow being 1, the chord is found 6. Or from the chord and arrow the diameter is deduced 10.

209—211. Rule :[1] three stanzas. By 103923, 84853, 70534, 60000, 52055, 45922, and 41031, multiply the diameter of a circle, and divide the respective products by 120000; the quotients are severally, in their order, the sides of polygons from the triangle to the enneagon (inscribed) within the circle.[2]

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[1] To find the sides of regular inscribed polygons.—Gan. and Súr.

[2] Describe a circle with any radius at pleasure, divide the circumference into three equal parts and mark the points; and with these points (A, B, C) as centres and with the same radius, describe three circles, which will be equal in circumference to the first circle; and it is thus manifest that the side of the regular hexagon within the circle is half a diameter.

The side of an equilateral triangle (inscribed) in a circle is the upright, the diameter is hypotenuse and the side of the hexagon is side of a right-angled triangle. See above figure. Therefore the square root of the difference of the squares of the semi-diameter and diameter is the side of the (inscribed) equilateral triangle: *viz.*, for the proposed diameter (120000), 103923

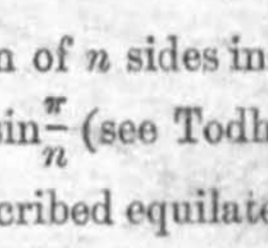
[In this rule the author gives the fractions by which the diameter of a circle is to be severally multiplied, in order to get the sides of inscribed regular figures from the triangle to the enneagon. Ganesa shows by a purely geometrical method how the fractions are arrived at, in the case of the triangle, the square, the hexagon, and the octagon ; and remarks that a similar proof cannot be given in the case of the pentagon, the heptagon and the enneagon. See foot-note. Súryadása tries to supply the proofs in these cases, but his attempt is a failure ; for the proofs he gives are not at all rigid and satisfactory, and it is not worth while to give them in the foot-note, as Colebrooke does. By the help of trigonometrical tables, proofs in all the cases may be given in a general manner as follows :—

Let a denote the side of a regular polygon of n sides inscribed in a circle of radius r. Then $a = 2r \sin\frac{\pi}{n}$ (see Todhunter's Trigonometry, Art. 255). Hence, side of inscribed equilateral triangle $= 2r \sin 60° = 2r\frac{\sqrt{3}}{2} = 2r\frac{1{\cdot}7320508...}{2} = 2r \times {\cdot}8660254...$ Now the fraction $\frac{103923}{120000} = {\cdot}866025$; thus the approximation is very close.

The side of the inscribed square $= 2r \sin\frac{\pi}{4} = 2r\frac{\sqrt{2}}{2} = 2r \times$ ${\cdot}7071067....$

Now the fraction $\frac{84853}{120000} = {\cdot}707108\dot{3}$, thus the fraction is a little too large.

The side of a square is hypotenuse, and the semi-diameter is upright and side. Wherefore the square root of twice the square of the semi-diameter is the side of the (inscribed) square : *viz.*, for the diameter assumed, 84853.



The side of the regular octagon (see above figure) is hypotenuse, half the side of the square is upright, and the difference between that and the semi-diameter is the side. Wherefore the square root of the sum of the squares of half the side of the square and the semi-diameter less half the side of the square is the side of the (inscribed) regular octagon : *viz.*, for the diameter as put, 45922.

The proof of the sides of the regular pentagon, heptagon and enneagon cannot be shown in a similar manner.—Gan.

The side of the inscribed pentagon $=2r\sin 36^\circ=2r\times\cdot 587785$ from a table of natural sines. Now the fraction $\frac{70534}{120000}=\cdot 58778\dot{3}$; thus the fraction is a little too small.

The side of the inscribed hexagon is equal to the radius.

The side of the inscribed heptagon $=2r\sin(25^\circ\ 42'\ 51''$ nearly $=2r\times\cdot 4338819$ from the tables and the theory of proportional parts. Now the fraction $\frac{52055}{120000}=\cdot 433791\dot{6}$; thus the fraction is a little too small.

The side of the inscribed octagon $=2r\sin 22\frac{1}{2}^\circ=2r\times\cdot 382683$ from the tables. Now the fraction $\frac{45922}{120000}=\cdot 38268\dot{3}$; thus the approximation is very close.

The side of the inscribed nonagon $=2r\sin 20^\circ=2r\times\cdot 342020$ from the tables. Now the fraction $\frac{41031}{120000}=\cdot 341925$; thus the fraction is a little too small.

In the appendix to the *Goládhyáya*, called *Jyotpatti*, Bháskara has given an elaborate method of constructing the sines of various angles, adopting the old definition of the sine. (See Todhunter's Trigonometry, Art. 71.) The values deduced by his method closely approximate the values given in our modern tables, there being slight discrepancies in some cases, which account for the discrepancies noticed above between the values of the sides of some of the inscribed regular polygons as given in the text, and their values as calculated from the tables. A table of sines and versed sines of certain angles in arithmetical progression is also given in the *Súrya-siddhánta*, II. 15—27, the values there stated being less accurate than those deducible from Bháskara's method. See *Súrya-siddhánta*, Bápú Deva Sástrí's translation, II. 16, foot-note. The decimal notation is nowhere used either by Bháskara or in the *Súrya-siddhánta*. See note to § 138.

212. Example. Within a circle of which the diameter is two thousand, tell me severally the sides of the inscribed equilateral triangle and other polygons.

Statement: diameter 2000.

Answer: side of the triangle, $1732\frac{1}{20}$; of the tetragon, $1414\frac{13}{60}$; of the pentagon, $1175\frac{17}{30}$; of the hexagon, 1000; of the heptagon, $867\frac{7}{12}$; of the octagon, $765\frac{11}{30}$; of the nonagon, $683\frac{17}{20}$.

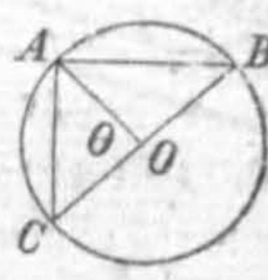
From variously assumed diameters, other chords are deducible, as will be shown by us under the head of construction of sines *(Jyotpatti)* in the treatise on Spherics.

[See *Goládhyáya*, appendix, Wilkinson's translation.]

The following rule teaches a short method of finding the gross chords.

213. Rule. The circumference less the arc being multiplied by the arc, the product is termed first.[1] From the quarter of the square of the circumference multiplied by five, subtract that first product, and by the remainder divide the first product taken into four times the diameter. The quotient will be the chord.

[This rule, as the author himself observes, gives a method of finding approximately the chords of given arcs of a circle. The commentators give an unsatisfactory and almost fanciful demonstration of the rule. The nature of the approximation may be shown thus:—



Let AB be the given arc whose chord is to be found. Draw the diameter BOC, and join AC, AO. Let θ denote the angle AOC, r the radius, and c the circumference of the circle. Then the value of the chord AB as given by the rule

$$=\frac{\text{arc } ACB\times\text{arc } AB\times 8r}{\frac{5}{4}c^2-\text{arc } ACB\times\text{arc } AB}$$

$$=\frac{8r\left\{(\frac{1}{2}c)^2-(\text{arc } CA)^2\right\}}{\frac{5}{4}c^2-(\frac{1}{2}c)^2+(\text{arc } CA)^2}=\frac{2r\,(4\pi^2-4\theta^2)}{4\pi^2+\theta^2}$$

$$=2r.\frac{1-\left(\frac{\theta}{\pi}\right)^2}{1+\left(\frac{\theta}{2\pi}\right)^2}=2r.\left\{1-\left(\frac{\theta}{\pi}\right)^2\right\}\left\{1-\left(\frac{\theta}{2\pi}\right)^2+\&c.\right\}$$

[1] *Prathama, ádya*, first (product).

$= 2r. \left(1 - \frac{5\theta^2}{4\pi^2}\right)$, neglecting powers of $\frac{\theta}{\pi}$ beyond the second The accurate value of the chord $AB = 2r \cos\frac{\theta}{2} = 2r \left(1 - \frac{\theta^2}{8} + \&c.\right)$. Taking $\pi = \frac{22}{7}$, the value of the fraction $\frac{5}{4\pi^2}$ will be found to be very slightly greater than $\frac{1}{8}$. This shows the nature of the approximation. It is worthy of notice that the rule gives the exact value of the chord when the arc is a sixth part of the circumference. The rule appears to have been obtained empirically after repeated trials.]

214. Example. Where the semi-diameter is a hundred and twenty, and the arc of the circle is measured by an eighteenth multiplied by one and so forth (up to nine[1]), tell quickly the chords of those arcs.

Statement : diameter 240.

Here the circumference is 754 (nearly).

Arcs being taken which are multiples of an eighteenth of the circumference, the (corresponding) chords are to be sought.

Or for the sake of facility, abridging both circumference and arcs by the eighteenth part of the circumference, the same chords are found. Thus, circumference, 18 ; arcs, 1, 2, 3, 4, 5, 6, 7, 8, 9. Proceeding as directed, the chords come out 42, 82, 120, 154, 184, 208, 226, 236, 240.

In like manner, with other diameters (chords of assigned arcs may be found).[2]

[We can easily test the accuracy of the values of the chords given above by calculating their actual values from a table of

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[1] Up to nine, or half the number of arcs ; for the chords of the eighth and tenth will be the same, and so will those of the seventh and eleventh, and so forth.—Gan.

[2] Gang. &c.

natural sines. It will be found that the values given are in some cases less, and in others greater than the true values.]

215. Rule.[1] The square of the circumference is multiplied by a quarter of the chord and by five, and divided by the chord added to four times the diameter; the quotient being subtracted from a quarter of the square of the circumference, the square root of the remainder, taken from half the circumference, will leave the arc.

[This rule is derived from the preceding one. Denoting the arc AB (see figure in the note to §213) by x, we get $AB = \dfrac{8r(c-x)x}{\frac{5}{4}c^2-(c-x)x}$ (§ 213), whence $x^2 - cx + \dfrac{5\,AB.\,c^2}{4(8r+AB)} = 0$;

$\therefore \dfrac{c}{2} - x = \sqrt{\left\{\dfrac{c^2}{4} - \dfrac{5\,AB\,c^2}{4(8r+AB)}\right\}}$, the upper sign being taken, as x is supposed to be less than the semi-circumference. Hence the reason for the rule is obvious.

The following empirical and approximate rule for finding the arc is cited by Ganesa from Áryabhatta:—Six times the square of the arrow being added to the square of the chord, the square root of the sum is the arc. If 2θ denote the angle subtended by the arc at the centre, and r the radius, the expression for the arc as given by the rule

$= \sqrt{\{4r^2 \sin^2\theta + 6r^2(1-\cos\theta)^2\}}$

$= r\sqrt{\{10 + 2\cos^2\theta - 12\cos\theta\}}$

$= r\sqrt{\left\{10 + 2\left(1 - \dfrac{\theta^2}{2} + \ldots\ldots\right)^2 - 12\left(1 - \dfrac{\theta^2}{2} + \ldots\ldots\right)\right\}}$

$= r\,.\,2\theta$, (neglecting higher powers of θ), which is the true value of the arc. This shows the nature of the approximation involved in the rule. In the case of the semi-circle, the rule gives for the length of the arc the expression $\sqrt{10}\,r$, so that the value of π is assumed to be $\sqrt{10}$.]

