EUCLID'S ELEMENTS.

BOOK I.

WITH NOTES, QUESTIONS,
GEOMETRICAL EXERCISES, &c.

One Shilling.

Mir Laik Ali Khan B

INTERNATIONAL EXHIBITION,
1862.

A Medal has been awarded to R. Potts,
"For the excellence of his Works on Geometry."

Jury Awards. Class XXIX. p. 313.

EUCLID'S
ELEMENTS OF GEOMETRY,

THE FIRST BOOK,

CHIEFLY FROM THE TEXT OF DR. SIMSON,
WITH EXPLANATORY NOTES;
A SERIES OF QUESTIONS ON EACH BOOK;
AND A SELECTION OF GEOMETRICAL EXERCISES FROM THE
SENATE-HOUSE AND COLLEGE EXAMINATION
PAPERS; WITH HINTS, ETC.

BY

ROBERT POTTS, M.A.,
TRINITY COLLEGE.

CORRECTED AND IMPROVED.

LONDON:
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1876.

PREFACE TO THE THIRD EDITION.

SOME time after the publication of an Octavo Edition of Euclid's Elements with Geometrical Exercises, &c., designed for the use of Academical Students; at the request of some schoolmasters of eminence, a duodecimo Edition of the Six Books was put forth on the same plan for the use of Schools. Soon after its appearance, Professor Christie, the Secretary of the Royal Society, in the Preface to his Treatise on Descriptive Geometry for the use of the Royal Military Academy, was pleased to notice these works in the following terms:—

"When the greater Portion of this Part of the Course was printed, and had for some time been in use in the Academy, a new Edition of Euclid's Elements, by Mr. Robert Potts, M.A., of Trinity College, Cambridge, which is likely to supersede most others, to the extent, at least, of the Six Books, was published. From the manner of arranging the Demonstrations, this edition has the advantages of the symbolical form, and it is at the same time free from the manifold objections to which that form is open. The duodecimo edition of this Work, comprising only the first Six Books of Euclid, with Deductions from them, having been introduced at this Institution as a text-book, now renders any other Treatise on Plane Geometry unnecessary in our course of Mathematics."

For the very favourable reception which both Editions have met with, the Editor's grateful acknowledgements are due. It has been his desire in putting forth a revised Edition of the School Euclid, to render the work in some degree more worthy of the favour which the former editions have received. In the present Edition several errors and oversights have been corrected and some additions made to the notes: the questions on each book have been considerably augmented and a better arrangement of the Geometrical Exercises has been attempted: and lastly, some hints and remarks on them have been given to assist the learner. The additions made to the present Edition amount to more than fifty pages, and, it is hoped, that they will render the work more useful to the learner.

And here an occasion may be taken to quote the opinions of some able men respecting the use and importance of the Mathematical Sciences.

On the subject of Education in its most extensive sense, an ancient writer "directs the aspirant after excellence to commence with the Science of Moral Culture; to proceed next to Logic; next to Mathematics; next to Physics; and lastly, to Theology." Another writer on Education would place Mathematics before Logic, which (he remarks) "seems the preferable course: for by practising itself in the former, the mind becomes stored with distinctions; the faculties of constancy and firmness are established; and its rule is always to distinguish between cavilling and investigation—between *close reasoning* and *cross reasoning;* for the contrary of all which habits, those are for the most part noted, who apply themselves to Logic without studying in some department of Mathematics; taking noise and wrangling for proficiency, and thinking refutation accomplished by the instancing of a doubt. This will explain the inscription placed by Plato over the door of his house: 'Whoso knows not Geometry, let him not enter here.' On the precedence of Moral Culture, however, to all the other Sciences, the acknowledgement is general, and the agreement entire." The same writer recommends the study of the Mathematics, for the cure of "compound ignorance." "Of this," he proceeds to say, "the essence is opinion not agreeable to fact; and it necessarily involves another opinion, namely, that we are already possessed of knowledge. So that besides not knowing, we know not that we know not; and hence its designation of compound ignorance. In like manner, as of many chronic complaints and established maladies, no cure can be effected by physicians of the body: of this, no cure can be effected by physicians of the mind: for with a pre-supposal of knowledge in our own regard, the pursuit and acquirement of further knowledge is not to be looked for. The approximate cure, and one from which in the main much benefit may be anticipated, is to engage the patient in the study of measures (Geometry, computation, &c.); for in such pursuits the true and the false are separated by the clearest interval, and no room is left for the intrusions of fancy. From these the mind may discover the delight of certainty; and when, on returning to his own opinions, it finds in them no such sort of repose and gratification, it may discover their erroneous character, its ignorance may become simple, and a capacity for the acquirement of truth and virtue be obtained."

Lord Bacon, the founder of Inductive Philosophy, was not insensible of the high importance of the Mathematical Sciences, as appears in the following passage from his work on "The Advancement of Learning."

"The Mathematics are either pure or mixed. To the pure Mathematics are those sciences belonging which handle quantity determinate, merely severed from any axioms of natural philosophy; and these are two, Geometry, and Arithmetic; the one handling quantity continued, and the other dissevered. Mixed hath for subject some axioms or parts of natural philosophy, and considereth quantity determined, as it is auxiliary and incident unto them. For many parts of nature can

neither be invented with sufficient subtlety, nor demonstrated with sufficient perspicuity, nor accommodated unto use with sufficient dexterity, without the aid and intervening of the Mathematics: of which sort are perspective, music, astronomy, cosmography, architecture, enginery, and divers others.

"In the Mathematics I can report no deficience, except it be that men do not sufficiently understand the excellent use of the pure Mathematics, in that they do remedy and cure many defects in the wit and faculties intellectual. For, if the wit be dull, they sharpen it; if too wandering, they fix it; if too inherent in the sense, they abstract it. So that as tennis is a game of no use in itself, but of great use in respect that it maketh a quick eye, and a body ready to put itself into all postures; so in the Mathematics, that use which is collateral and intervenient, is no less worthy than that which is principal and intended. And as for the mixed Mathematics, I may only make this prediction, that there cannot fail to be more kinds of them, as nature grows further disclosed."

How truly has this prediction been fulfilled in the subsequent advancement of the Mixed Sciences, and in the applications of the pure Mathematics to Natural Philosophy!

Dr. Whewell, in his "Thoughts on the Study of Mathematics," has maintained, that mathematical studies judiciously pursued, form one of the most effective means of developing and cultivating the reason: and that "the object of a *liberal education* is to develope the whole mental system of man;—to make his speculative inferences coincide with his practical convictions;—to enable him to render a reason for the belief that is in him, and not to leave him in the condition of Solomon's sluggard, who is wiser in his own conceit than seven men that *can* render a reason." And in his more recent work entitled, "Of a Liberal Education, &c." he has more fully shewn the importance of Geometry as one of the most effectual instruments of intellectual education. In page 55 he thus proceeds:—"But besides the value of Mathematical Studies in Education, as a perfect example and complete exercise of demonstrative reasoning; Mathematical Truths have this additional recommendation, that they have always been referred to, by each successive generation of thoughtful and cultivated men, as examples of truth and of demonstration; and have thus become standard points of reference, among cultivated men, whenever they speak of truth, knowledge, or proof. Thus Mathematics has not only a disciplinal but an historical interest. This is peculiarly the case with those portions of Mathematics which we have mentioned. We find geometrical proof adduced in illustration of the nature of reasoning, in the earliest speculations on this subject, th Dialogues of Plato; we find geometrical proof one of the main subjects of discussion in some of the most recent of such speculations, as those of Dugald Stewart and his contemporaries. The recollection of the truths of Elementary Geometry has, in all ages, given a meanin and a reality to the best attempts to explain man's power of arrivin at truth. Other branches of Mathematics have, in like manner, become recognized examples, among educated men, of man's powers of attaining truth."

Dr. Pemberton, in the preface to his view of Sir Isaac Newton's Discoveries, makes mention of the circumstance, "that Newton to speak with regret of his mistake, at the beginning of his M matical Studies, in having applied himself to the works of Desc and other Algebraical writers, before he had considered the Eleme of Euclid with the attention they deserve."

To these we may subjoin the opinion of Mr. John Stuart Mill, which he has recorded in his invaluable System of Logic, (Vol. II. p. 180) in the following terms. "The value of Mathematical instruction as a preparation for those more difficult investigations (physiology, society, government, &c.) consists in the applicability not of its doctrines, but of its method. Mathematics will ever remain the most perfect type of the Deductive Method in general; and the applications of Mathematics to the simpler branches of physics, furnish the only school in which philosophers can effectually learn the most difficult and important portion of their art, the employment of the laws of simpler phenomena for explaining and predicting those of the more complex. These grounds are quite sufficient for deeming mathematical training an indispensable basis of real scientific education, and regarding, with Plato, one who is ἀγεωμέτρητος, as wanting in one of the most essential qualifications for the successful cultivation of the higher branches of philosophy."

In addition to these authorities it may be remarked, that the new Regulations which were confirmed by a Grace of the Senate on the 11th of May, 1846, assign to Geometry and to Geometrical methods a more important place in the Examinations both for Honors and for the Ordinary Degree in this University.

R. P.

Trinity College,
March 1, 1850.

The supplement to the School Euclid (about forty-eight pages) has been incorporated with this impression of the Fifth Edition.

Trinity College,
October, 1863.

EUCLID'S
ELEMENTS OF GEOMETRY.

BOOK I.

DEFINITIONS.

I.

A **point** is that which has no parts, or which has no magnitude.

II.

A line is length without breadth.

III.

The extremities of a line are points.

IV.

A straight line is that which lies evenly between its extreme points.

V.

A superficies is that which has only length and breadth.

VI.

The extremities of a superficies are lines.

VII.

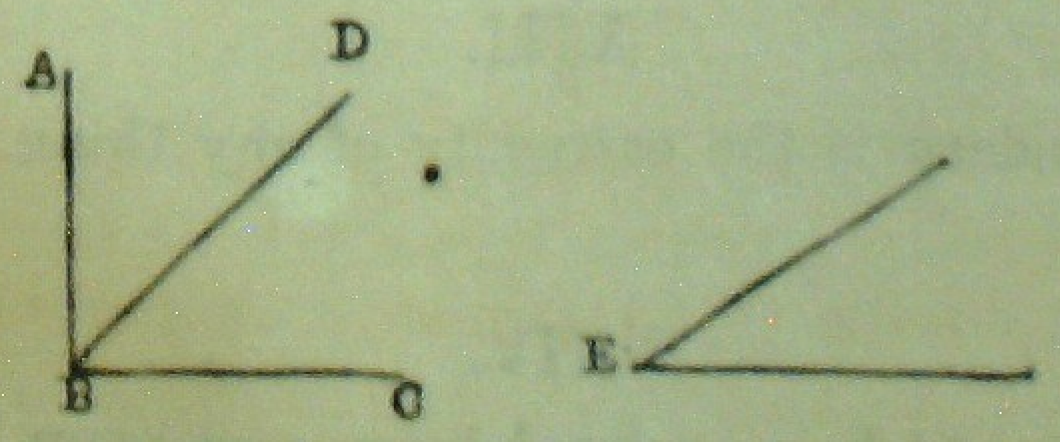
A plane superficies is that in which any two points being taken, the straight line between them lies wholly in that superficies.

VIII.

A plane angle is the inclination of two lines to each other in a plane, which meet together, but are not in the same direction.

IX.

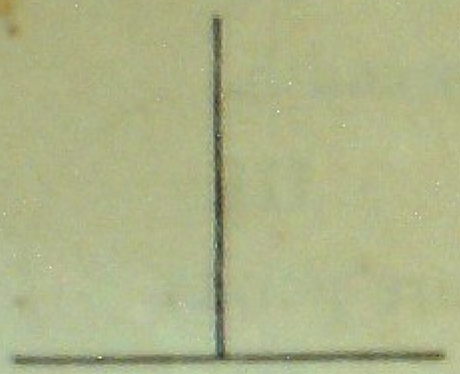
A plane rectilineal angle is the inclination of two straight lines to one another, which meet together, but are not in the same straight line.



N.B. If there be only one angle at a point, it may be expresse a letter placed at that point, as the angle at E: but when several ang are at one point B, either of them is expressed by three letters, of wh the letter that is at the vertex of the angle, that is, at the point in wh the straight lines that contain the angle meet one another, is put betw the other two letters, and one of these two is somewhere upon one these straight lines, and the other upon the other line. Thus the which is contained by the straight lines AB, CB, is named the ABC, or CBA; that which is contained by AB, DB, is named th ABD, or DBA; and that which is contained by DB, CB, is ca angle DBC, or CBD.

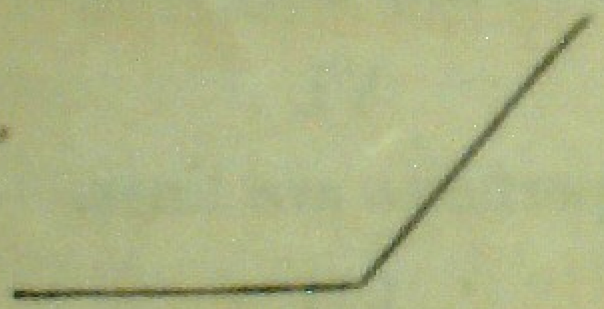
X.

When a straight line standing on another straight line, makes the adjacent angles equal to one another, each of these angles is called a right angle; and the straight line which stands on the other is called a perpendicular to it.



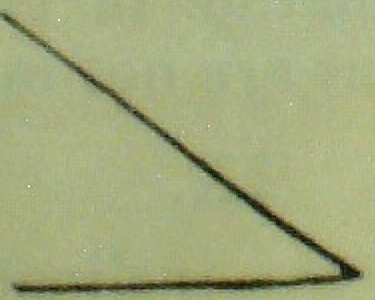
XI.

An obtuse angle is that which is greater than a right angle.



XII.

An acute angle is that which is less than a right angle.



XIII.

A term or boundary is the extremity of any thing.

XIV.

A figure is that which is enclosed by one or more boundaries.

XV.

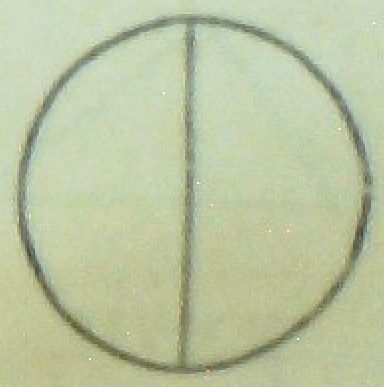
A circle is a plane figure contained by one line, which is called the circumference, and is such that all straight lines drawn from a certain point within the figure to the circumference, are equal to one another.

XVI.

And this point is called the center of the circle.

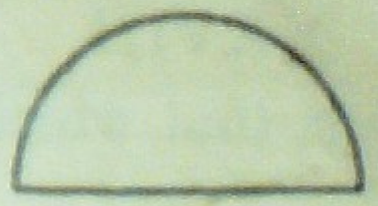
XVII.

A diameter of a circle is a straight line drawn through the center, and terminated both ways by the circumference.



XVIII.

A semicircle is the figure contained by a diameter and the part of the circumference cut off by the diameter.



XIX.

The center of a semicircle is the same with that of the circle.

XX.

Rectilineal figures are those which are contained by straight lines.

XXI.

Trilateral figures, or triangles, by three straight lines.

XXII.

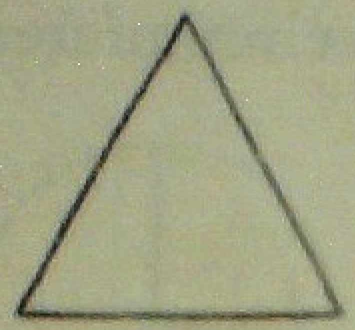
Quadrilateral, by four straight lines.

XXIII.

Multilateral figures, or polygons, by more than four straight lines.

XXIV.

√ Of three-sided figures, an equilateral triangle is that which h three equal sides.

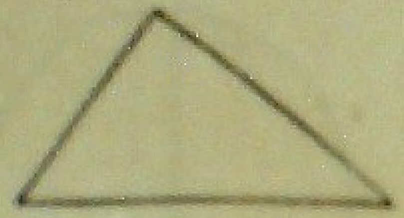


XXV.

√ An isosceles triangle is that which has two sides equal.

XXVI.

√ A scalene triangle is that which has three unequal sides.

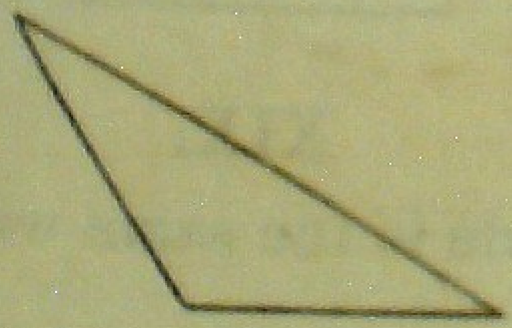


XXVII.

A right-angled triangle is that which has a right angle.

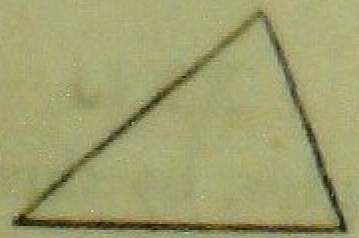
XXVIII.

An obtuse-angled triangle is that which has an obtuse angle.



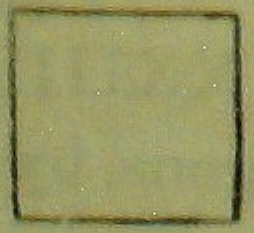
XXIX.

An acute-angled triangle is that which has three acute angles.



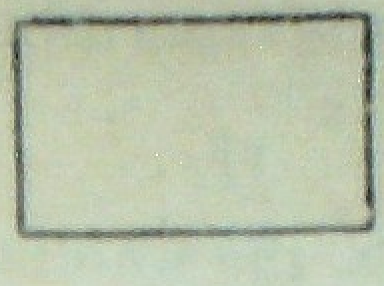
XXX.

Of quadrilateral or four-sided figures, a square has all its sides equal and all its angles right angles.



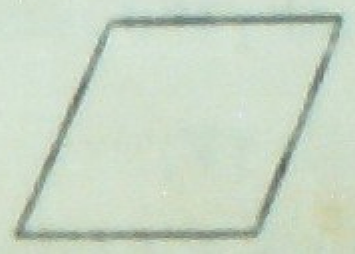
XXXI.

An oblong is that which has all its angles right angles, but has not all its sides equal.



XXXII.

A rhombus has all its sides equal, but its angles are not right angles.



XXXIII.

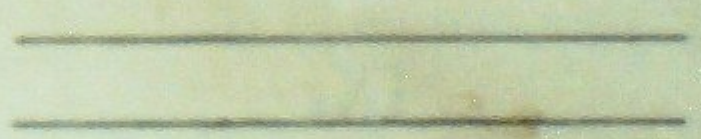
A rhomboid has its opposite sides equal to each other, but all its sides are not equal, nor its angles right angles.

XXXIV.

All other four-sided figures besides these, are called Trapeziums.

XXXV.

Parallel straight lines are such as are in the same plane, and which being produced ever so far both ways, do not meet.



A.

A parallelogram is a four-sided figure, of which the opposite sides are parallel: and the diameter, or the diagonal is the straight line joining two of its opposite angles.

POSTULATES.

I.

Let it be granted that a straight line may be drawn from any one point to any other point.

II.

That a terminated straight line may be produced to any length in a straight line.

III.

And that a circle may be described from any center, at any distance from that center.

AXIOMS.

I.

Things which are equal to the same thing are equal to one another.

II.

If equals be added to equals, the wholes are equal.

III.

If equals be taken from equals, the remainders are equal.

IV.

If equals be added to unequals, the wholes are unequal.

V.

If equals be taken from unequals, the remainders are unequal.

VI.

Things which are double of the same, are equal to one another.

VII.

Things which are halves of the same, are equal to one another.

VIII.

Magnitudes which coincide with one another, that is, which exactly fill the same space, are equal to one another.

IX.

The whole is greater than its part.

X.

Two straight lines cannot enclose a space.

XI.

All right angles are equal to one another.

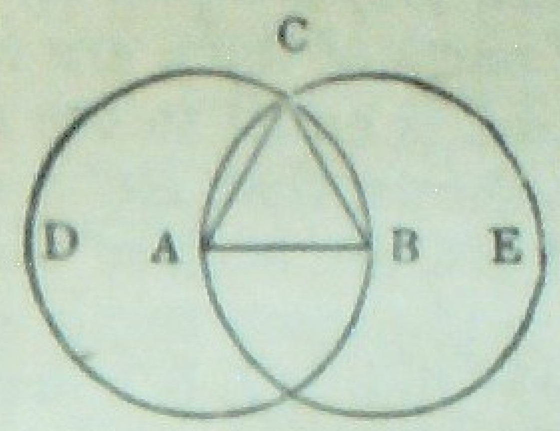
XII.

If a straight line meets two straight lines, so as to make the two interior angles on the same side of it taken together less than two right angles; these straight lines being continually produced, shall at length meet upon that side on which are the angles which are less than two right angles.

PROPOSITION I. PROBLEM.

To describe an equilateral triangle upon a given finite straight line.

Let AB be the given straight line.
It is required to describe an equilateral triangle upon AB.

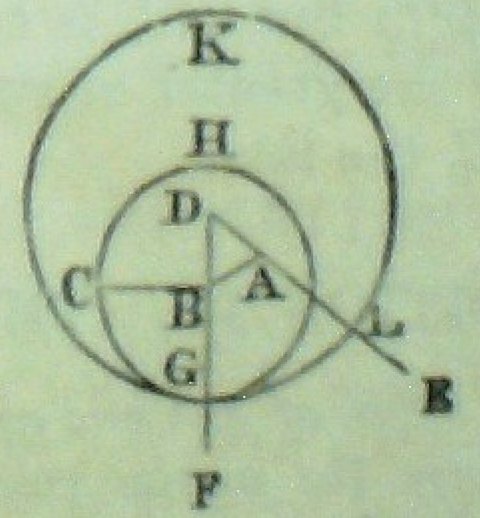


From the center A, at the distance AB, describe the circle BCD; (post. 3.)
from the center B, at the distance BA, describe the circle ACE;
and from C, one of the points in which the circles cut one another, draw the straight lines CA, CB to the points A, B. (post. 1.)
Then ABC shall be an equilateral triangle.
Because the point A is the center of the circle BCD,
therefore AC is equal to AB; (def. 15.)
and because the point B is the center of the circle ACE,
therefore BC is equal to AB;
but it has been proved that AC is equal to AB;
therefore AC, BC are each of them equal to AB;
but things which are equal to the same thing are equal to one another;
therefore AC is equal to BC; (ax. 1.)
wherefore AB, BC, CA are equal to one another:
and the triangle ABC is therefore equilateral,
and it is described upon the given straight line AB.
Which was required to be done.