216. Example. From the chords which have been here found, now tell the length of the arcs, if, mathe-

[1] To find the arc from the chord given.

matician, thou have skill in computing the relation of arc and chord.

Statement: chords, 42, 82, 120, 154, 184, 208, 226, 236, 240.

Circumference abridged 18. The arcs thence found are, 1, 2, 3, 4, 5, 6, 7, 8, 9. They must be multiplied by the eighteenth part of the circumference.[1]

[Bháskara has given no rule for finding the area of a bow or segment of a circle. Ganesa cites in his commentary two rules for this purpose, which are practically the same. One of them, due to his father Kesava, is as follows:—The arrow being multiplied by half the sum of the chord and arrow, and a twentieth part of the product being added, the sum is the area of the segment. The other rule due to Srídhara is as follows:—The square of the arrow multiplied by the square of half the sum of the chord and arrow, being multiplied by ten and divided by nine, the square root of the product is the area of the bow. Since the fraction $\frac{21}{20}$ is very nearly equal to the fraction $\frac{\sqrt{10}}{3}$, we see that these two rules are practically the same. They both appear to be empirical and give very rough results, as may be readily seen by applying them to one or two particular cases. Thus, taking the first rule and applying it to the case when the segment is a semi-circle, we get for the area, the expression $\frac{21}{20}\,\frac{3r^2}{2}$, the true area being $\frac{\pi r^2}{2}$, r denoting the radius; so that in this case the value of π is taken to be $\frac{63}{20}$, which is greater than the true value.

[1] Súryadása and Gangádhara notice other figures omitted by the author, *e. g.*, *gaja-danta* or elephant's tusk, which may be treated as a triangle according to Srídhara; *bálendu* or crescent, which may be considered as composed of two triangles, according to the same author; *yava* or barleycorn, a convex lens, treated as consisting either of two triangles or two bows, accordi g to Gangádhara; *nemi* or felloe; *vajra* or thunderbolt, treated as a quadrilateral with two bows, according to Gangádhara; *sankha* or conch; *mridanga* or great drum; and several others.

the same rule to the case when the arc of the seg-
[d]rant, we get for the area, the expression $\frac{21}{20} \cdot \frac{r^2}{4}$,
[are]a being $\frac{(\pi - 2)\, r^2}{4}$; so that in this case the value of
[take]n to be $\frac{61}{20}$, which is less than the true value. Ganesa
[. . .] gives the accurate method of finding the area of the
[segme]nt, namely, by subtracting the area of the triangle formed
[by th]e radii and the chord of the segment from the area of the
[secto]r. The same rule is also given in the *Manoranjana*.]

CHAPTER VII.
EXCAVATIONS[1] AND CONTENTS OF SOLI

217—218. Rule[2]: a couplet and a half. Taki the breadth in several places,[3] let the sum of the me sures be divided by the number of places: the qu tient is the mean measure. So likewise with the lengt and depth.[4] The area of the plane figure, mul' ,lie by the depth, will be the number of solid cubits co tained in the excavation.

[The rule is very rough, giving a result much smaller th the true one. It is curious that such a rough rule was giv when the author intended to lay down the correct rule imm diately afterwards (§221). The tank contemplated is no dou an ordinary one with slant sides, and we need not take tl measurements in several places; the length and breadth of tl mouth and bottom, and the depth of the bottom from the mout being sufficient for finding the volume accurately. See §221

219—220. Example: two stanzas. Where tl length of the cavity, owing to the slant of the side

[1] *Khâta-vyavahâra*. The author treats first of excavations, secondly stacks of bricks and the like, thirdly of sawing of timber, and fourthly stores of grain, in as many distinct chapters.

[2] For measuring an excavation, the sides of which are trapezia.—Gan.

[3] *Vistâra*, breadth; *dairghya*, length; *bedha*, depth. *Khâta*, an exca tion; *sama-khâta*, a cavity in the form of a rectangular parallelopi cylinder, &c.; *vishama-khâta*, a cavity in the form of an irregular so *súchí-khâta*, an acute one, a pyramid or cone. *Sama-miti*, mean mea *Ghana-phala*, the content of the excavation.

[4] The irregular solid is reduced to a regular one, to find its content.—

is measured by ten, eleven and twelve cubits in three several places, its breadth by six, five and seven, and its depth by two, four and three : tell me, friend, how many solid cubits are contained in that excavation.

Statement:	lengths,	12	11	10 ;
	breadths,	7	5	6 ;
	depths,	3	4	2.

Here finding the mean measure, the breadth is 6 cubits, the length 11, and the depth 3. The number of solid cubits is found, 198.

221. Rule[1] : a couplet and a half. The aggregate of the areas at the top and at the bottom, and of that resulting from the sum (of the sides of the summit and base), being divided by six, the quotient is the mean area : that multiplied by the depth is the neat[2] content.[3] A third part of the content of the regular equal solid is the content of the acute one.[4]

[This rule gives the exact volume. The tank contemplated is an ordinary one with uniformly slanting sides. Let *ABCD* be the mouth of the tank, and *EFGH* its base, both being supposed rectangular. Suppose the mouth of the tank to be covered by a plane. Draw perpendiculars on this plane from *E*, *F*, *G*, *H*, and from the feet of these perpendiculars, draw perpendiculars

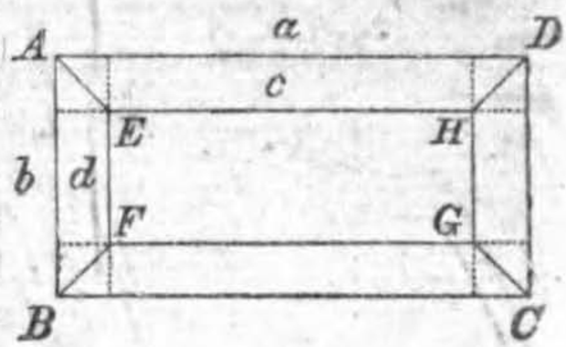
[1] To find the content of a prism, pyramid, cylinder and cone.

[2] Contrasted with the result of the preceding rule, which gave a gross or approximate measure.

[3] Half the sum of the breadths at the mouth and bottom is the mean breadth ; and half the sum of the lengths at the mouth and bottom is the mean length : their product is the area at the middle of the tank. (Four times that is the product of the sums of the length and breadth.) This, added to once the area at the mouth and once the area at the bottom, is six times the mean area.—Gan.

[4] As the bottom of the acute excavation is deep, by finding an area for it in the manner before directed, the regular equal solid is produced ; wherefore proportion is made : if such be th content, assuming three places, what is the content taking one? Thus the content of the regular equal solid, divided by three, is that of the acute one.—Súr.

on AB, BC, CD, DA, as in the figure. Let $AD=a$, $AB=b$, $EH=c$, $EF=d$, and $z=$ vertical depth of the tank. Then it will be easily seen that the tank is divided into a rectangular parallelopiped whose volume is cdz; four triangular prisms, two and two being equal, the united volume



being $\frac{1}{2}z\left\{(a-c)\ d+(b-d)c\right\}$; and four equal pyramids on square bases, one at each corner, the united volume being $\frac{1}{3}z\ (a-c)\ (b-d)$. Hence the volume of the tank

$$=z\left\{cd+\tfrac{1}{2}(a-c)\,d+\tfrac{1}{2}(b-d)\,c+\tfrac{1}{3}(a-c)\,(b-d)\right\}$$
$$=z\times\tfrac{1}{6}\left\{ab+cd+(a+c)\,(b+d)\right\}.$$

The last expression stated in words leads to the rule. The last part of the rule relating to the volumes of pyramids and cones is well known. Ganesa and Súryadása give curious demonstrations of the rule. See the foot-notes.]

222. Example. Tell the quantity of the excavation in a tank, of which the length and breadth are equal to twelve and ten cubits at its mouth, and half as much at the bottom, and of which the depth, friend, is seven cubits.

Statement: length 12; breadth 10; depth 7. The area at the mouth is 120; at the bottom 30; reckoned by the sum of the sides 270. Total 420. Mean area 70. Solid content 490.

223. Example. In a quadrangular excavation, the side being equal to twelve cubits, what is the content, if the depth be measured by nine? and in a round one, of which the diameter is ten and depth five? and tell me separately, friend, the content of both acute solids.

Statement of quadrangular tank: side 12; depth 9. Proceeding as directed, the solid content comes out

1296. The content of the acute solid (quadrangular pyramid) is 432.

Statement of round tank : diameter 10 ; depth 5. The content nearly exact is $\frac{3927}{10}$; of the acute solid (cone), $\frac{1309}{10}$. Or gross content of the cylindrical tank is $\frac{2750}{7}$; of the cone, $\frac{2750}{21}$.

[The value of π is taken to be $\frac{3927}{1250}$. (See §201).]

CHAPTER VIII.

STACKS.[1]

224—225. Rule[2]: a stanza and a half. The area of the plane figure (or base) of the stack,[3] multiplied by the height,[4] will be the solid content. The content of the whole pile, being divided by that of one brick, the number of bricks is found. The height of the stack, being divided by that of one brick, gives the number of layers.[5] So likewise with piles of stones.

[The stack is supposed to be in the form of a rectangular parallelopiped, and the reason for the rule is obvious. Bricks are, however, usually arranged in a pile so as to form a frustum of a quadrangular pyramid.]

226—227. Example : two stanzas. The bricks of a pile being eighteen fingers long, twelve broad and three high, and the stack being five cubits broad, eight long and three high, say what the solid content of the pile is; and what the number of bricks, and how many the layers.

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[1] *Chiti-vyavahára.*

[2] To find the solid content of a pile of bricks, or of stones or other things of uniform dimensions: also the number of bricks and of strata contained in the stack.

[3] *Chiti*, a pile or stack.

[4] *Uchchhráya, uchchhriti, auchchya*, height.

[5] *Stara*, layer or stratum.

Statement: length of pile, 8 ; breadth, 5 ; height, 3. Bricks, $\frac{3}{4}$ by $\frac{1}{2}$ by $\frac{1}{8}$.

Answer. Solid content of the brick, $\frac{8}{64}$; of the stack, 120. Number of bricks, 2560. Number of layers, 24.

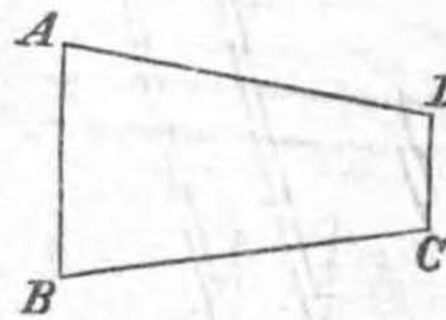
So likewise in the case of a pile of stones.

CHAPTER IX.

SAW.[1]

228. **Rule: two half stanzas.[2] Half the sum of the thickness at both extremities, multiplied by the length in fingers, and the product again multiplied by the number of sections of the timber, and divided by five hundred and seventy-six,[3] will be the measure in cubits.**

[The faces of the timber to which the sections are parallel, are supposed to be trapeziums, and the ends are supposed to be rectangular. Let $ABCD$ represent one of the sections. Then its area $= \frac{1}{2}(AB + CD) \times$ perpendicular distance between AB and $CD = \frac{1}{2}(AB + CD) \times AD$, nearly. The object of the reckoning is to settle the sawyer's charge which is at a certain rate for each square cubit along which the sawing is made. Hence the above area must be multiplied by the number of sections to get the total area for which the charge is to be reckoned.]