PROPOSITION II. PROBLEM.

From a given point, to draw a straight line equal to a given straight line.

Let A be the given point, and BC the given straight line.
It is required to draw from the point A, a straight line equal to BC.



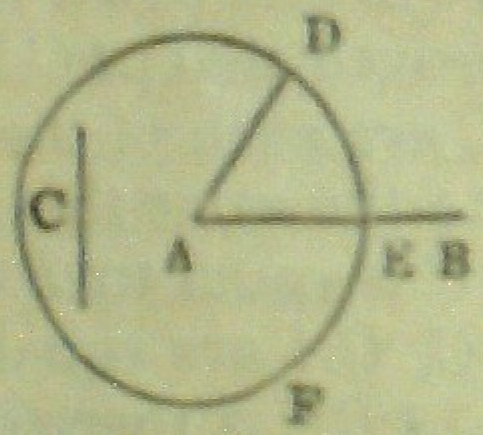
From the point A to B draw the straight line AB; (post. 1.)
upon AB describe the equilateral triangle ABD, (1. 1.)
and produce the straight lines DA, DB to E and F; (post. 2.)
from the center B, at the distance BC, describe the circle CGH, (post. 3.) cutting DF in the point G:
and from the center D, at the distance DG, describe the circle GKL, cutting AE in the point L.

Then the straight line AL shall be equal to BC.
Because the point B is the center of the circle CGH,
therefore BC is equal to BG; (def. 15.)
and because D is the center of the circle GKL,
therefore DL is equal to DG,
and DA, DB parts of them are equal; (I. 1.)
therefore the remainder AL is equal to the remainder BG; (ax. 3.)
but it has been shewn that BC is equal to BG,
wherefore AL and BC are each of them equal to BG;
and things that are equal to the same thing are equal to one another;
therefore the straight line AL is equal to BC. (ax. 1.)
Wherefore from the given point A, a straight line AL has been drawn equal to the given straight line BC. Which was to be done.

PROPOSITION III. PROBLEM.

From the greater of two given straight lines to cut off a part equal to the less.

Let AB and C be the two given straight lines, of which AB is the greater.
It is required to cut off from AB the greater, a part equal to C, the less.



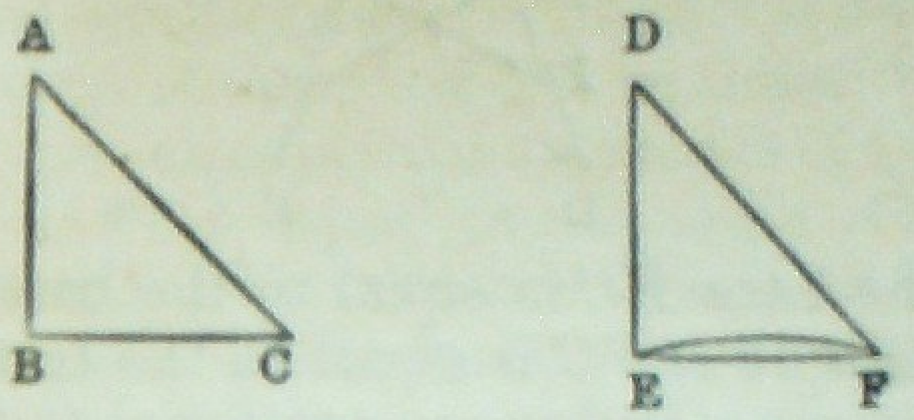
From the point A draw the straight line AD equal to C; (I. 2.)
and from the center A, at the distance AD, describe the circle DEF (post. 3.) cutting AB in the point E.
Then AE shall be equal to C.
Because A is the center of the circle DEF,
therefore AE is equal to AD; (def. 15.)
but the straight line C is equal to AD; (constr.)
whence AE and C are each of them equal to AD;
wherefore the straight line AE is equal to C. (ax. 1.)
And therefore from AB the greater of two straight lines, a part AE has been cut off equal to C, the less. Which was to be done.

PROPOSITION IV. THEOREM.

If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise the angles contained by those sides equal t each other; they shall likewise have their bases or third sides equal, an the two triangles shall be equal, and their other angles shall be equal, eac to each, viz. those to which the equal sides are opposite.

Let ABC, DEF be two triangles, which have the two sides AB, AC equal to the two sides DE, DF, each to each, viz. AB to DE, and AC to DF, and the included angle BAC equal to the included angl EDF.

Then shall the base BC be equal to the base EF; and the triangle ABC to the triangle DEF; and the other angles to which the equal sides are opposite shall be equal, each to each, viz. the angle ABC to the angle DEF, and the angle ACB to the angle DFE.

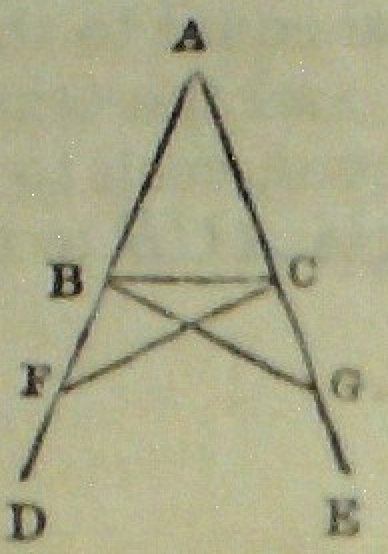


For, if the triangle ABC be applied to the triangle DEF,
so that the point A may be on D, and the straight line AB on DE;
then the point B shall coincide with the point E,
because AB is equal to DE;
and AB coinciding with DE,
the straight line AC shall fall on DF,
because the angle BAC is equal to the angle EDF;
therefore also the point C shall coincide with the point F,
because AC is equal to DF;
but the point B was shewn to coincide with the point E;
wherefore the base BC shall coincide with the base EF;
because the point B coinciding with E, and C with F,
if the base BC do not coincide with the base EF, the two straight lines BC and EF would enclose a space, which is impossible. (ax. 10.)
Therefore the base BC does coincide with EF, and is equal to it;
and the whole triangle ABC coincides with the whole triangle DEF, and is equal to it;
also the remaining angles of one triangle coincide with the remaining angles of the other, and are equal to them,
viz. the angle ABC to the angle DEF,
and the angle ACB to DFE.
Therefore, if two triangles have two sides of the one equal to two sides, &c. Which was to be demonstrated.

PROPOSITION V. THEOREM.

The angles at the base of an isosceles triangle are equal to each other; and if the equal sides be produced, the angles on the other side of the base shall be equal.

Let ABC be an isosceles triangle of which the side AB is equal to AC,
and let the equal sides AB, AC be produced to D and E.
Then the angle ABC shall be equal to the angle ACB,
and the angle DBC to the angle ECB.
In BD take any point F;
from AE the greater, cut off AG equal to AF the less, (I. 3.)
and join FC, GB.
Because AF is equal to AG, (constr.) and AB to AC; (hyp.)
the two sides FA, AC are equal to the two GA, AB, each to each;
and they contain the angle FAG common to the two triangles
AFC, AGB;



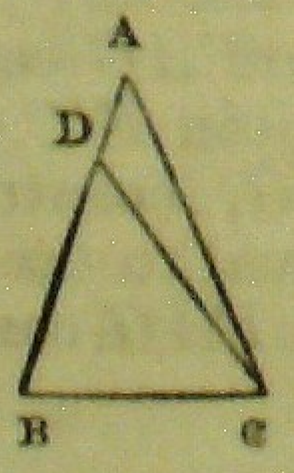
therefore the base FC is equal to the base GB, (I. 4.)
and the triangle AFC is equal to the triangle AGB,
also the remaining angles of the one are equal to the remaining angles
of the other, each to each, to which the equal sides are opposite;
viz. the angle ACF to the angle ABG,
and the angle AFC to the angle AGB.
And because the whole AF is equal to the whole AG,
of which the parts AB, AC, are equal;
therefore the remainder BF is equal to the remainder CG; (ax. 3.)
and FC has been proved to be equal to GB;
hence, because the two sides BF, FC are equal to the two CG, GB,
each to each;
and the angle BFC has been proved to be equal to the angle CGB,
also the base BC is common to the two triangles BFC, CGB;
wherefore these triangles are equal, (I. 4.)
and their remaining angles, each to each, to which the equal sides
are opposite;
therefore the angle FBC is equal to the angle GCB,
and the angle BCF to the angle CBG.
And, since it has been demonstrated,
that the whole angle ABG is equal to the whole ACF,
the parts of which, the angles CBG, BCF are also equal;
therefore the remaining angle ABC is equal to the remaining angle ACB,
which are the angles at the base of the triangle ABC;
and it has also been proved,
that the angle FBC is equal to the angle GCB,
which are the angles upon the other side of the base.
Therefore the angles at the base, &c. Q.E.D.

COR. Hence an equilateral triangle is also equiangular.

PROPOSITION VI. THEOREM.

If two angles of a triangle be equal to each other; the sides also which subtend, or are opposite to, *the equal angles, shall be equal to one another.*

Let ABC be a triangle having the angle ABC equal to the angle ACB.
Then the side AB shall be equal to the side AC.



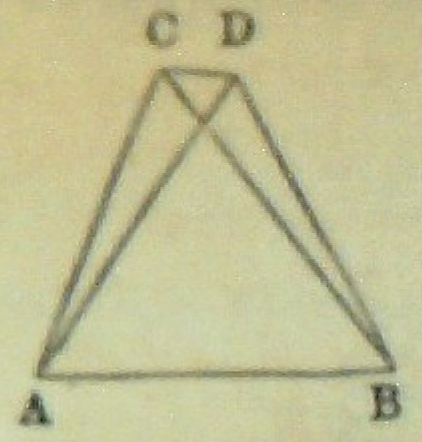
For, if AB be not equal to AC,
one of them is greater than the other.
If possible, let AB be greater than AC;
and from BA cut off BD equal to CA the less, (I. 3.) and join DC.
Then, in the triangles DBC, ABC,
because DB is equal to AC, and BC is common to both triangles,
the two sides DB, BC are equal to the two sides AC, CB, each to each;
and the angle DBC is equal to the angle ACB; (hyp.)
therefore the base DC is equal to the base AB, (I. 4.)
and the triangle DBC is equal to the triangle ABC,
the less equal to the greater, which is absurd. (ax. 9.)
Therefore AB is not unequal to AC, that is, AB is equal to AC.
Wherefore, if two angles, &c. Q.E.D.

COR. Hence an equiangular triangle is also equilateral.

PROPOSITION VII. THEOREM.

Upon the same base, and on the same side of it, there cannot be two triangles that have their sides which are terminated in one extremity of the base, equal to one another, and likewise those which are terminated in the other extremity.

If it be possible, on the same base AB, and upon the same side of it, let there be two triangles ACB, ADB, which have their sides CA, DA, terminated in the extremity A of the base, equal to one another, and likewise their sides CB, DB, that are terminated in B.

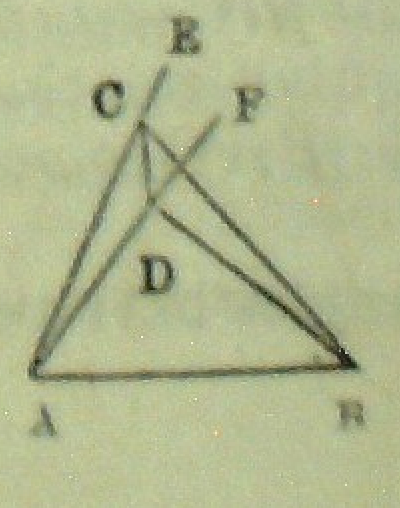


Join CD.

First. When the vertex of each of the triangles is without the other triangle.

Because AC is equal to AD in the triangle ACD,
therefore the angle ADC is equal to the angle ACD; (I. 5.)
but the angle ACD is greater than the angle BCD; (ax. 9.)
therefore also the angle ADC is greater than BCD;
much more therefore is the angle BDC greater than BCD.
Again, because the side BC is equal to BD in the triangle BCD, (hyp.)
therefore the angle BDC is equal to the angle BCD; (I. 5.)
but the angle BDC was proved greater than the angle BCD,
hence the angle BDC is both equal to, and greater than the angle BCD;
which is impossible.

Secondly. Let the vertex D of the triangle ADB fall within the triangle ACB.



Produce AC to E, and AD to F, and join CD.
Then because AC is equal to AD in the triangle ACD,
therefore the angles ECD, FDC upon the other side of the base CD, are equal to one another; (I. 5.)
but the angle ECD is greater than the angle BCD; (ax. 9.)
therefore also the angle FDC is greater than the angle BCD;
much more then is the angle BDC greater than the angle BCD.
Again, because BC is equal to BD in the triangle BCD,
therefore the angle BDC is equal to the angle BCD. (I. 5.)
but the angle BDC has been proved greater than BCD,
wherefore the angle BDC is both equal to, and greater than the angle BCD; which is impossible.

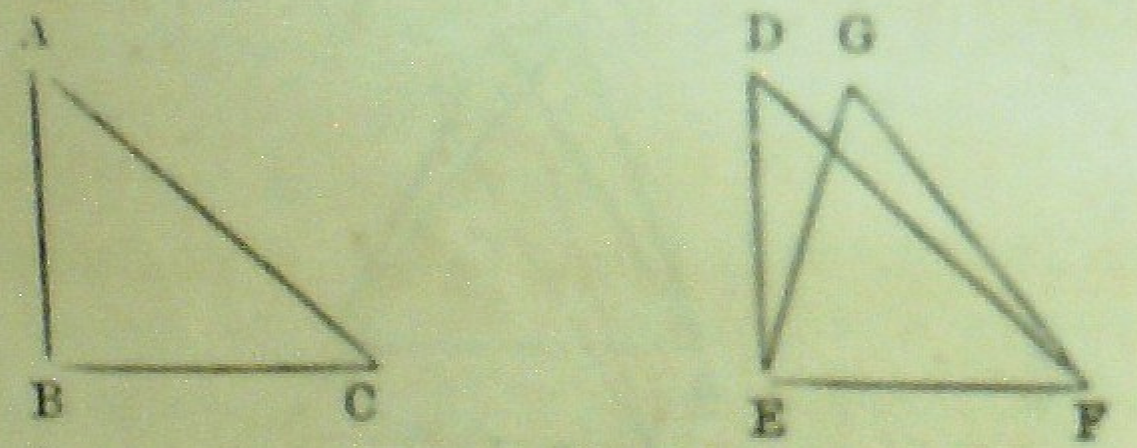
Thirdly. The case in which the vertex of one triangle is upon a side of the other, needs no demonstration.

Therefore, upon the same base and on the same side of it, &c. Q.E.D.

PROPOSITION VIII. THEOREM.

If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal; the angle which is contained by the two sides of the one shall be equal to the angle contained by the two sides equal to them, of the other.

Let ABC, DEF be two triangles, having the two sides AB, AC, equal to the two sides DE, DF, each to each, viz. AB to DE, and AC to DF, and also the base BC equal to the base EF.



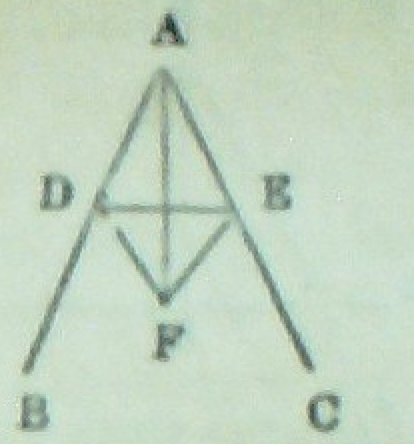
Then the angle BAC shall be equal to the angle EDF.
For, if the triangle ABC be applied to DEF,
so that the point B be on E, and the straight line BC on EF;
then because BC is equal to EF, (hyp.)
therefore the point C shall coincide with the point F.
wherefore BC coinciding with EF,
BA and AC shall coincide with ED, DF;
for, if the base BC coincide with the base EF, but the sides BA, AC, do not coincide with the sides ED, DF, but have a different situation as EG, GF:
then, upon the same base, and upon the same side of it, there can be two triangles which have their sides which are terminated in one extremity of the base, equal to one another, and likewise those sides which are terminated in the other extremity; but this is impossible. (I. 7.)
Therefore, if the base BC coincide with the base EF,
the sides BA, AC cannot but coincide with the sides ED, DF;
wherefore likewise the angle BAC coincides with the angle EDF, and is equal to it. (ax. 8.)

Therefore if two triangles have two sides, &c. Q.E.D.

PROPOSITION IX. PROBLEM.

To bisect a given rectilineal angle, that is, to divide it into two equal angles.

Let BAC be the given rectilineal angle.
It is required to bisect it.

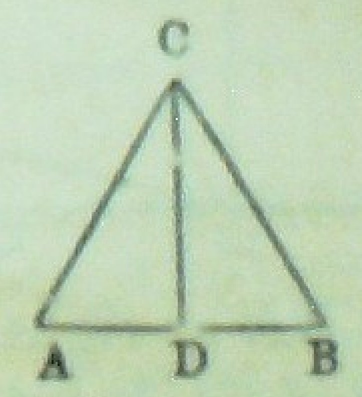


In AB take any point D;
from AC cut off AE equal to AD, (I. 3.) and join DE;
on the side of DE remote from A,
describe the equilateral triangle DEF (I. 1.), and join AF.
Then the straight line AF shall bisect the angle BAC.
Because AD is equal to AE, (constr.)
and AF is common to the two triangles DAF, EAF;
the two sides DA, AF, are equal to the two sides EA, AF, each to each;
and the base DF is equal to the base EF: (constr.)
therefore the angle DAF is equal to the angle EAF. (I. 8.)
Wherefore the angle BAC is bisected by the straight line AF. Q.E.F.

PROPOSITION X. PROBLEM.

To bisect a given finite straight line, that is, to divide it into two equal parts.

Let AB be the given straight line.
It is required to divide AB into two equal parts.
Upon AB describe the equilateral triangle ABC; (I. 1.)



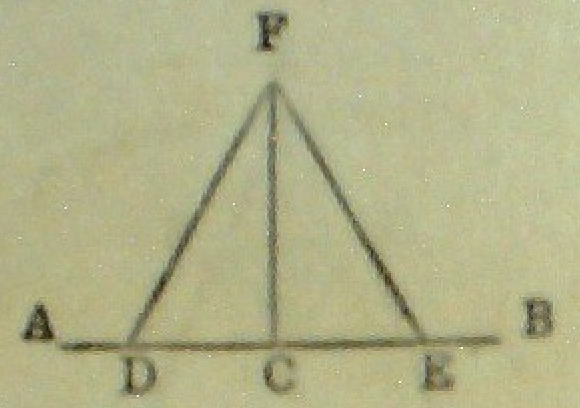
and bisect the angle ACB by the straight line CD meeting AB in the point D. (I. 9.)
Then AB shall be cut into two equal parts in the point D.
Because AC is equal to CB, (constr.)
and CD is common to the two triangles ACD, BCD;
the two sides AC, CD are equal to the two BC, CD, each to each;
and the angle ACD is equal to BCD; (constr.)
therefore the base AD is equal to the base BD. (I. 4.)
Wherefore the straight line AB is divided into two equal parts in the point D. Q.E.F.

PROPOSITION XI. PROBLEM.

To draw a straight line at right angles to a given straight line, from a given point in the same.

Let AB be the given straight line, and C a given point in it.

It is required to draw a straight line from the point C at right angles to AB



In AC take any point D, and make CE equal to CD; (I. 3.)

upon DE describe the equilateral triangle DEF (I. 1,) and join CF.

Then CF drawn from the point C, shall be at right angles to AB.

Because DC is equal to EC, and FC is common to the two triangles DCF, ECF;

the two sides DC, CF are equal to the two sides EC, CF, each to each;

and the base DF is equal to the base EF; (constr.)

therefore the angle DCF is equal to the angle ECF: (I. 8.)

and these two angles are adjacent angles.

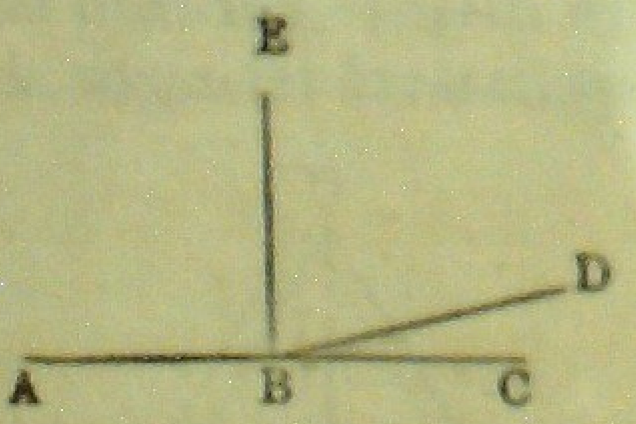
But when the two adjacent angles which one straight line makes with another straight line, are equal to one another, each of them is called a right angle: (def. 10.)

therefore each of the angles DCF, ECF is a right angle.

Wherefore from the given point C, in the given straight line AB, FC has been drawn at right angles to AB. Q.E.F.

Cor. By help of this problem, it may be demonstrated that two straight lines cannot have a common segment.

If it be possible, let the segment AB be common to the two straight lines ABC, ABD.



From the point B, draw BE at right angles to AB; (I. 11.)

then because ABC is a straight line,

therefore the angle ABE is equal to the angle EBC. (def. 10.)

Similarly, because ABD is a straight line,

therefore the angle ABE is equal to the angle EBD;

but the angle ABE is equal to the angle EBC,

wherefore the angle EBD is equal to the angle EBC, (ax. 1.)

the less equal to the greater angle, which is impossible.