229. **Example. Tell me quickly, friend, what the reckoning will be in cubits, for a timber the thickness**

[1] *Krakacha-vyavahára*, determination of the reckoning concerning the saw (*krakacha*) or iron instrument with a jagged edge for cutting wood.—Súr.

[2] The concluding half of one stanza begun in the preceding rule (§ 225), and the first half of another stanza of like metre completed in the following rule (§ 230).

[3] To reduce superficial fingers to superficial cubits.

of which is twenty fingers at the root, and sixteen fingers at the tip, and the length a hundred fingers, and which is cut by four sections.

Statement : length 100 ; thicknesses 20 and 16. Number of sections, 4.

Half the sum of the thicknesses at the two extremities, 18, multiplied by the length, makes 1800; this multiplied by the number of sections, gives 7200 ; divided by 576, gives the quotient in cubits, $\frac{25}{2}$.

230. Rule: half a stanza. But when the wood is cut across, the superficial measure is found by the multiplication of the thickness and breadth, in the mode above mentioned.[1]

[The reason for the rule is obvious.]

231. Maxim. The price for the stack of bricks or the pile of stones, or for excavation and sawing, is settled by the agreement of the workman, according to the softness or hardness of the materials.[2]

232. Example. Tell me what the superficial measure in cubits will be, for nine cross sections of a timber, of which the breadth is thirty-two fingers, and thickness sixteen.

Statement : breadth 32 ; thickness 16. Number of sections, 9.

Answer : 8 superficial cubits.

[The timber is supposed to be in the form of a rectangular parallelopiped.]

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[1] If the breadth be unequal, the mean breadth must be taken.—Gan. and Súr.

[2] This is levelled at certain preceding writers who have given rules for computing specific prices or wages, as A'rya-bhatta quoted by Ganesa, and as Brahmagupta (XII, 49); particularly in the instance of sawyer's work, by varying the divisors according to the difference of the timber.

CHAPTER X.

MOUND[1] OF GRAIN.

233. Rule. The tenth part of the circumference is equal to the depth (height[2]) in the case of coarse grain; the eleventh part, in that of fine; and the ninth, in the instance of bearded corn.[3] A sixth of the circumference being squared and multiplied by the depth (height), the product will be the solid cubits:[4] and they are *khárís* of *Magadha*.[5]

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[1] *Rási-vyavahára*, determination of a mound (of grain).

[2] *Bedha*, depth. Here it is the height in the middle from the ground to the summit of the mound.—Súr.

[3] *Anu, súkshma-dhánya*, fine grain, as mustard seed, &c.—Gan. As Paspalum Kora, &c.—*Mano.* As wheat, &c.—Súr.

Ananu, sthúla-dhánya, coarse grain, as chiches (cicer arietinum).—Gan. and Súr. As wheat, &c.—*Mano.* Barley, &c.—Chaturveda on Brahm.

Súkin, súka-dhánya, bearded corn, as rice, &c.

The coarser the grain, the higher the mound. The rule is founded on trial and experience; and for other sorts of grain, other proportions may be taken, as $9\frac{1}{2}$ or $10\frac{1}{2}$ or 12 times the height, equal to the circumference.—Gan. and Súr. The rule is taken from Brahmagupta, XII, 50.

[4] This is a rough calculation, in which the diameter is taken at one-third of the circumference. The content may be found with greater precision by taking a more nearly correct proportion between the circumference and diameter.—Gan.

[5] See § 7. The proportion of the *khárí* or other dry measure of any province to the solid cubit being determined, a rule may be readily formed for computing the number of such measures in a conical mound of grain. Ganesa accordingly delivers rules by him devised for the *khárí* of *Nandigráma* and for that of *Devagiri*: 'the circumference measured by the human cubit, squared and divided by sixteen, gives the *khárí* of *Nandigráma*; and by sixty, that of *Devagiri*.' (*Devagiri*, lit. mountain of the gods, is better

[The mound is supposed to be conical, the height being stated arbitrarily. The circumference of the base being given, the height will of course depend on the vertical angle of the cone. The rule is very rough, the value of π being taken equal to 3, as Ganesa remarks.

Let r denote the radius of the base, and h the height. Then the volume of the mound $= \frac{1}{3}\pi r^2 h = \frac{(2\pi r)^2}{3 \times 4\pi} \times h = \left(\frac{\text{circumference}}{6}\right)^2 . h$, supposing $\pi = 3$.]

234. Example. Mathematician, tell me quickly how many *khárís* are contained in a mound of coarse grain standing on even ground, the circumference of which (mound) measures sixty cubits ; and separately say how many (are there) in a like mound of fine grain and in one of bearded corn.

Statement : circumference 60 ; height 6.

Answer : 600 *khárís* of coarse grain. But of fine grain, the height is $\frac{60}{11}$, and quantity thence deduced, $\frac{6000}{11}$. So, of bearded corn, the height is $\frac{60}{9}$, and quantity $\frac{6000}{9}$ *khárís*.

235. Rule. In the case of a mound piled against the side of a wall, or against the inside or outside of a corner of it, the product is to be sought with the circumference multiplied by two, four, and one and a third ; and is to be divided by its own multiplier.[1]

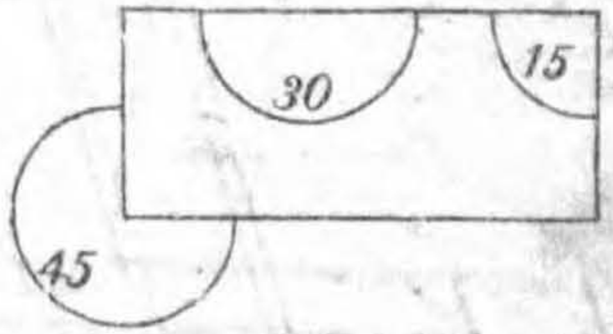
known by the name of *Daulatabad*, which the Emperor Muhammad conferred on it in the 14th century. *Nandigráma*, lit. the town or village of *Nandi*, Siva's bull and vehicle, retains the antique name, and is situated about 65 miles west of *Devagiri*.) He further observes that the cubit intended by the text is a measure in use with artisans, called in vulgar speech *gaj* ; and a *khárí* equal to such a solid cubit will contain twenty-five *manas* and three quarters.

[1] Against the wall, the mound is half a cone ; in the inner corner, a quarter of a cone ; and against the outer corner, three quarters. The circumference intended is a like portion of a circular base ; and the rule finds the content of a complete cone, and then divides it in the proportion of the part. See Gan., &c.

[The reason for the rule is clear from the foot-note. Th circumferences that are supposed to be given in the three case are respectively half, one-fourth, and three-fourths of that o the base of the complete conical mound.]

236—237. Example : two stanzas. Tell me promptly, friend, the number of solid cubits contained in a mound of grain, which rests against the side of a wall, and the circumference of which measures thirty cubits ; and that contained in one piled in the inner corner and measuring fifteen cubits ; as also in one raised against the outer corner and measuring nine times five cubits.

Statement :



Twice the first mentioned circumference is 60. Four times the next is 60. The last multiplied by one and a third is likewise 60. With these the product is alike 600. This being divided by the respective multipliers, gives the several answers,[1] 300, 150 and 450.

[1] For coarse grain : but the product is $\frac{6000}{11}$ for fine, and $\frac{6000}{9}$ for bearded corn ; and the answers are $\frac{3000}{11}$, $\frac{1500}{11}$, $\frac{4500}{11}$; and $\frac{3000}{9}$, $\frac{1500}{9}$, $\frac{4500}{9}$.—Gan &c.

CHAPTER XI.

SHADOW[1] OF A GNOMON.

238. **Rule.[2] The number five hundred and seventy-six being divided by the difference of the squares of the differences of both shadows and of the two hypotenuses,[3] and the quotient being added to one, the difference of the hypotenuses is multiplied by the square root of that sum ; and the product being added to, and subtracted from, the difference of the shadows, the moieties of the sum and difference are the shadows.**

[The translation of the last sentence is not quite correct. It should be, "and the difference of the shadows being added to and subtracted from the product, the moieties, &c."

The rule, as the author hints in the example which follows (§239), is founded on the algebraic solution of a quadratic equation. Ganesa gives it at length after the manner of the author's *Víja-ganita*. It is however very long and not at all

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[1] *Chháyá-vyavahára*, determination of shadow; that is measurement by means of a gnomon.

[2] The difference of the shadows and difference of the hypotenuses being given, to find the length of the shadows and hypotenuses.—Súr.

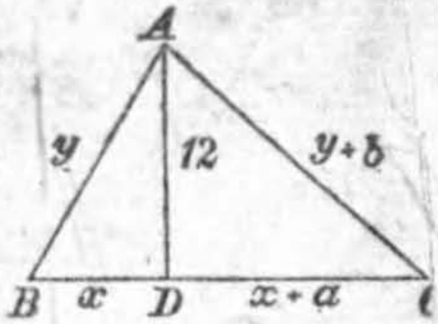
This rule is the first in the chapter, according to all the commentators except Súryadása, who begins with the next, §240, and places this after §244.

[3] *Chháyá*, *bhá*, *prabhá*, shadow.

Sanku, *nara*, *nri*, a gnomon, usually 12 fingers long.

Karna, hypotenuse of the triangle, of which the gnomon is the perpendicular, and the shadow the base.

clear, and so it has not been given in the foot-note. The r may be demonstrated after the manner of modern algebra follows :—



Let ABC be a triangle, and AD perpendicular to BC. Then BD, DC are called the shadows of the *sanku* or gnomon AD, which is supposed to be 12 fingers long ; BD, DC being supposed to be the shadows on a horizontal plane of the gnomon AD, produced either by sunlight at two different hours of the day or by artificial light. The object of the rule is to find these shadows, the difference of AC, AB, and of CD, DB being supposed known.

Let $BD=x$, $DC=x+a$, $AB=y$, $AC=y+b$, a and b being known, and the measurements being in fingers.

Then $y^2-x^2=(y+b)^2-(x+a)^2=144$;

$$\therefore\ by=ax+\frac{a^2-b^2}{2},$$

whence by squaring and substituting x^2+144 for y^2, we obtain the quadratic

$$x^2+ax+\left(\frac{a^2-b^2}{4}-\frac{144b^2}{a^2-b^2}\right)=0,$$

solving which we get $x=\frac{1}{2}\left\{-a+b\sqrt{\left(1+\frac{576}{a^2-b^2}\right)}\right\}$,

(the upper sign only being admissible),

and $x+a=\frac{1}{2}\left\{a+b\sqrt{\left(1+\frac{576}{a^2-b^2}\right)}\right\}$.

These results stated in words lead to the rule. The rule is not of much importance.]

239. Example. The ingenious man, who tells the shadows of which the difference is measured by nineteen, and the difference of hypotenuses by thirteen, I take to be thoroughly acquainted with the whole of algebra as well as arithmetic.

Statement : difference of shadows, 19 ; difference of hypotenuses, 13. (Gnomon 12.)

Difference of their squares 192. By this divide 576 : quotient 3. Add one. Sum 4. Square root 2. By this multiply the difference of hypotenuses 13 : product 26. Add it to, and subtract it from, the difference of the shadows 19.