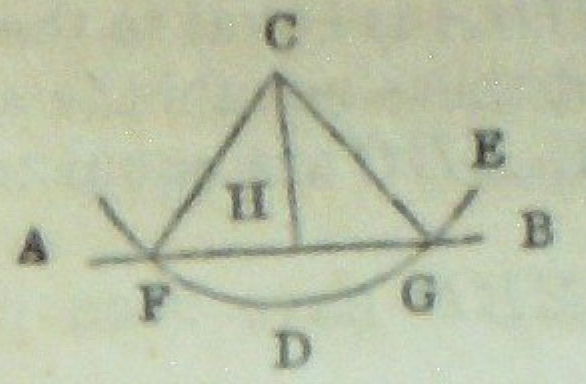
Therefore two straight lines cannot have a common segment.

PROPOSITION XII. PROBLEM.

To draw a straight line perpendicular to a given straight line of unlimited length, from a given point without it.

Let AB be the given straight line, which may be produced any length both ways, and let C be a point without it.

It is required to draw a straight line perpendicular to AB from the point C.



Upon the other side of AB take any point D,

and from the center C, at the distance CD, describe the circle EGF

meeting AB, produced if necessary, in F and G: (post. 3.)

bisect FG in H (I. 10.), and join CH.

Then the straight line CH drawn from the given point C, shall be perpendicular to the given straight line AB.

Join FC, and CG.

Because FH is equal to HG, (constr.)

and HC is common to the triangles FHC, GHC;

the two sides FH, HC, are equal to the two GH, HC, each to each;

and the base CF is equal to the base CG; (def. 15.)

therefore the angle FHC is equal to the angle GHC; (I. 8.)

and these are adjacent angles.

But when a straight line standing on another straight line, makes the adjacent angles equal to one another, each of them is a right angle, and the straight line which stands upon the other is called a perpendicular to it. (def. 10.)

Therefore from the given point C, a perpendicular CH has been drawn to the given straight line AB. Q.E.F.

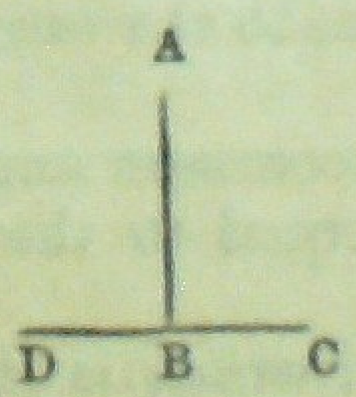
PROPOSITION XIII. THEOREM.

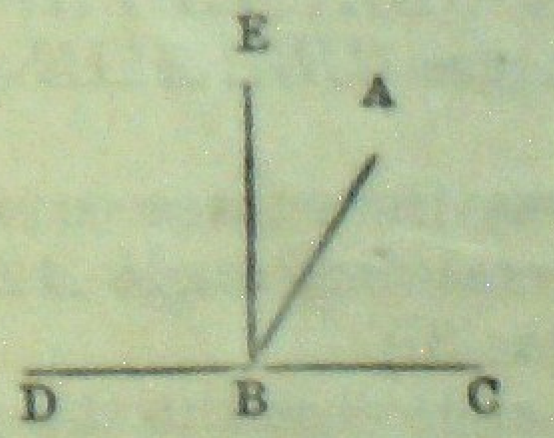
The angles which one straight line makes with another upon one side of it, are either two right angles, or are together equal to two right angles.

Let the straight line AB make with CD, upon one side of it, the angles CBA, ABD.

Then these shall be either two right angles,

or, shall be together, equal to two right angles.





For if the angle CBA be equal to the angle ABD,

each of them is a right angle. (def. 10.)

But if the angle CBA be not equal to the angle ABD,

from the point B draw BE at right angles to CD. (I. 11.)

Then the angles CBE, EBD are two right angles. (def. 10.)

And because the angle CBE is equal to the angles CBA, ABE,

add the angle EBD to each of these equals;

therefore the angles CBE, EBD are equal to the three angles CBA, ABE, EBD. (ax. 2.)

Again, because the angle DBA is equal to the two angles DBE, EBA,

add to each of these equals the angle ABC;

therefore the angles DBA, ABC are equal to the three angles DBE, EBA, ABC.

But the angles CBE, EBD have been proved equal to the same three angles;

and things which are equal to the same thing are equal to one another;

therefore the angles CBE, EBD are equal to the angles DBA, ABC;

but the angles CBE, EBD are two right angles;

therefore the angles DBA, ABC are together equal to two right angles. (ax. 1.)

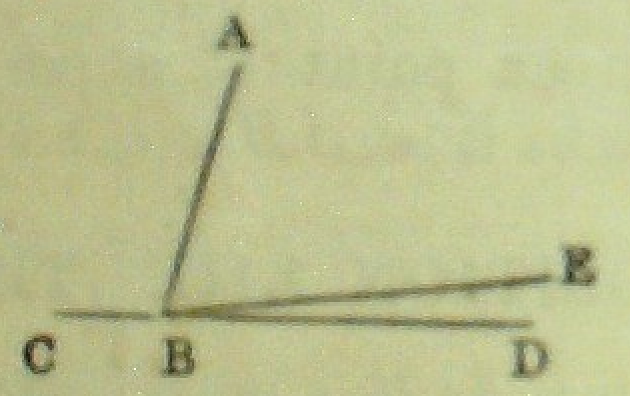
Wherefore, when a straight line, &c. Q.E.D.

PROPOSITION XIV. THEOREM.

If at a point in a straight line, two other straight lines, upon the opposite sides of it, make the adjacent angles together equal to two right angles; then these two straight lines shall be in one and the same straight line.

At the point B in the straight line AB, let the two straight lines BC, BD upon the opposite sides of AB, make the adjacent angles ABC, ABD together equal to two right angles.

Then BD shall be in the same straight line with BC.



For, if BD be not in the same straight line with BC,

if possible, let BE be in the same straight line with it.

Then because AB meets the straight line CBE;

therefore the adjacent angles CBA, ABE are equal to two right angles; (I. 13.)

but the angles CBA, ABD are equal to two right angles; (hyp.)

therefore the angles CBA, ABE are equal to the angles CBA, ABD: (ax. 1.)

take away from these equals the common angle CBA,

therefore the remaining angle ABE is equal to the remaining angle ABD; (ax. 3.)

the less angle equal to the greater, which is impossible:

therefore BE is not in the same straight line with BC.

And in the same manner it may be demonstrated, that no other can be in the same straight line with it but BD, which therefore is in the same straight line with BC.

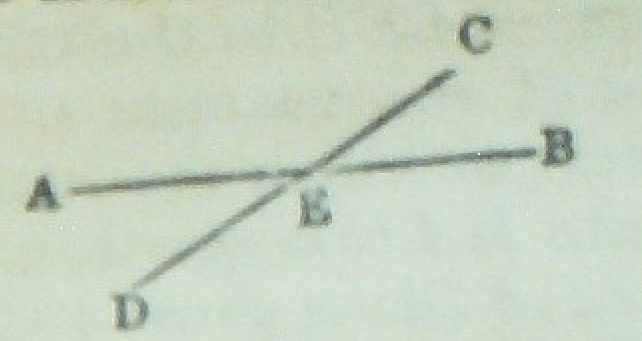
Wherefore, if at a point, &c. Q.E.D.

PROPOSITION XV. THEOREM.

If two straight lines cut one another, the vertical, or opposite angles shall be equal.

Let the two straight lines AB, CD cut one another in the point E.

Then the angle AEC shall be equal to the angle DEB, and the angle CEB to the angle AED.



Because the straight line AE makes with CD at the point E, the adjacent angles CEA, AED;

these angles are together equal to two right angles. (I. 13.)

Again, because the straight line DE makes with AB at the point E, the adjacent angles AED, DEB;

these angles also are equal to two right angles;

but the angles CEA, AED have been shewn to be equal to two right angles;

wherefore the angles CEA, AED are equal to the angles AED, DEB:

take away from each the common angle AED,

and the remaining angle CEA is equal to the remaining angle DEB. (ax. 3.)

In the same manner it may be demonstrated, that the angle CEB is equal to the angle AED.

Therefore, if two straight lines cut one another, &c. Q.E.D.

Cor. 1. From this it is manifest, that, if two straight lines cut each other, the angles which they make at the point where they cut, are together equal to four right angles.

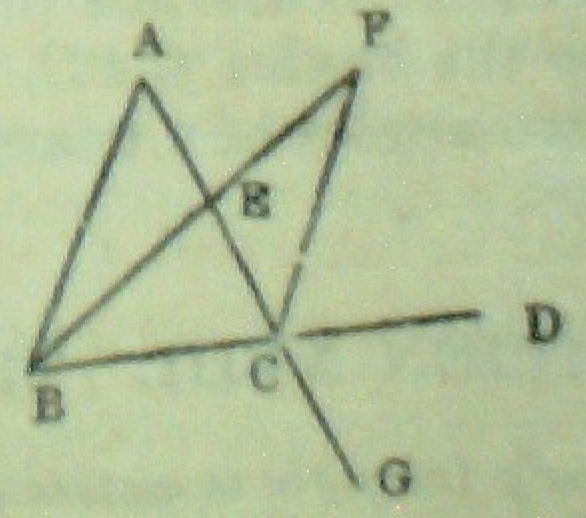
Cor. 2. And consequently that all the angles made by any number of lines meeting in one point, are together equal to four right angles.

PROPOSITION XVI. THEOREM.

If one side of a triangle be produced, the exterior angle is greater than either of the interior opposite angles.

Let ABC be a triangle, and let the side BC be produced to D.

Then the exterior angle ACD shall be greater than either of the interior opposite angles CBA or BAC.



Bisect AC in E, (I. 10.) and join BE;

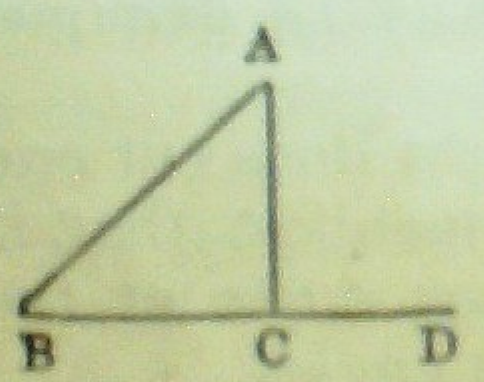
produce BE to F, making EF equal to BE, (I. 3.) and join FC.

Because AE is equal to EC, and BE to EF; (constr.)
the two sides AE, EB are equal to the two CE, EF, each to each, in the triangles ABE, CFE;
and the angle AEB is equal to the angle CEF,
because they are opposite vertical angles; (I. 15.)
therefore the base AB is equal to the base CF, (I. 4.)
and the triangle AEB to the triangle CEF,
and the remaining angles of one triangle to the remaining angles of the other, each to each, to which the equal sides are opposite;
wherefore the angle BAE is equal to the angle ECF;
but the angle ECD or ACD is greater than the angle ECF;
therefore the angle ACD is greater than the angle BAE or BAC.
In the same manner, if the side BC be bisected, and AC be produced to G; it may be demonstrated that the angle BCG, that is, the angle ACD, (I. 15.) is greater than the angle ABC.
Therefore, if one side of a triangle, &c. Q. E. D.

PROPOSITION XVII. THEOREM.

Any two angles of a triangle are together less than two right angles.

Let ABC be any triangle.
Then any two of its angles together shall be less than two right angles.



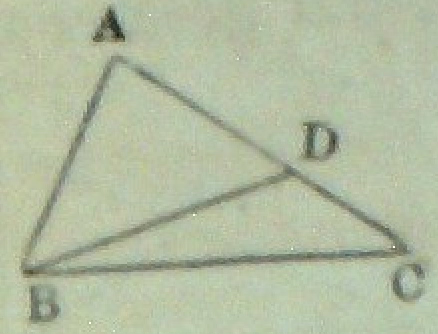
Produce any side BC to D.
Then because ACD is the exterior angle of the triangle ABC;
therefore the angle ACD is greater than the interior and opposite angle ABC; (I. 16.)
to each of these unequals add the angle ACB;
therefore the angles ACD, ACB are greater than the angles ABC, ACB;
but the angles ACD, ACB are equal to two right angles; (I. 13.)
therefore the angles ABC, ACB are less than two right angles.
In like manner it may be demonstrated,
that the angles BAC, ACB are less than two right angles,
as also the angles CAB, ABC.
Therefore any two angles of a triangle, &c. Q. E. D.

PROPOSITION XVIII. THEOREM.

The greater side of every triangle is opposite to the greater angle.

Let ABC be a triangle, of which the side AC is greater than the side AB.

Then the angle ABC shall be greater than the angle ACB.

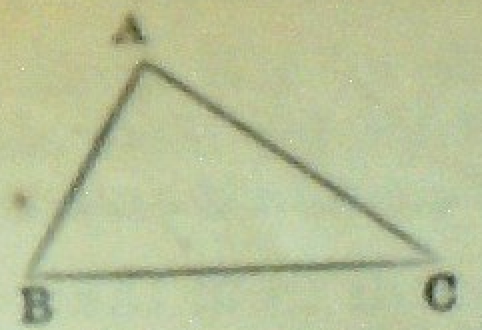


Since the side AC is greater than the side AB, (hyp.)
make AD equal to AB, (I. 3.) and join BD.
Then, because AD is equal to AB, in the triangle ABD,
therefore the angle ABD is equal to the angle ADB, (I. 5.)
but because the side CD of the triangle BDC is produced to A,
therefore the exterior angle ADB is greater than the interior and opposite angle DCB; (I. 16.)
but the angle ADB has been proved equal to the angle ABD,
therefore the angle ABD is greater than the angle DCB;
wherefore much more is the angle ABC greater than the angle ACB.
Therefore the greater side, &c. Q. E. D.

PROPOSITION XIX. THEOREM.

The greater angle of every triangle is subtended by the greater side, or, has the greater side opposite to it.

Let ABC be a triangle of which the angle ABC is greater than the angle BCA.
Then the side AC shall be greater than the side AB.



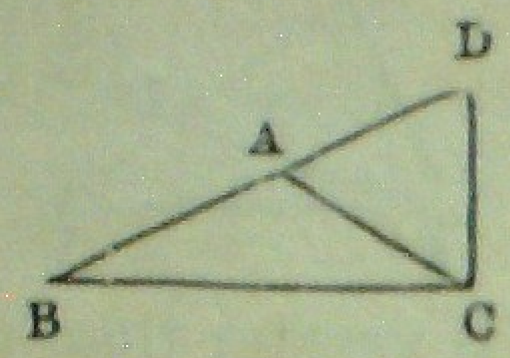
For, if AC be not greater than AB,
AC must either be equal to, or less than AB;
if AC were equal to AB,
then the angle ABC would be equal to the angle ACB; (I. 5.)
but it is not equal; (hyp.)
therefore the side AC is not equal to AB.
Again, if AC were less than AB,
then the angle ABC would be less than the angle ACB; (I. 18.)
but it is not less, (hyp.)
therefore the side AC is not less than AB;
and AC has been shewn to be not equal to AB;
therefore AC is greater than AB.
Wherefore the greater angle, &c. Q. E. D.

PROPOSITION XX. THEOREM.

Any two sides of a triangle are together greater than the third side.

Let ABC be a triangle.
Then any two sides of it together shall be greater than the third side,
viz. the sides BA, AC greater than the side BC;

AB, *BC* greater than *AC*;
and *BC*, *CA* greater than *AB*.

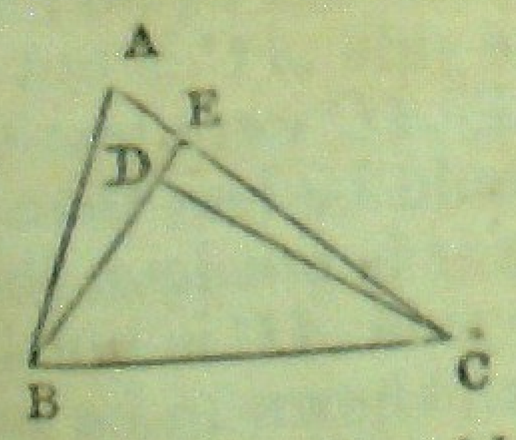


Produce the side *BA* to the point *D*,
make *AD* equal to *AC*, (I. 3.) and join *DC*.
Then because *AD* is equal to *AC*, (constr.)
therefore the angle *ACD* is equal to the angle *ADC*; (I. 5.)
but the angle *BCD* is greater than the angle *ACD*; (ax. 9.)
therefore also the angle *BCD* is greater than the angle *ADC*.
And because in the triangle *DBC*,
the angle *BCD* is greater than the angle *BDC*,
and that the greater angle is subtended by the greater side; (I. 19.)
therefore the side *DB* is greater than the side *BC*;
but *DB* is equal to *BA* and *AC*,
therefore the sides *BA* and *AC* are greater than *BC*.
In the same manner it may be demonstrated,
that the sides *AB*, *BC* are greater than *CA*;
also that *BC*, *CA* are greater than *AB*.
Therefore any two sides, &c. Q. E. D.

PROPOSITION XXI. THEOREM.

If from the ends of a side of a triangle, there be drawn two straight lines to a point within the triangle; these shall be less than the other two sides of the triangle, but shall contain a greater angle.

Let *ABC* be a triangle, and from the points *B*, *C*, the ends of the side *BC*, let the two straight lines *BD*, *CD* be drawn to a point *D* within the triangle.
Then *BD* and *DC* shall be less than *BA* and *AC* the other two sides of the triangle,
but shall contain an angle *BDC* greater than the angle *BAC*.



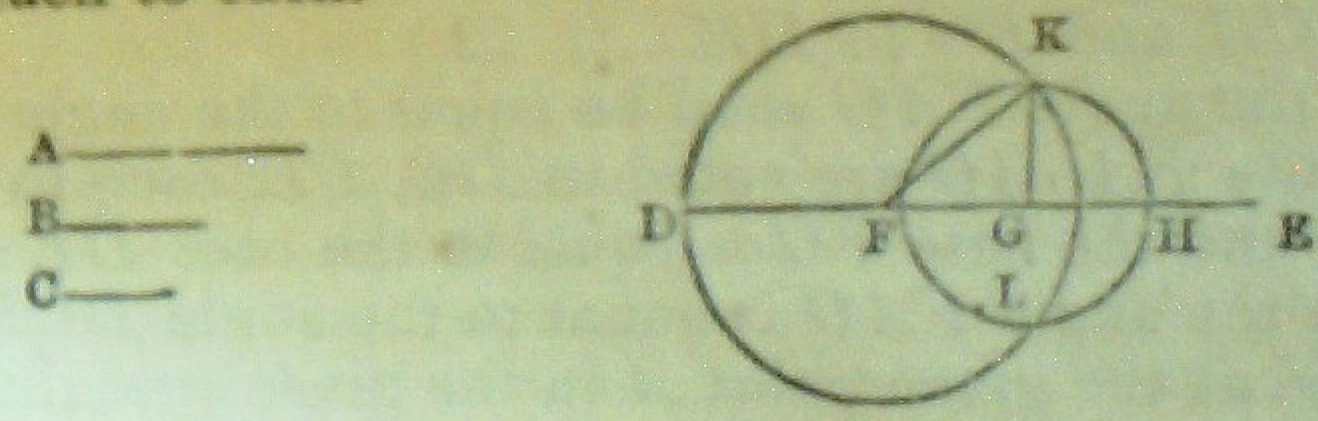
Produce *BD* to meet the side *AC* in *E*.
Because two sides of a triangle are greater than the third side, (I. 20.)
therefore the two sides *BA*, *AE* of the triangle *ABE* are greater than *BE*;
to each of these unequals add *EC*;
therefore the sides *BA*, *AC* are greater than *BE*, *EC*. (ax. 4.)
Again, because the two sides *CE*, *ED* of the triangle *CED* are greater than *DC*; (I. 20.)
add *DB* to each of these unequals;

therefore the sides *CE*, *EB* are greater than *CD*, *DB*. (ax. 4.)
But it has been shewn that *BA*, *AC* are greater than *BE*, *EC*;
much more then are *BA*, *AC* greater than *BD*, *DC*.
Again, because the exterior angle of a triangle is greater than the interior and opposite angle; (I. 16.)
therefore the exterior angle *BDC* of the triangle *CDE* is graeter than the interior and opposite angle *CED*;
for the same reason, the exterior angle *CED* of the triangle *ABE* is greater than the interior and opposite angle *BAC*;
and it has been demonstrated,
that the angle *BDC* is greater than the angle *CEB*;
much more therefore is the angle *BDC* greater than the angle *BAC*.
Therefore, if from the ends of the side, &c. Q. E. D.

PROPOSITION XXII. PROBLEM.

To make a triangle of which the sides shall be equal to three given straight lines, but any two whatever of these must be greater than the third.

Let *A*, *B*, *C* be the three given straight lines,
of which any two whatever are greater than the third, (I. 20.)
namely, *A* and *B* greater than *C*;
A and *C* greater than *B*;
and *B* and *C* greater than *A*.
It is required to make a triangle of which the sides shall be equal to *A*, *B*, *C*, each to each.



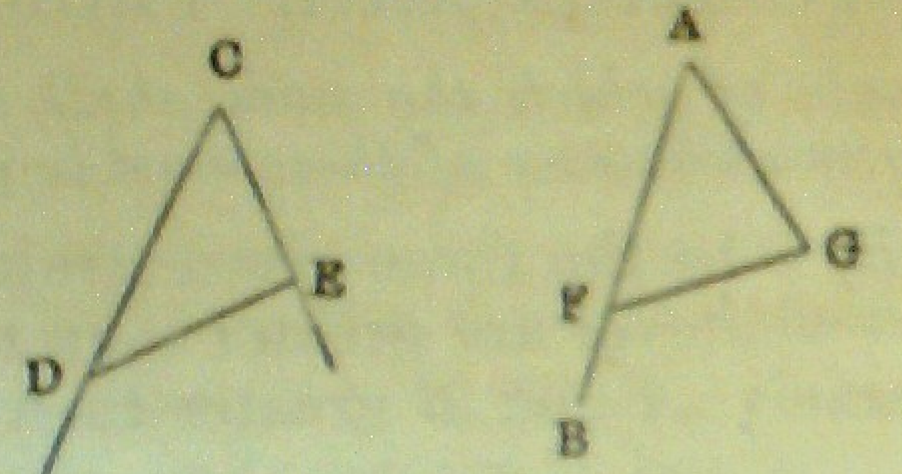
Take a straight line *DE* terminated at the point *D*, but unlimited towards *E*,
make *DF* equal to *A*, *FG* equal to *B*, and *GH* equal to *C*; (I. 3.)
from the center *F*, at the distance *FD*, describe the circle *DKL*; (post. 3.)
from the center *G*, at the distance *GH*, describe the circle *HLK*;
from *K* where the circles cut each other, draw *KF*, *KG* to the points *F*, *G*;
Then the triangle *KFG* shall have its sides equal to the three straight lines *A*, *B*, *C*.
Because the point *F* is the center of the circle *DKL*,
therefore *FD* is equal to *FK*; (def. 15.)
but *FD* is equal to the straight line *A*;
therefore *FK* is equal to *A*.
Again, because *G* is the center of the circle *HKL*,
therefore *GH* is equal to *GK*, (def. 15.)
but *GH* is equal to *C*;
therefore also *GK* is equal to *C*; (ax. 1.)
and *FG* is equal to *B*;

therefore the three straight lines KF, FG, GK, are respectively
equal to the three, A, B, C:
and therefore the triangle KFG has its three sides KF, FG, GK,
equal to the three given straight lines A, B, C. Q.E.F.