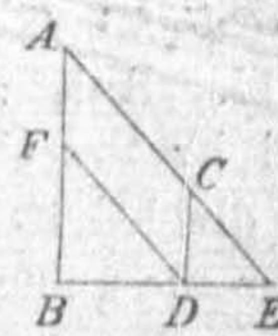
[The translation here is incorrect; it should be, "add to it, and subtract from it, the difference, &c." See note to § 238.]

Half the sum and difference are the shadows, *viz.*, $\frac{45}{2}$ and $\frac{7}{2}$.

Under the rule in § 134, the gnomon being the upright, and the shadow the side, the square root of the sum of their squares is the hypotenuse. Thus the hypotenuses are $\frac{51}{2}$ and $\frac{25}{2}$.

240. Rule[1]: half a stanza. The gnomon multiplied by the distance of its foot from the foot of the light, and divided by the height of the torch's flame less the gnomon, will be the shadow.

[The rule follows from similar triangles as explained by Súryadása, as follows :—



Let A be the position of the light, CD the gnomon, and DE its shadow. From A draw AB perpendicular to ED produced. Through D draw DF parallel to AE. Then from the similar triangles CDE, FBD, we get $\frac{DE}{DC}=\frac{BD}{BF}$, whence $DE=\frac{BD \,.\, DC}{AB-CD}$, $\because CD = AF$. Hence the rule.]

241. Example. If the base between the gnomon and torch be three cubits, and the elevation of the light, three cubits and a half, say quickly, friend, how

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[1] The elevation of the light and (horizontal) distance of its foot from the foot of the gnomon being given, to find the shadow.— Gan.

much the shadow of a gnomon will be which measures twelve fingers.

Statement : gnomon $\frac{1}{2}$; distance between gnomon and torch, $\frac{7}{2}$; elevation of the light, 3.

Answer. Proceeding as directed, the shadow comes out 12 fingers.

242. Rule[1]: half a stanza. The gnomon being multiplied by the distance between it and the light, and divided by the shadow, and the quotient being added to the gnomon, the sum is the elevation of the torch.

[As Súryadása remarks, this rule also follows from similar triangles. See figure in the note to § 240.]

243. Example. If the base between the torch and gnomon be three cubits, and the shadow be equal to sixteen fingers, how much will be the elevation of the torch ? And tell me what the distance is between the torch and gnomon (if the elevation be given.)

Statement : distance between torch and gnomon, 3; shadow $\frac{2}{3}$.

Answer : height of the torch $\frac{11}{4}$.

244. Rule[2] : half a stanza. The shadow, multiplied by the elevation of the light less the gnomon and divided by the gnomon, will be the interval between the gnomon and light.

[This like the preceding rule also follows from similar triangles.]

Example, as before proposed (§ 243.)

Answer : distance 3 cubits.

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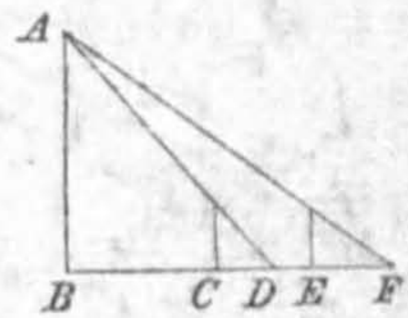
[1] To find the elevation of the torch, the length of the shadow, and the (horizontal) distance being given.—Súr.

[2] To find the (horizontal) distance, the elevation of the torch and length of the shadow being given.—Gan. and Súr.

245. Rule[1] : a stanza and a half. The length of a shadow multiplied by the distance between the terminations of the shadows and divided by the difference of the lengths of the shadows, will be the base. The product of the base and the gnomon, divided by the length of the shadow, gives the elevation of the torch's flame.[2]

In like manner is all this, which has been before declared, pervaded by the Rule of Three with its variations, as the universe is by the Deity.[3]



[Let A be the position of the light, C, E, the positions of the foot of the gnomon, and CD, EF, the corresponding shadows. Let $BC = x$, $BA = y$, $CD = a$, $EF = b$, $CE = c$, the measurements being in fingers.

Then from similar triangles we have,

$$\frac{y}{x+a} = \frac{12}{a}, \quad \frac{y}{x+b+c} = \frac{12}{b};$$

whence $x = \dfrac{ac}{b-a}$.

$$\therefore x + a = \frac{a(b+c-a)}{b-a} \quad . \; . \; . \; . \; . \; (1),$$

$$\text{and } y = \frac{12\,(x+a)}{a} = \frac{12\,(x+b+c)}{b},$$

whence the reason for the rule is obvious. Bháskara's own explanation is practically the same as the above, but it is not clearly put. He at once states a proportion which is equivalent to equation (1) above, but he does not explain how it is obtained.]

[1] The gnomon being set up successively in two places, the distance between which is known, and the length of the two shadows being given, to find the elevation of the light, and the base.—Gan and Súr.

[2] The rule is borrowed from Brahmagupta (XII, 54).

[3] The author intimates that the whole preceding system of computation, as well as the rules contained under the present head, as those before delivered, is founded on the rule of proportion.—Gan.

246. Example. The shadow of a gnomon measuring twelve fingers being found to be eight, and that of the same placed on a spot two cubits further in the same direction, being measured twelve fingers, say, intelligent mathematician, how much the distance of the shadow[1] from the torch is, and the height of the light, if thou be conversant with computation, as it is termed, of shadow.

Statement: shadows, 8, 12; interval between the positions of the foot of gnomon, 48.

Here the interval between the termination of the shadows is in fingers 52. The first shadow 8, multiplied by the interval 52, and divided by the difference of the length of the shadows, *viz.*, 4, gives the length of the base 104. It is the distance between the foot of the torch and the tip of the first shadow. So the length of the base to the tip of the second shadow is 156.

The product of the base and gnomon, divided by the shadow, gives both ways the same elevation of the light, *viz.*, $6\frac{1}{2}$ cubits.

"In like manner, &c."[2]—Under the present head of measurement of shadow, the solution is obtained by putting a proportion: *viz.*, if so much of the shadow, as is the excess of the second above the first, give the base intercepted between the tips of the shadows, what will the first give? The distances of the terminations of the shadows from the foot of the torch are in this manner severally found. Then a second pro-

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[1] All the commentators appear to have read 'gnomon' in this place; but one copy of the text exhibits 'shadow' as the reading: and this seems to be correct.

[2] Reference to the text, §245. [The author here purports to explain fully what he has hinted at before.—Ed.]

rtion is put: if, the shadow being the side, the gnoon be the upright; then, the base being the side, hat will be the upright? The elevation of the torch thus found; and is both ways (that is, computed ith either shadow,) alike.

[See note to § 245.]

So the whole sets of five or more terms are exlained by twice putting three terms and so forth.

As the Being, who relieves the minds of his worippers from suffering, and who is the sole cause of h roduction of this universe, pervades the whole, id does so with his various manifestations, as worlds, aradises,[1] mountains, rivers, gods, demons, men, trees,[2] ad cities; so is all this collection of instructions for omputations pervaded by the rule of three terms. hen why has it been set forth by so many different writers[3], with much labour and at great length)? he answer is :—

247. Whatever is computed either in algebra or this (arithmetic) by means of a multiplier and a ivisor, may be comprehended by the sagacious learned s the rule of three terms. Yet has it been composed y wise instructors in miscellaneous and other manild rules, teaching its easy variations, thinking therey to increase the intelligence of such dull compreensions as ours.

[1] *Bhuvana*, worlds. *Bhavana*, paradises, the abodes of Brahmá and the t of the gods. [The reading here adopted by Colebrooke is apparently ferent from that in Pandit Jívánanda Vidyásàgara's edition in which we re *sakala-bhuvana-bhávanena*, which rendered becomes, 'who is the creator all the worlds,' the words, 'worlds' and 'paradises' in Colebrooke's nslation, being omitted in that case.—Ed.]

[2] *Naga*, either tree or mountain. The term, however, is read in the text none of the commentators besides Ganesa.

[3] As Sridhara and the rest.—*Mano.* As Brahmagupta and others.—*Gang.*

CHAPTER XII.

PULVERIZER.

248—252. Rule: five stanzas.

248. In the first place, as preparatory to the in vestigation of a pulverizer,[1] the dividend, divisor an

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[1] *Kuttaka-vyavahára* or *kuttakádhyáya*, determination of a grinding c pulverizing multiplier, or quantity such that a given number being mult plied by it, and the roduct added o a given quantity, the sum (or, if th additive be negative, the difference) may be divisible by a given diviso without remainder. *Kuttaka* or *kutta* from *kutt*, to grind or pulverize; (t multiply: all verbs importing tendency to destruction also signifying multi plication.—Gan.) The derivative import of the word is retained in th present version to distinguish it from multiplier in general; *kuttaka* bein restricted to the particular multiplier of the problem in question.

According to the remark of Ganesa, this chapter as well as the followin chapter on combination belongs to algebra rather than arithmetic; an they are here introduced, as he observes, and treated without employin algebraic forms, to gratify such as are unacquainted with analysis. Se *Víja-ganita*, Chap. II, from which the present chapter is borrowed, th contents being copied, with some variation of the order, nearly word fo word.

Ganesa notices an objection, namely, that this subject ought not to hav been introduced into a treatise on arithmetic, while a passage of A'rya bhatta expressly distinguishes it from both arithmetic and algebra: "th multifarious doctrine of the planets, arithmetic, the pulverizer (*kuttaka*) and analysis (*víja*) and the rest of the science treating of seen (or physical objects." He answers the objection by saying that mathematics (*ganita* consists of two branches treating of known and of unknown quanti (*vyakta-ganita* and *avyakta-ganita*); that the investigation of the pulve rizer is comprehended in algebra; and that the separate mention of thi subject by A'ryabhatta and other ancient authors is intended to indicate it difficulty and importance. In Brahmagupta's work, the whole of algebr is comprised under the title of *kuttakádhyáya*, chapter on the pulverize See Brahm., Ch. XVIII.

additive quantity[1] are, if practicable, to be reduced by some number.[2] If the number, by which the dividend and divisor are both measured, do not also measure the additive quantity, the question is an ill put (or impossible) one.

249—251. The last remainder, when the dividend and divisor are mutually divided, is their common measure.[2] Being divided by that common measure, they are termed reduced quantities.[3] Divide mutually the reduced dividend and divisor, until unity be the remainder in the dividend. Place the quotients one under the other, and the additive quantity beneath them, and cipher at the bottom. By the penult multiply the number next above it and add the lowest term. Then reject the last and repeat the operation until a pair of numbers be left. The uppermost of these being abraded[4] by the reduced dividend, the remainder is the quotient. The other (or lowermost) being in like manner abraded by the reduced divisor, the remainder is the multiplier.

252. Thus precisely is the operation when the number of quotients is even. But if the number be

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[1] *Kshepa* or *yuti*, additive; from *kship* to cast or throw in, and from *yu* to mix. *Visuddhi*, subtractive quantity.

[2] *Apavartana*, abridgment.—Gan. Reduction to least terms, division without remainder; also the number which serves to divide without residue, the common measure.—Súr. [It is really the greatest common measure.—Ed.]

[3] *Dridha*, firm; reduced by the common divisor to the least term. The word is applicable to the reduced additive, as well as to the dividend and divisor.

[4] *Tashta* abraded; from *taksh*, to pare or abrade: divided, but the residue taken, disregarding the quotient.—Súr. As it were a residue after repeated subtractions.—Gang.

Takshana, the abrader; the divisor employed in such operation.

odd, the numbers as found must be subtracted from their respective abraders, and the residues will be the true quotient and multiplier.

[This Chapter, as Ganesa remarks, properly belongs to Algebra and not to Arithmetic. We have already seen, however, that the present treatise deals with both Arithmetic and Algebra.

The whole of this Chapter is occupied with problems producing indeterminate equations of the first degree, and the object of the several rules is to find positive integral solutions of such equations. The reason for the above rule will be best understood from the example in § 253. Let y denote the multiplier and x the integral quotient. Then we get $\frac{221x + 65}{195} = x$, and the object of the rule is to find positive integral solutions of this equation.

Dividing by the common measure 13, we get $\frac{17y + 5}{15} = x$,

$\therefore\ 15x - 17y = 5 \ldots\ldots (1)$.

This equation is of the type $Ax - By = C$, A being less than B, and the rule refers to cases of this class. Convert $\frac{B}{A}$ into a continued fraction, and suppose the result is $\frac{B}{A} = a + \frac{1}{b+} \frac{1}{c+\&c.}$

Let $\frac{q}{p}$ be the convergent immediately preceding $\frac{B}{A}$. Then we know that $x = qC$, $y = pC$, or $x = (B-q)\ C$, $y = (A-p)\ C$, is one solution of equation (1), according as $Aq - Bp = \pm\ 1$; and that the general solution is $x = \alpha + Bt$, $y = \beta + At$, where α and β are one set of values of x and y, and where we may give to t any positive integral value, and also such negative integral values as make Bt and At numerically less than α and β respectively. (See Todhunter's Algebra, Arts. 630, 631.) Now the succesive convergents to $\frac{B}{A}$ are $\frac{a}{1}$, $\frac{ab+1}{b}$, $\frac{abc+a+c}{bc+1}$, &c., the law of formation being well known (see Todhunter's Algebra, Art. 604); and the object of the rule in § 251 is to find the value of the convergent $\frac{q}{p}$, and thence the value of the quantities qC, pC.

The rule, however, is very vaguely and obscurely expressed, and it is difficult to understand its working. An explanation of the rule by means of a particular example is given by Krishna in his commentary on the *Vija-ganita*, from which we may deduce the following general explanation. Let us first consider the case where the number of quotients exclusive of the last one is two, *viz.*, a, b, the additive being C. Then according to the rule we get the series a, b, C, 0. The rule next directs us to multiply the penultimate (C) by the number preceding it, *viz.*, b, and to add the last term, *viz.*, 0, to the product. We thus get $bC+0$ or bC, and we have now to replace the previous multiplier b by this quantity bC, and reject the last term 0. We thus have the new series, a, bC, C, with which we have to repeat the above operation: that is, we have to multiply the penultimate bC, by a, and add the last term C to the product; whence we get $abC+C$, by which we have to replace the multiplier a, and we have to reject C. The series thus becomes $abC+C$, bC, and this consisting only of two terms, we infer according to the rule that $qC=abC+C$, and $pC=bC$. But we know as matter of fact that in the case we are considering, $q=ab+1$, and $p=b$; thus the rule holds good in this case. We may now prove by induction that the rule holds universally. For supposing the number of quotients exclusive of the last one to be three, *viz.*, a, b, c, all we have got to do is to take the additional quotient c, so that we have to write $b+\frac{1}{c}$ for b in the above expressions for qC, pC; and we get

in this case $\dfrac{qC}{pC}=\dfrac{a\left(b+\frac{1}{c}\right)C+C}{\left(b+\frac{1}{c}\right)C}=\dfrac{a(bcC+C)+cC}{bcC+C}$, so that

$qC=a\ (bcC+C)+cC$, and $pC=bcC+C$. But it is easy to see from the very nature of these expressions that they are precisely what we would get by means of the rule, if we take into consideration the quotient c. Thus we see that the rule is universally true. In the present example, $a=1$, $b=7$, $c=2$, and 7 is the last quotient but one. Hence, $q=ab+1=7+1$, and $qC=7\times5+5$; $p=b=7$, and $pC=7\times5$. And $\because$ in this case $Aq-Bp$

$=1$, $\therefore x=40$, $y=35$, is one solution. Hence putting $\alpha=40$, $\beta=35$, and $t=-2$, in the general expressions for x and y, we get $x=40-17\times 2=6$, $y=35-15\times 2=5$. These are the least positive integral values. Putting $t=-1$, we get $x=23$, $y=20$, and so on. Or taking $x=6$, $y=5$, as one solution, we may get others from the expressions $6+17\,t$, $5+15\,t$, by giving to t any positive integral value. Thus putting $t=1$, we get $x=23$, $y=20$; putting $t=2$, we get $x=40$, $y=35$; and so on. Thus the reason for the process in § 253, as well as that for the rule in § 262, is clear. The meaning of the term *abraded* as used by the author is also clear.]

253. Example. Say quickly, mathematician, what that multiplier is, by which two hundred and twenty-one being multiplied, and sixty-five added to the product, the sum divided by a hundred and ninety-five becomes exhausted.

Statement : dividend 221
divisor 195 . Additive 65.

Here the dividend and divisor being mutually divided, the last of the remainders (or divisors) is 13. By this common measure, the dividend, divisor and additive, being reduced to their least terms, are dividend 17, divisor 15, additive 5. The reduced dividend and divisor being divided reciprocally, and the quotients put one under the other, the additive under them, and cipher at the bottom, the series[1] which results is 1
7
5
0

Then multiplying by the penult the number above it and proceeding as directed, the two quantities are obtained 40
35. These being abraded by the reduced

[1] *Valli*, series.

dividend and divisor 17 and 15, the quotient and multiplier are obtained 6 and 5. Or, by the subsequent rule (§ 262), adding them to their abraders multiplied by an assumed number, the quotient and multiplier (putting 1) are 23 and 20 ; or putting 2, they are 40 and 35; and so forth.

254. Rule. The multiplier is also found by the method of the pulverizer, the additive quantity and dividend, being either reduced by a common measure (or used unreduced[1]). But if the additive and divisor be so reduced, the multiplier found, being multiplied by the common measure, is the true one.

[The reason for the rule will appear from the solution of the example in § 255.]

255. Example. If thou be expert in the investigation of such questions, tell me the precise multiplier by which a hundred being multiplied, with ninety added to the product, or subtracted from it,[2] the sum or the difference may be divisible by sixty-three without a remainder.

Statement: dividend 100. Additive 90.
divisor 63

The quotient and multiplier are found by proceeding as before, 30 and 18.

Or, the dividend and additive being reduced by the common measure ten, we get dividend 10, divisor 63, additive 9. Placing the quotients of reciprocal division, the additive quantity and cipher, one under the other,

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[1] Gan.

[2] An example of the subsequent rule in § 256.

the series is $\begin{array}{c}0\\6\\3\\9\\0\end{array}$. And the multiplier is found by the former process 45. The quotient[1] (3) is here not to be taken ; and the number of quotients (of the series) being odd, the mult plier 45 is to be subtracted from its own abrader 63; the true multiplier is thus found 18. The dividend being multiplied by that multiplier, and the additive quantity being added, and the sum divided by the divisor, the quotient is found 30.

Or, the divisor and additive quantity being reduced by the common measure nine, we get dividend 100, divisor 7, additive 10. Here the quotients, the additive and cipher make the series $\begin{array}{c}14\\3\\10\\0\end{array}$. The multiplier is found 2, which multiplied by the common measure 9, gives the true multiplier 18.

Or, the dividend and additive being reduced, and further the divisor and additive, by common measures, we get dividend 10, divisor 7, additive 1. Proceeding as before the series is $\begin{array}{c}1\\2\\1\\0\end{array}$. The multiplier hence deduced is 2, which taken into the common measure 9, gives 18 ; and hence, by multiplication and division, the quotient comes out 30.

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[1] [This probably means the *last* quotient. There is, however, no force in the remark; the last quotient being always excluded under the rule in §§ 250—251.—ED.]

Or, adding the quotient and multiplier as found, to (multiples of) their respective divisors, the quotient and multiplier are 130 and 81 ; or 230 and 144 ; and so forth.

[Putting $y=$ multiplier, and $x=$ quotient, we get the equation $63x-100y=90$......(1). If we convert $\frac{100}{63}$ into a continued fraction, the number of quotients will be found to be large, and so it will be tedious to form $\frac{q}{p}$. To make the process shorter, let us first consider the equation $63x-10y=9$...... (2). Now if from (2) we find sets of positive integral values of x and y, it is clear that those values of y and 10 times the corresponding values of x will be sets of positive integral values of x and y satisfying (1). The general solution of (2) will be found to be $x=(10-3)9+10t$, $y=(63-19)9+63t$. Putting $t=-6$, we get $x=3$, $y=18$, as the least values. Thus $x=30$, $y=18$, are the least values satisfying (1). In the text, the least value of y is found from the expression $63-(19\times 9+63t')$, by putting $t'=-2$, and the reason for this is stated to be the fact that the number of quotients (exclusive of the last one) is odd. The explanation is that the number of quotients exclusive of the last one being odd, $\frac{q}{p}$ is less than $\frac{B}{A}$ (Todhunter's Algebra, Art. 603), and so $Aq-Bp=-1$, and $\therefore$ the general solution is $x=(B-q)C+Bt$, $y=(A-p)C+At$......(3). In the text, the value of y is derived from the expression $A-(pC+At')$...(4). Supposing $\therefore$ that the expressions in (3) and (4) give the same values of y for certain values of t and t', the relation between such values will be given by $t+t'=1-C$. Thus it is clear that we can derive values of y from both these expressions, but not from the expression $pC+At$, when the number of quotients exclusive of the last one is odd. This also explains the statement in §252.

Similarly, if we find values of x and y from the equation $7x-100y=10$, these values of x and 9 times the corresponding values of y will satisfy (1).

Lastly, if we find values of x and y from the equation $7x-10y=1$, it is easy to see that 10 times these values of x, and 9 times the corresponding values of y will satisfy (1). Thus the reason for the rule in § 254 is obvious.]

256. Rule[1] : half a stanza. The multiplier and quotient, as found for an additive quantity, being subtracted from their respective abraders, answer for the same as a subtractive quantity.

Here the quotient and multiplier as found for the additive quantity ninety in the preceding example, namely, 30 and 18, being subtracted from their respective abraders, namely, 100 and 63, the remainders are the quotient and multiplier which answer when ninety is subtractive : *viz.*, 70 and 45.

Or, these being added to arbitrary multiples of their respective abraders, the quotient and multiplier are 170 and 108, or 270 and 171, &c.

[Let C be the additive or subtractive quantity. Then the corresponding equations will be $Ax-By=\ \ C$......(1).

$$Ax-By=-C\text{......(2)}.$$

Let $x=\alpha$, $y=\beta$, be a solution of (1). Then $A\alpha-B\beta=C$.

$\therefore\ A(B-\alpha)-B(A-\beta)=-C.$

$\therefore\ x=B-\alpha$, $y=A-\beta$, is a solution of (2), whence the reason for the rule is clear. It will be readily seen that the general solution of (2) is $x=(B-\alpha)+Bt$, $y=(A-\beta)+At$.]