PROPOSITION XXIII. PROBLEM.

At a given point in a given straight line, to make a rectilineal angle equal to a given rectilineal angle.

Let AB be the given straight line, and A the given point in it,
and DCE the given rectilineal angle.
It is required, at the given point A in the given straight line AB, to make an angle that shall be equal to the given rectilineal angle DCE.

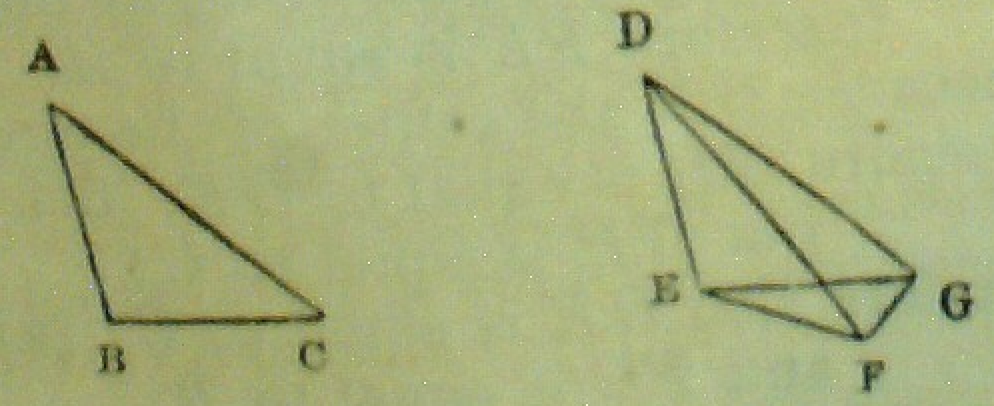


In CD, CE, take any points D, E, and join DE;
on AB, make the triangle AFG, the sides of which shall be equal to the three straight lines CD, DE, EC, so that AF be equal to CD, AG to CE, and FG to DE. (I. 22.)
Then the angle FAG shall be equal to the angle DCE.
Because FA, AG are equal to DC, CE, each to each,
and the base FG is equal to the base DE;
therefore the angle FAG is equal to the angle DCE. (I. 8.)
Wherefore, at the given point A in the given straight line AB, the angle FAG is made equal to the given rectilineal angle DCE. Q.E.F.

PROPOSITION XXIV. THEOREM.

If two triangles have two sides of the one equal to two sides of the other, each to each, but the angle contained by the two sides of one of them greater than the angle contained by the two sides equal to them, of the other; the base of that which has the greater angle, shall be greater than the base of the other.

Let ABC, DEF be two triangles, which have the two sides AB, AC, equal to the two DE, DF, each to each, namely, AB equal to DE, and AC to DF; but the angle BAC greater than the angle EDF.
Then the base BC shall be greater than the base EF.

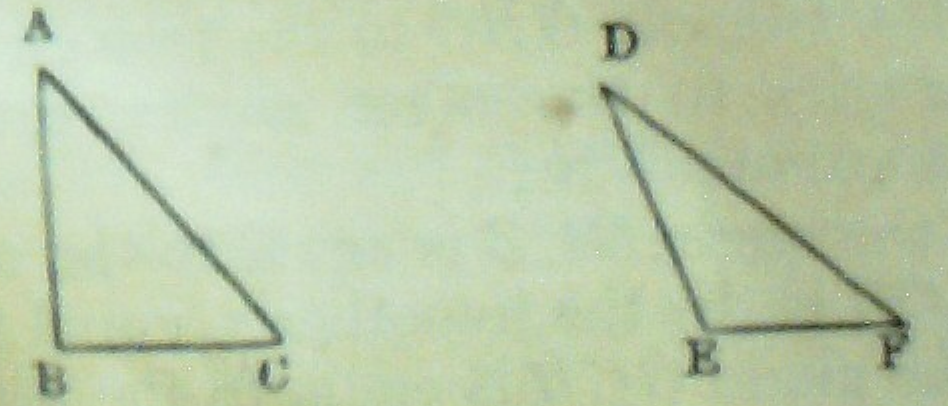


Of the two sides DE, DF, let DE be not greater than DF.
at the point D, in the line DE, and on the same side of it as DF,
make the angle EDG equal to the angle BAC; (I. 23.)
make DG equal to DF or AC, (I. 3.) and join EG, GF.
Then, because DE is equal to AB, and DG to AC,
the two sides DE, DG are equal to the two AB, AC, each to each,
and the angle EDG is equal to the angle BAC;
therefore the base EG is equal to the base BC. (I. 4.)
And because DG is equal to DF in the triangle DFG,
therefore the angle DFG is equal to the angle DGF; (I. 5.)
but the angle DGF is greater than the angle EGF; (ax. 9.)
therefore the angle DFG is also greater than the angle EGF;
much more therefore is the angle EFG greater than the angle EGF.
And because in the triangle EFG, the angle EFG is greater than the angle EGF,
and that the greater angle is subtended by the greater side; (I. 19.)
therefore the side EG is greater than the side EF;
but EG was proved equal to BC;
therefore BC is greater than EF.
Wherefore, if two triangles, &c. Q.E.D.

PROPOSITION XXV. THEOREM.

If two triangles have two sides of the one equal to two sides of the other, each to each, but the base of one greater than the base of the other; the angle contained by the sides of the one which has the greater base, shall be greater than the angle contained by the sides, equal to them, of the other.

Let ABC, DEF be two triangles which have the two sides AB, AC, equal to the two sides DE, DF, each to each, namely, AB equal to DE, and AC to DF; but the base BC greater than the base EF.
Then the angle BAC shall be greater than the angle EDF.



For, if the angle BAC be not greater than the angle EDF,
it must either be equal to it, or less than it.
If the angle BAC were equal to the angle EDF,
then the base BC would be equal to the base EF; (I. 4.)
but it is not equal, (hyp.)
therefore the angle BAC is not equal to the angle EDF.
Again, if the angle BAC were less than the angle EDF,
then the base BC would be less than the base EF; (I. 24.)
but it is not less, (hyp.)
therefore the angle BAC is not less than the angle EDF;
and it has been shewn, that the angle BAC is not equal to the angle EDF;
therefore the angle BAC is greater than the angle EDF.
Wherefore, if two triangles, &c. Q.E.D.

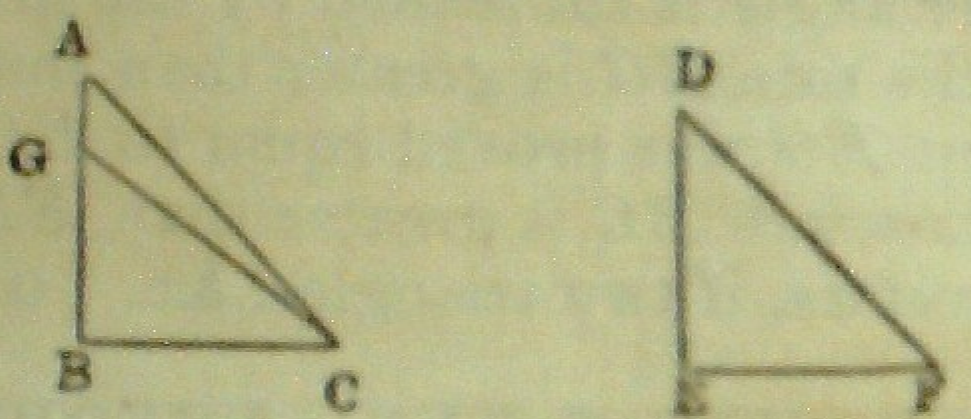
PROPOSITION XXVI. THEOREM.

If two triangles have two angles of the one equal to two angles of t other, each to each, and one side equal to one side, viz. either the sides adj cent to the equal angles in each, or the sides opposite to them; then shall t other sides be equal, each to each, and also the third angle of the one equ to the third angle of the other.

Let *ABC*, *DEF* be two triangles which have the angles *AB* *BCA*, equal to the angles *DEF*, *EFD*, each to each, namely, *AB* to *DEF*, and *BCA* to *EFD*; also one side equal to one side.

First, let those sides be equal which are adjacent to the angles t are equal in the two triangles, namely, *BC* to *EF*.

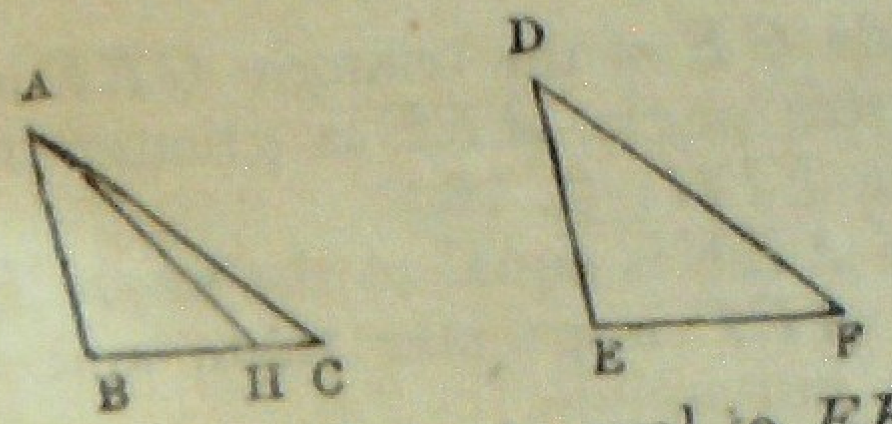
Then the other sides shall be equal, each to each, namely, *AB* t *DE*, and *AC* to *DF*, and the third angle *BAC* to the third angle *EDF*.