257. Another example[2]. Tell me, mathematician, the multipliers severally, by which sixty being multiplied, and sixteen being added to the product, or subtracted from it, the sum or difference may be divisible by thirteen without a remainder.

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[1] The rule serves when the additive quantity is negative.—Gan. and Súr.

[2] This additional example is unnoticed by Ganesa, but expounded by the rest of the commentators, and found in all copies of the text that have been collated.

Statement : dividend 60 / divisor 13 . Additive 16.

The series found as before, is 4.
1
1
1
1
16
0

Hence the multiplier and quotient are deduced 2 and 8. But the number of quotients (of the series) is here uneven ; wherefore the multiplier and quotient must be subtracted from their abraders 13 and 60 ; and the multiplier and quotient, answering to the additive quantity sixteen, are 11 and 52. These being subtracted from the abraders, the multiplier and quotient, corresponding to the subtractive quantity sixteen, are 2 and 8.

258. Rule[1] : a stanza and a half. The intelligent calculator should take a like quotient (of both divisions) in the abrading of the numbers for the multiplier and quotient (sought). But the multiplier and quotient may be found as before, the additive quantity being (first) abraded by the divisor ; the quotient, however, must have added to it the quotient obtained in the abrading of the additive. But in the case of a subtractive quantity, it is subtracted.

[The reason for the rule will appear from the solution of the example which follows.]

259. Example. What is the multiplier, by which five being multiplied and twenty-three added to the

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[1] Applicable when the additive quantity exceeds the dividend and divisor. —Gan.

product, or subtracted from it, the sum or difference may be divided by three without remainder?

Statement : dividend 5 / divisor 3. Additive 23.

Here the series is 1 and the pair of numbers found
1
23
0

as before 46 / 23. They are abraded by the dividend and divisor, respectively. The lower number being abraded by 3, the quotient is 7 (and residue 2). The upper number being abraded by 5, the quotient would be 9 (and residue 1); nine, however, is not to be taken; but, under the rule for taking like quotients, seven only, (and the residue consequently is 11). Thus the multiplier and quotient come out 2 and 11.

And by the former rule (§ 256) the multiplier and quotient answering to the same as a negative quantity are found, 1 and the negative quantity—6. Added to arbitrary multiples of their abraders, double for example, so that the quotient may be positive, the multiplier and quotient are 7 and 4. So in every (similar) case.

Or, statement for the second (part of the) rule: dividend 5 / divisor 3. Additive abraded[1] 2.

The multiplier and quotient hence found as before are 2 and 4. These subtracted from their respective divisors, give 1 and 1, as answering to the subtractive quantity. The quotient obtained in the abrading of the additive, (*viz.* 7) being added in one instance and

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[1] 23, abraded by the divisor 3, gives the quotient 7 and residue 2.

subtracted in the other, the results are 2 and 11 answering to the additive quantity, and 1 and -6 answering to the subtractive: or, to obtain a positive quotient, add to the latter twice their divisors, and the result is 7 and 4.

[Putting y = multiplier, and x = quotient, we get the equations $3x-5y=\pm 23 \ldots (1)$. Taking the upper sign, the general solution will be found to be $x=2\times 23+5t$, $y=1\times 23+3t$. The least positive integral values are got by putting $t=-7$, *viz.*, $x=11$, $y=2$. The meaning of the first part of the rule in §258 is that the *same negative* value is to be given to t in the expressions for x and y. To explain the second part of the rule, we observe that equation (1) may be written $\frac{5y+2}{3}=x\mp 7=X$ suppose. We may then solve $3X-5y=2 \ldots\ldots (2)$, the values of y being the same in (1) and (2), and the values of x being deducible from those of X, from the re[illegible]tion [illegible] $X\pm 7$.]

260. Rule[1]: one stanza. If [illegible]o additive qu[illegible] or if the additive be [illegible]e divisor, the multiplier [illegible]ered a[illegible] the quotient as the additive divided by the divisor.[2]

[The rule is clear enough.]

261. Example. Tell me promptly, mathematician, the multiplier by which five being multiplied and added to cipher, or added to sixty-five, the division by thirteen shall in both cases be without remainder.

Statement: dividend 5 / divisor 13. Additive 0.

There being no additive, the multiplier and quotient are 0 and 0; or 13 and 5; or 26 and 10; and so forth.

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[1] Applicable if there be no additive, or if it be divisible by the divisor without remainder.

[2] It is so in the latter case; but in the former (where the additive is null), the quotient is cipher.—Súr.

Statement : dividend 5 / divisor 13 . Additive 65.

By the rule (§ 260), the multiplier and quotient come out 0 and 5 ; or 13 and 10 ; or 26 and 15 ; and so forth.

[Putting y = multiplier, and x = quotient, we get in the first case the equation $\frac{5y+0}{13}=x$; and in the second case, the equation $\frac{5y+65}{13}=x$. The general solution in positive integers of the first equation will be readily found to be $x=5r$, $y=13r$; and that of the second, $x=5\ (1+r)$, $y=13r$, where r may be zero or any positive integer.]

Rule.[1] Or, the dividend and additive being abraded by the divisor, the multiplier may thence be found as before ; and the quotient from it, by multiplying the dividend, a[illegible] the additive, and dividing by the divisor.

In the [illegible] (§ 253), the red[illegible] dividend, divisor a[illegible] additive respectively are, 17, 15, 5. Abraded by the divisor (15) the additive and dividend become 5 and 2 ; and the statement is :—

dividend 2 / divisor 15 . Additive 5.

Proceeding as before the two terms found are 5, 35. The second one, abraded by the divisor (15), gives the multiplier 5 ; whence, by multiplying with it the dividend (17) and adding (the additive), and dividing (by the divisor), the quotient comes out 6.

[The reason for the above rule is clear. Let the equation be $Ax-By=C$, and suppose B greater than A, and C less than

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[1] This is found in one copy of the text, and is expounded only by Gangádhara, being unnoticed by the other commentators. It occurs, however, in the similar chapter of the *Víja-ganita*, §62.

A. Divide B by A; let K denote the quotient, and B' the remainder. Thus $B=KA+B'$, and $\therefore \frac{B}{A}=K+\frac{B'}{A}$. Convert $\frac{B}{A}$ and $\frac{B'}{A}$ into continued fractions, and let $\frac{q}{p}$ and $\frac{q'}{p'}$ be the convergents immediately preceding $\frac{B}{A}$ and $\frac{B'}{A}$. Then $\frac{q}{p}=K+\frac{q'}{p'}=\frac{Kp'+q'}{p}$. Thus $p=p'$, and $Aq-Bp=A(Kp'+q')-(KA+B')p'=Aq'-B'p'$. Hence it is evident that the values of y found from $Ax-B'y=C$, will be the same as those found from $Ax-By=C$; and y being known, x is of course known from the equation $x=\frac{By+C}{A}$. It must be remarked here that the above rule applies only when the additive C is less than the divisor A, so that the additive abraded by the divisor remains unchanged.]

262. Rule for finding divers multipliers and quotients in every case: half a stanza. The multiplier and quotient, being added to their respective (abrading) divisors multiplied by assumed numbers, become manifold.

The influence and operation of this rule have been already shown in various instances.

[See note to § 252.]

263. Rule for a constant pulverizer[1]: one stanza. Unity being taken for the additive quantity, or for the subtractive, the multiplier and quotient, which may be thence deduced, being severally multiplied by an arbitrary additive or subtractive, and abraded by the respective divisors, will be the multiplier and quotient for such assumed quantity.

In the first example (§253), the reduced dividend and divisor with additive unity furnish this state-

[1] *Sthira-kuttaka*, steady pulverizer.

ment : $\begin{matrix} \text{dividend } 17 \\ \text{divisor } 15 \end{matrix}$. Additive 1.

Here the multiplier and quotient (found in the usual manner) are 7 and 8. These multiplied by an assumed additive five, and abraded by the respective divisors 15 and 17, give the multiplier and quotient 5 and 6, for that additive.

Next, unity being the subtractive quantity, the multiplier and quotient thence deduced are 8 and 9. These multiplied by five and abraded by the respective divisors, give 10 and 11.

So in every (similar) case.

Of this method of investigation great use is made in the computation of planets. On that account something is here said (by way of instance.)

[The above rule is not a new one. The equation is supposed to be $Ax-By=\pm 1 \ldots (1)$, and from what we have already seen, the general solution of this is $x=q+Bt$, or $=(B-q)+Bt$, and $y=p+At$, or $=(A-p)+At$. If now the additive or subtractive be any integer whatever, *i. e.*, if the equation be $Ax-By=\pm C \ldots (2)$, we have only to multiply q or $B-q$, and p or $A-p$ by C in the above expressions for x and y, in order to get the general solution of equation (2). We may thus regard the general value of y found from (1) as a *steady* quantity from which we may derive the general value of y satisfying (2). This shows the propriety of the expression *constant pulverizer*.]

264. A stanza and a half. Let the remainder of seconds be made the subtractive quantity,[1] sixty the dividend, and terrestrial days[2] the divisor. The quo-

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[1] The present rule is for finding a planet's place and the elapsed time, when the fraction above seconds is alone given.—Gan.

[2] The number of terrestrial days in a *kalpa* is stated at 1577916450000. See the *Ganitádhyáya* of the *Siddhánta-siromani*, I, 20—21. [By a *terrestrial* day is meant the *mean solar* day, when it is taken for the purpose of

tient deduced therefrom will be the seconds ; and the multiplier will be the remainder of minutes. From this again the minutes and remainder of degrees are found ; and so on upwards. In like manner, from the remainder of exceeding months and deficient days,[1] may be found the solar and lunar days.

The finding of (the place of) the planet and the elapsed days, from the remainder of seconds in the planet's place, is thus shown. Sixty is there made the dividend ; terrestrial days, the divisor ; and the remainder of seconds, the subtractive quantity : with which the multiplier and quotient are to be found. The quotient will be seconds ; and the multiplier, the remainder of minutes. From this remainder of minutes taken (as the subtractive quantity), the quotient deduced will be minutes ; and the multiplier, the remainder of degrees. The residue of degrees is next the subtractive quantity; terrestrial days, the divisor ; and thirty, the dividend : the quotient will be degrees ; and the multiplier, the remainder of signs. Then twelve is made the dividend; terrestrial days, the divisor ; and the remainder of signs the subtractive quantity : the quotient will be signs ; and the multiplier, the remainder of revolutions. Lastly, the revolutions in a *kalpa* become the dividend ; terrestrial days, the divisor ; and the remainder of re-

astronomical measurement ; but for practical purposes, it is taken as the time from sunrise to sunrise, which would make its duration variable. See *Goládhyáya*, Wilkinson's translation, II, 3, Bápú Deva Sástrí's note : *Súrya-siddhánta*, Burgess's translation, I, 34—40, note.—Ed.]

[1] *Adhi-mása*, additive months ; *Avamadina*, subtractive days. See *Ganitádhyáya*, I, 42. [See also *Goládhyáya*, IV, 10—16, note : *Súrya-siddhánta*, I, 47—50, note.—Ed.] The *adhimásas* in a *kalpa* are 1593300000 $=1602999000000 \div 30 - 4320000000 \times 12$], being the excess of the lunar over the solar months. The *avamas* in a *kalpa* are 25082550000, being the excess of the lunar days over the terrestrial days.

volutions, the subtractive quantity : the quotient will be the elapsed revolutions ; and the multiplier, the number of elapsed days.[1] Examples of this occur (in the *Siromani*) in the chapter of the problems[2] (*Triprasnádhyáya*).

In like manner, the exceeding months in a *kalpa* are made the dividend ; solar days,[3] the divisor ; and the remainder of exceeding months, the subtractive quantity : the quotient will be the elapsed additional months; and the multiplier, the elapsed solar days. So the deficient days in a *kalpa* are made the dividend ; lunar days,[4] the divisor ; and the remainder of deficient days, the subtractive quantity : the quotient will be the elapsed fewer days ; and the multiplier the elapsed lunar days.

[The reason for the rule for finding a planet's place and the elapsed time will be best understood from the illustration given by Ganesa and Gangádhara in arbitrary numbers. Put the terrestrial days in a *kalpa* 19, the revolutions of the planet in the *kalpa* 10, the elapsed days 12. Then we evidently get the proportion, 19 : 12 :: 10 : number of revolutions already performed by the planet, whence the revolutions = $6\frac{6}{19}$. Thus the planet has performed 6 complete revolutions, and $\frac{6}{19}$ of a revolution, so that to find the planet's place, we must reduce the

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[1] The elapsed days of the *kalpa* to the time for which the planet's place is found. See *Ganitádhyáyá*, I, 47—49.

[2] [See also *Goládhyáya*, Chap. XIII.—Ed.]

[3] The solar days in a *kalpa* are 1555200000000 [=4320000000 × 360]. See *Ganitádhyáya*, I, 40.

[The number of solar years in a *kalpa* is 4320000000. See *Súrya-siddhánta*, I, 19, note.—Ed.]

[4] The lunar days, reckoning thirty to the month or synodical revolution are 1602999000000 in the *kalpa*. See *Ganitádhyáya*, I, 40. [See also *Goládhyáya*, II, 3, note. The terrestrial days in a *kalpa* are 1577916450000 See *ibid.*, II, 3, note. These two numbers as given in the *Súrya-siddhánta* (I, 37) are slightly different.—Ed.]

fraction $\frac{6}{19}$ to signs (*rásis*), degrees, minutes and seconds. Now as there are 12 signs in one revolution, 30 degrees in one sign, 60 minutes in one degree, and 60 seconds in one minute, we get $\frac{6}{19}$ of a revolution = 3 signs, $23^\circ. 41'.3''\frac{3}{19}$, and this result indicates the planet's place. Now suppose the remainder of seconds after division by 19, *i.e.*, 3, is alone given, and we have to find the planet's place by an inverse process. Let y denote the remainder of minutes, and x the integral number of seconds. Then it is clear from the process which we adopted in reducing the fraction $\frac{6}{19}$, that $\frac{60y-3}{19}=x$, the general solution of which is given (§256) by $x=60-(57+60t)$, $y=19-(18+19t)$. The only positive integral solution is got by putting $t=0$; then $x=3$, $y=1$. The quotient x is the number of seconds, *viz.*, 3; and the multiplier y is the remainder of minutes, *viz.*, 1. It is easy to see that there can be only one positive integral solution satisfying the conditions of the problem. For x must obviously be less than 60, and y less than 19; so that $57+60t$ must be positive and less than 60, and $18+19t$ must be positive and less than 19. Hence there can be only *one* value of t satisfying these conditions, and consequently only *one* positive integral solution satisfying the problem. Making the necessary changes in the coefficient of y and in the subtractive quantity, and repeating the above process, we clearly obtain the number of minutes, degrees and signs indicating the planet's place, and the elapsed days. Thus the reason for the rule is clear.

Similarly, to find the number of solar days which have elapsed from the beginning of a *kalpa* up to any given epoch, suppose S, y, denote respectively the *saura* days in the *kalpa* and the *saura* days elapsed, and A, a, the corresponding additive months. Now since one additive month occurs in every $32\frac{1}{2}$ solar months (*Goládhyáya*, IV, 9,10), we evidently get the proportion, $S : A :: y : a$, whence $a=\frac{Ay}{S}=x+\frac{b}{S}$ suppose, x being an integer, and $\frac{b}{S}$ a proper fraction. Consequently, $\frac{Ay-b}{S}=x$, and a positive integral solution of this equation will give the solar days and the integral number of additive months that have elapsed. Since by supposition, y is less than S, and x less than A, we can show as above that there can be only *one* positive integral solution, satisfying the conditions of the problem.

The case of finding the elapsed lunar days from the given remainder of deficient days or *avamas*, is precisely similar to the above, it being observed that an *avama* occurs in $64\frac{1}{11}$ lunar days (*Goládhyáya*, IV, 12).

In a period of $32\frac{1}{2}$ solar months there are $33\frac{1}{2}$ lunar months very nearly ; this excess of the number of lunar months, *viz.*, one lunar month is called an *adhimása* or additive month, because a proportionate multiple of it is to be added to the solar months in any given period in order to convert them into lunar months. Again, in a period of $64\frac{1}{11}$ lunar days there are $63\frac{1}{11}$ terrestrial or mean solar days very nearly ; this difference between the two numbers, *viz.*, one mean solar day, is called an *avama* or subtractive day, because a proportionate multiple of it is to be subtracted from the lunar days in any given period in order to convert them into mean solar days.]

265. Rule for a conjunct pulverizer.[1] If the divisor be the same and the multipliers various, then, making the sum of those multipliers the dividend, and the sum of the remainders a single remainder, and applying the foregoing method of investigation, the precise multiplier so found is denominated a conjunct one.

[The reason for the rule will appear from the solution of the example which follows.]

266. Example. What quantity is it, which multiplied by five, and divided by sixty-three, gives a residue of seven ; and the same multiplied by ten

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[1] *Sanslishta-kuttaka* or *sanslishtasphuta-kuttaka*, a distinct pulverizing multiplier belonging to conjunct residues.—Gan. A multiplier deduced from the sum of multipliers and that of remainders.—Súr.

and divided by sixty-three, a remainder of fourteen? Declare the number.[1]

Here the sum of the multipliers is made the dividend, and the sum of the residues, a subtractive quantity; and the statement is as follows :—

$\begin{matrix} \text{dividend} & 15 \\ \text{divisor} & 63 \end{matrix}$. Subtractive 21. Or reduced to least terms :—

$\begin{matrix} \text{dividend} & 5 \\ \text{divisor} & 21 \end{matrix}$. Subtractive 7.

Proceeding as before,[2] the multiplier is found 14.

[In this example we have two simultaneous equations involving three unknown quantities. Let y = quantity required. Then we have evidently, $5y=63m+7$, $10y=63n+14$, where m and n are certain positive integers. Put $m+n=x$; thus, $63x-15y=-21$, whence y can be found by §256, and the reason for the rule in §265 is obvious.]

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[1] [See *Goládhyáya*, XIII, 13—15.—Ed.]

[2] The quotient as it comes out in this operation is not to be taken: but it is to be separately sought with the several original multipliers applied to this quantity and divided by the divisor as given.—Gan.

CHAPTER XIII.

COMBINATION OF DIGITS.[1]

267. Rule.[2] The product of multiplication of the arithmetical series beginning and increasing by unity and continued to the number of places, will be the variations of number with specific figures: that divided by the number of digits and multiplied by the sum of the digits, being repeated in the places of figures and added together, will be the sum of the permutations.

[Let there be n digits. Then evidently there are $\underline{|n}$ numbers which can be formed with all these digits. Consider any one of these digits, and denote it by d. In $\underline{|n-1}$ cases, d is in the units' place, in as many cases d is in the tens' place, in as many cases d is in the hundreds' place, and so on. Thus the sum arising from the d alone is $\underline{|n-1}\{d+10d+100d+\ldots .10^{n-1}d\}$. Proceeding similarly with the other digits, we get the sum of all the numbers $=\underline{|n-1}\times$ sum of digits $\times(10^{n-1}+\ldots+10+1)$ $=\dfrac{\underline{|n}}{n}\times$ sum of digits $\times(10^{n-1}+\ldots+10+1)$, which stated in words leads to the rule. The meaning of the phrase, *being repeated in the places of figures and added together*, is obvious. See foot-note to §13.]

[1] *Anka-pása-vyavahára*, concatenation of digits: a mutual mixing of the numbers, as it were a rope of numerals, their variations being likened to a coil. See Gan. and Súr.

[2] To find the number of the permutations and the sum or amount of them, with specific numbers.—Gan. and Súr.

268. Example. How many variations of number can there be with two and eight, or with three, nine and eight, or with the continued series from two to nine? Tell promptly the several sums of these numbers.

Statement of the first example: 2, 8. Here the number of places is 2. The product of the series from 1 to the number of places and increasing by unity, will be 2. Thus the permutations of number are found 2. That product 2, multiplied by the sum of the figures 10, is 20; and divided by the number of digits 2, is 10. This repeated in the places of figures $(^{10}_{10})$ and added together, is 110, the sum of the numbers.

Statement of the second example: 3, 9, 8.

The arithmetical series is 1, 2, 3, of which the product is 6; and so many are the variations of number. That multiplied by the sum 20, is 120; which divided by the number of digits 3, gives 40; and this, repeated in the three places of figures and summed, makes 4440, the sum of the numbers.

Statement of the third example: 2, 3, 4, 5, 6, 7, 8, 9. The arithmetical series beginning and increasing by unity is 1, 2, 3, 4, 5, 6, 7, 8. The product gives the permutation of numbers, 40320. This, multiplied by the sum of the figures 44, is 1774080, which divided by the number of terms 8, is 221760; and the quotient being repeated in the eight places of figures and summed, the total is the sum of the numbers, 2463999975360.

269. Example. How many are the variations of form of the god *Sambhu* by the exchange of his ten attributes held reciprocally in his several hands, namely, the rope, the elephant's hook, the serpent, the tabor, the skull, the trident, the bedstead, the dagger, the

arrow, and the bow[1] as those of *Hari* by the exchange of the mace, the discus, the lotus and the conch?

Statement: number of places 10.

In the same mode, as above shown, the variations of form are found 3628800. So the variations of form of *Hari* are 24.

270. Rule.[2] The permutations found as before, being divided by the permutations separately computed for as many places as are filled by like digits, will be the variations of number, from which the sum of the numbers will be found as before.

[Let there be n digits; and suppose p of them to be d_1, q of them to be d_2, and the rest unlike, namely, d_3, d_4, &c. Then the variations of number will clearly be $= \frac{|\underline{n}}{|\underline{p}\,|\underline{q}}$. (See Todhunter's Algebra, Art. 497.) The number of cases in which d_1 is in the units' or tens' or hundreds' &c. place is $\frac{|\underline{n-1}}{|\underline{p-1}\,|\underline{q}}$ (Todhunter's Algebra, Art. 497); and hence the sum arising from d_1

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[1] *Sambhu* or *Siva* is represented with ten arms, and holding in his ten hands the ten weapons or symbols here specified; and, by changing the several attributes from one hand to another, a variation may be effected in the representation of the idol, in the same manner as the image of *Hari* or *Vishnu* is varied by the exchange of his four symbols in his four hands. The twenty-four different representations of *Vishnu*, arising from this diversity in the manner of placing the weapons or attributes in his four hands, are distinguished by as many discriminative titles of the god allotted to those figures in the theogonies or *Puránas*. It does not appear that distinct titles have been in like manner assigned to any of the more than three millions of varied representations of *Siva*.

The ten attributes of *Siva* are:—1st, *pása*, a rope or chain for binding an elephant; 2nd, *ankusa*, a hook for guiding an elephant; 3rd, *ahi*, a serpent; 4th, *damaru*, a tabor; 5th, *kapála*, a human skull; 6th, *trisúla*, a trident; 7th, *khatwánga*, a bedstead, or a club in the form of the foot of one; [it may also mean a club having a skull at the top.—ED.] 8th, *sakti*, a dagger; 9th, *sara*, an arrow; 10th, *chápa*, a bow.

[2] Special; being applicable when two or more of the digits are alike.

alone is $\frac{(n-1)!}{(p-1)!\,q!}(10^{n-1}+\ldots\ldots+10+1)\,d_1$. Similarly the sum arising from d_2 is $\frac{(n-1)!}{p!\,(q-1)!}(10^{n-1}+\ldots\ldots+10+1)d_2$, and that arising from d_3, d_4, &c., is

$$\frac{(n-1)!}{p!\,q!}(10^{n-1}+\ldots\ldots+10+1)\,(d_3+d_4+\&c.)$$

Hence the sum of all the numbers is

$$\frac{n!}{n}(10^{n-1}+\ldots\ldots+10+1)\left\{\frac{d_1}{(p-1)!\,q!}+\frac{d_2}{p!\,(q-1)!}+\frac{d_3+d_4+\&c.}{p!\,q!}\right\}$$

$$=\frac{n!}{n\,p!\,q!}\;(pd_1+qd_2+d_3+d_4+\&c.)\;(10^{n-1}+\ldots+10+1)$$

$=\frac{n!}{n\,p!\,q!}\times$sum of digits$\times(10^{n-1}+\ldots+10+1)$, whence the rule.]

271. Example. How many are the numbers with two, two, one and one? And tell me quickly, mathematician, their sum : also with four, eight, five, five and five, if thou be conversant with the rule of permutation of numbers.

Statement of the 1st example : 2, 2, 1, 1. Here the permutations found as before (§267) are 24. First, two places are filled by like digits (2, 2), and the permutations for that number of places are 2. Next two other places are filled by like digits (1, 1), and the permutations for these places are also 2. Total 4. The permutations 24 divided by 4 give 6 for the variations of number : *viz.*,[1] 2211, 2121, 2112, 1212, 1221, 1122. The sum[2] of the numbers is found as before 9999.

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[1] The enumeration of the possible combinations is termed *prastára*.

[2] The variations 6, multiplied by the sum of the figures 6, and divided by the number of digits 4, give 9; which being repeated in four places of figures and summed, makes 9999.

Statement of the 2nd example: 4, 8, 5, 5, 5. Here the permutations found as before are 120, which, divided by the permutations for three places, *viz.*, 6, give the variations 20: *viz.*, 48555, 84555, 54855, 58455, 55485, 55845, 55548, 55584, 45855, 45585, 45558, 85455, 85545, 85554, 54585, 58545, 55458, 55854, 54558, 58554. The sum[1] of the numbers comes out 1199988.

272. Rule[2]: half a stanza. The series of the numbers decreasing by unity from the last[3] to the number of places, being multiplied together, will be the variations of number, with dissimilar digits.

[This rule gives the ordinary formula for the number of permutations of n things taken r at a time, *viz.*, $n\ (n-1)\ (n-2)$ ......$(n-r+1)$.]

273. Example. How many are the variations of number with any digits except cipher exchanged in six places of figures? If thou know, declare them.

The last number is nine. Decreasing by unity, for as many as are the places of figures, the statement of the series is 9. 8. 7. 6. 5. 4. The product of these is 60480.

274. Rule[4]: two stanzas. If the sum of the digits be determinate, the arithmetical series of numbers from one less than the sum of the digits, decreasing by unity, and continued to one less than the places, being divided by one and so forth, and the quotients being

[1] The variations 20, multiplied by the sum of the figures 27, give 540, which, divided by the number of digits 5, makes 108: and this being repeated in five places of figures and summed, yields 1199988.

[2] To find the variations for a definite number of places with indeterminate digits.—Gan.

[3] That is, from nine [in the example which follows.]—Gan.

[4] To find the permutations with indeterminate digits for a definite sum and a specific number of places.—Gan.

multiplied together, the product will be equal to the variations of number. This rule must be understood to hold good, provided the sum of the digits be less than the number of places added to nine.

A compendium only has been here delivered for fear of prolixity, since the ocean of calculation has no bounds.

[Let $s =$ the sum of the digits,
$n =$ the number of digits,
and let $s = n + m$.

Then by supposition, $n+m < n+9$, or $m<9$, or $m+1$ not > 9, so that even if $n-1$ of the n digits be 1's, the remainder of the sum, $m+1$ being not > 9, can form the remaining digit.

Now let the n 1's composing n be denoted by 1^n, 1^{n-1}, 1^{n-2} ....1^1, and the m 1's composing m, by 1_1, 1_2, 1_3......1_m. Then, if we fix 1^n in the first place on the left, and take the different permutations of the remaining $n-1+m$ symbols 1^{n-1}, 1^{n-2},......1^1 and 1_1, 1_2, 1_3......1_m, of which the $n-1$ indexed 1's are considered to be alike and of one sort, and the other m 1's are considered to be alike and of another sort, and place each of these permutations to the right of 1^n, and regard the sum of each indexed 1 with the group of 1's with suffixes, if any, following it on its right as forming a digit of one of the required numbers,

we shall have a series of numbers like the following :—

$(1^n\ 1_1\ 1_2)\ (1^{n-1}\ 1_3\ 1_4\ 1_5) \ldots . (1^p \ldots . 1_{m-1}) \ldots (1^2\ 1_m)\ (1^1)$,

$(1^n\ 1_1\ 1_2\ 1_3\ 1_4)\ (1^{n-1}) \ldots \ldots (1^p \ldots . 1_n) \ldots \ldots (1^2)\ (1^1)$,

....

This series will evidently contain all the required numbers and those alone ; and the number of these numbers being the number required, the problem is reduced to finding the number of permutations of $n+m-1$ things taken all together, of which $n-1$ are alike and of one sort, and m are alike and of another sort.

And this number $= \dfrac{|\underline{n+m-1}}{|\underline{n-1}\ |\underline{m}}$

$$= \frac{(n+m-1)\,(n+m-1-1)\ldots(n+m-1-\overline{n-1})\,(m-1)\ldots1}{|\underline{n-1}\ |\underline{m}}$$

$$= \frac{(n+m-1)\,(n+m-2)\,\ldots\,(n+m-1-\overline{n-2})}{|\underline{n-1}}$$

$$= \frac{(s-1)\,(s-2)\,\ldots\,(s-\overline{n-1})}{1\,.\,2\ldots\ldots\ldots\ldots(n-1)},$$

which proves the rule.]

275. Example. How many various numbers are there, with digits standing in five places, the sum of which is thirteen? If thou know, declare them.

Here the sum of the digits less one is 12. The decreasing series from this to one less than the number of digits, divided by unity, &c. being exhibited, the statement is, $\frac{12}{1}.\ \frac{11}{2}.\ \frac{10}{3}.\ \frac{9}{4}$. The product of their multiplication ($\frac{11880}{24}$) is equal to the variations of the number[1], 495.

276. Though neither multiplier nor divisor be asked, nor square, nor cube, still presumptuous inexpert scholars in arithmetic will assuredly fail in (problems on) this combination of numbers.

277. Joy and happiness is indeed ever increasing in this world for those who have *Lílávatí* clasped to their throats[2], decorated as the members are with neat reduc-

[1] 91111, 52222, 13333, each five ways; 55111, 22333, each ten ways, 82111, 73111, 64111, 43222, 61222, each twenty ways; 72211, 53311, 44221, 44311, each thirty ways; 63211, 54211, 53221, 43321, each sixty ways. Thus the total is 495.

[2] By constant repetition of the text. This stanza, ambiguously expressed and bearing a double import, implies a simile: as a charming woman closely embraced, whose person is embellished by an assemblage of elegant qualities, who is pure and perfect in her conduct, and who utters agreeable discourse. See Gan.

[illegible] fractions, multiplication and involution, pure and [illegible] as are the solutions, and tasteful as is the speech wh[illegible] is exemplified.

[illegible] Pandit Jívánanda Vidyáságara's edition of the original, [illegible] is a stanza [illegible] showing the varied scholarship [illegible]kara, and stating that the present work was composed [illegible] It was probably added by some pupil of Bháskara, [illegible] has been omitted by Colebrooke. It runs as follows :— "[illegible] author of this (*Lílávatí*) is that illustrious Bhásk[illegible] (a [illegible]ar) of vast erudition, who thoroughly mastered eight works [illegible]mar, (*viz.*, those of Indra, Chandra, Kásakritsni, Ápisali, [illegible]na, Pánini, Amara, and Jainendra),[1] six works on medi[illegible]ce, (*viz.*, *Agnivesa-sanhitá*, *Bheda-sanhitá*, *Játúkarna*-[illegible] *Parásara-sanhitá*, *Sírapáni-sanhitá*, and *Hárita-sanhitá*),[2] [illegible] philosophical systems (*viz.*, *Sánkhya*, *Yoga*, *Nyáya*, *Vai*-[illegible], *Mímánsá* and *Vedánta*), five works on *ganita* (calcula[illegible] (*viz.*, *Paulisa-siddhánta*, *Romaka-siddhánta*, *Básishtha*-[illegible]ta, *Súrya-siddhánta* and *Paitámaha-siddhánt*[illegible] and [illegible] *Vedas* (*viz.*, the *Rik*, the *Yajush*, the *Sáman* and the [illegible]n); and who understood the three *Ratnas*, (*i.e.*, the [illegible] *Prasthánas* of the *Vedánta*, *viz.*, the *Sútras*, the *Upanishads* [illegible] *Prakaranas*), as well as the two *Mímánsás*,[4] and the [illegible]rnal *Brahman*, the aim and scope of both."]

[[1] See Bibliotheca Indica, Nirukta, Vol. IV, Appendix, page *jau*. Some of [illegible] authors composed dictionaries and not works on grammar. Thus the [illegible] *vyákaranáni* in the original has been rather loosely used.—Ed.]

[[2] These six ancient works form the basis of the later works of Charaka, Susruta and Bágbhata. The works of Charaka and Susruta are usually called *sanhitás*; that of Bágbhata is known under the name of *Ashtánga-hrida*[illegible].—Ed.]

[[3] See Varáha-mihira's *Vrihat-sanhitá*, Ch. II.—Ed.]

[[4] Namely, the *Púrva-mímánsá* of Jaimini, usually called the *Mímánsá*, and the *Uttara-mímánsá* of Vyása, usually called the *Vedánta*.—Ed.]

And this number $= \dfrac{\underline{|n+m-1}}{\underline{|n-1}\ \underline{|m}}$

$$= \frac{(n+m-1)\,(n+m-1-1)\ldots(n+m-1-\overline{n-1})\,(m-1)\ldots 1}{\underline{|n-1}\ \underline{|m}}$$

$$= \frac{(n+m-1)\,(n+m-2)\,\ldots\ldots\,(n+m-1-\overline{n-2})}{\underline{|n-1}}$$

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