For, if *AB* be not equal to *DE*,
one of them must be greater than the other.
If possible, let *AB* be greater than *DE*,
make *BG* equal to *ED*, (I. 3) and join *GC*.
Then in the two triangles *GBC*, *DEF*,
because *GB* is equal to *DE*, and *BC* to *EF*, (hyp.)
the two sides, *GB*, *BC* are equal to the two *DE*, *EF*, each to each;
and the angle *GBC* is equal to the angle *DEF*;
therefore the base *GC* is equal to the base *DF*, (I. 4.)
and the triangle *GBC* to the triangle *DEF*,
and the other angles to the other angles, each to each, to which
the equal sides are opposite;
therefore the angle *GCB* is equal to the angle *DFE*;
but the angle *ACB* is, by the hypothesis, equal to the angle *DFE*;
wherefore also the angle *GCB* is equal to the angle *ACB*; (ax. 1.)
the less angle equal to the greater, which is impossible;
therefore *AB* is not unequal to *DE*,
that is, *AB* is equal to *DE*.
Hence, in the triangles *ABC*, *DEF*;
because *AB* is equal to *DE*, and *BC* to *EF*, (hyp.)
and the angle *ABC* is equal to the angle *DEF*; (hyp.)
therefore the base *AC* is equal to the base *DF*, (I. 4.)
and the third angle *BAC* to the third angle *EDF*.

Secondly, let the sides which are opposite to one of the equal ang in each triangle be equal to one another, namely *AB* equal to *DE*.

Then in this case likewise the other sides shall be equal, *AC* to and *BC* to *EF* and also the third angle *BAC* to the third angle *E*



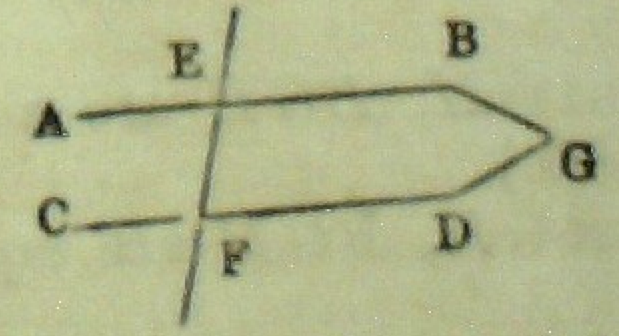
For if *BC* be not equal to *EF*,
one of them must be greater than the other.
If possible, let *BC* be greater than *EF*;
make *BH* equal to *EF*, (I. 3.) and join *AH*.
Then in the two triangles *ABH*, *DEF*,
because *AB* is equal to *DE*, and *BH* to *EF*,
and the angle *ABH* to the angle *DEF*; (hyp.)
therefore the base *AH* is equal to the base *DF*, (I. 4.)
and the triangle *ABH* to the triangle *DEF*,
and the other angles to the other angles, each to each, to which the
equal sides are opposite;
therefore the angle *BHA* is equal to the angle *EFD*;
but the angle *EFD* is equal to the angle *BCA*; (hyp.)
therefore the angle *BHA* is equal to the angle *BCA*, (ax. 1.)
that is, the exterior angle *BHA* of the triangle *AHC*, is
equal to its interior and opposite angle *BCA*;
which is impossible; (I. 16.)
wherefore *BC* is not unequal to *EF*,
that is, *BC* is equal to *EF*.
Hence, in the triangles *ABC*, *DEF*;
because *AB* is equal to *DE*, and *BC* to *EF*, (hyp.)
and the included angle *ABC* is equal to the included angle *DEF*; (hyp.)
therefore the base *AC* is equal to the base *DF*, (I. 4.)
and the third angle *BAC* to the third angle *EDF*.
Wherefore, if two triangles, &c. Q. E. D.

PROPOSITION XXVII. THEOREM.

If a straight line falling on two other straight lines, make the alternate angles equal to each other; these two straight lines shall be parallel.

Let the straight line *EF*, which falls upon the two straight lines *AB*, *CD*, make the alternate angles *AEF*, *EFD*, equal to one another.

Then *AB* shall be parallel to *CD*.



For, if *AB* be not parallel to *CD*,
then *AB* and *CD* being produced will meet, either towards *A* and *C*,
or towards *B* and *D*.
Let *AB*, *CD* be produced and meet, if possible, towards *B* and *D*,
in the point *G*,
then *GEF* is a triangle.

And because a side GE of the triangle GEF is produced to
therefore its exterior angle AEF is greater than the interior an
opposite angle EFG; (I. 16.)
but the angle AEF is equal to the angle EFG; (hyp.)
therefore the angle AEF is greater than, and equal to, the angl
EFG; which is impossible.

Therefore AB, CD being produced, do not meet towards B, D.

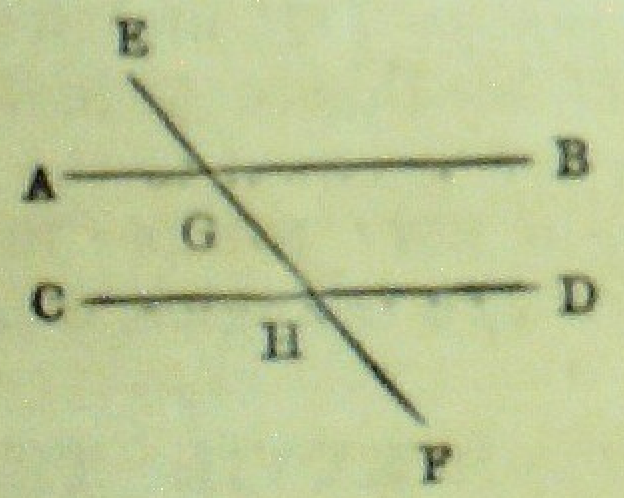
In like manner, it may be demonstrated, that they do not me
when produced towards A, C.

But those straight lines in the same plane, which meet neither w
though produced ever so far, are parallel to one another; (def. 35.)
therefore AB is parallel to CD.
Wherefore, if a straight line, &c. Q. E. D.

PROPOSITION XXVIII. THEOREM.

If a straight line falling upon two other straight lines, make the exter angle equal to the interior and opposite upon the same side of the line; make the interior angles upon the same side together equal to two ri angles; the two straight lines shall be parallel to one another.

Let the straight line EF, which falls upon the two straight li
AB, CD, make the exterior angle EGB equal to the interior a
opposite angle GHD, upon the same side of the line EF; or ma
the two interior angles BGH, GHD on the same side togeth
equal to two right angles.
Then AB shall be parallel to CD.



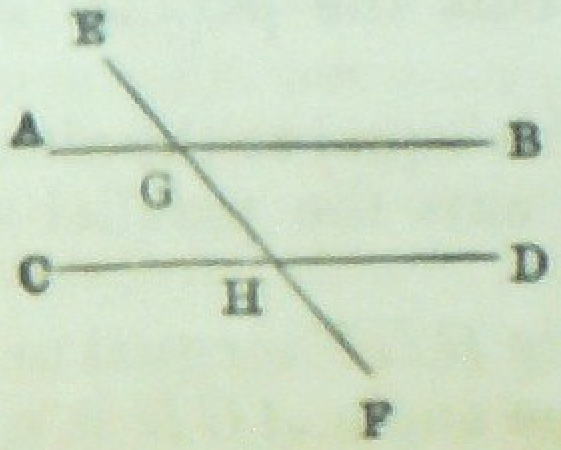
Because the angle EGB is equal to the angle GHD, (hyp.)
and the angle EGB is equal to the angle AGH, (I. 15.)
therefore the angle AGH is equal to the angle GHD; (ax. 1.)
and they are alternate angles,
therefore AB is parallel to CD. (I. 27.)

Again, because the angles BGH, GHD are together equal to
right angles, (hyp.)
and that the angles AGH, BGH are also together equal
right angles; (I. 13.)
therefore the angles AGH, BGH are equal to the angles
GHD; (ax. 1.)
take away from these equals, the common angle BGH;
therefore the remaining angle AGH is equal to the remaining a
GHD; (ax. 3.)
and they are alternate angles;
therefore AB is parallel to CD. (I. 27.)
Wherefore, if a straight line, &c. Q. E. D.

PROPOSITION XXIX. THEOREM.

If a straight line fall upon two parallel straight lines, it makes the alternate angles equal to one another; and the exterior angle equal to the interior and opposite upon the same side; and likewise the two interior angles upon the same side together equal to two right angles.

Let the straight line EF fall upon the parallel straight lines AB, CD.
Then the alternate angles AGH, GHD shall be equal to one another;
the exterior angle EGB shall be equal to the interior and opposite
angle GHD upon the same side of the line EF;
and the two interior angles BGH, GHD upon the same side of EF
shall be together equal to two right angles.



First. For, if the angle AGH be not equal to the alternate angle
GHD, one of them must be greater than the other;
if possible, let AGH be greater than GHD,
then because the angle AGH is greater than the angle GHD,
add to each of these unequals the angle BGH;
therefore the angles AGH, BGH are greater than the angles BGH,
GHD; (ax. 4.)
but the angles AGH, BGH are equal to two right angles; (I. 13.)
therefore the angles BGH, GHD are less than two right angles;
but those straight lines, which with another straight line falling upon
them, make the two interior angles on the same side less than two
right angles, will meet together if continually produced; (ax. 12.)
therefore the straight lines AB, CD, if produced far enough, will
meet towards B, D;
but they never meet, since they are parallel by the hypothesis;
therefore the angle AGH is not unequal to the angle GHD,
that is, the angle AGH is equal to the alternate angle GHD.

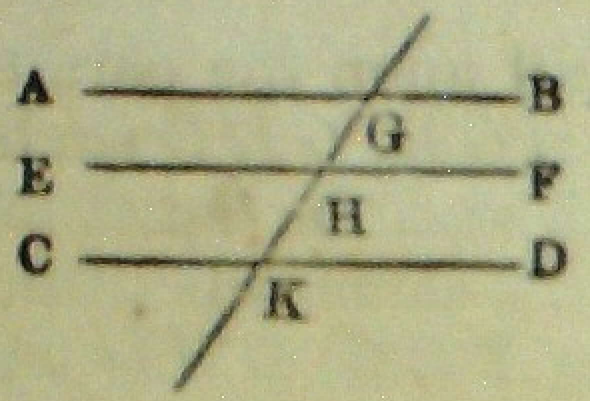
Secondly. Because the angle AGH is equal to the angle EGB, (I. 15.)
and the angle AGH is equal to the angle GHD,
therefore the exterior angle EGB is equal to the interior and opposite
angle GHD, on the same side of the line.

Thirdly. Because the angle EGB is equal to the angle GHD,
add to each of them the angle BGH;
therefore the angles EGB, BGH are equal to the angles BGH, GHD;
(ax. 2.)
but EGB, BGH are equal to two right angles; (I. 13.)
therefore also the two interior angles BGH, GHD on the same side
of the line are equal to two right angles. (ax. 1.)
Wherefore, if a straight line, &c. Q. E. D.

PROPOSITION XXX. THEOREM.

Straight lines which are parallel to the same straight line are parallel t each other.

Let the straight lines AB, CD, be each of them parallel to EF.
Then shall AB be also parallel to CD.

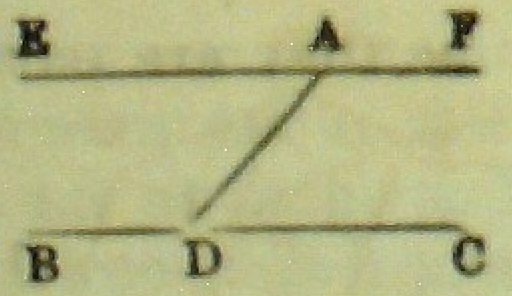


Let the straight line GHK cut AB, EF, CD.
Then because GHK cuts the parallel straight lines AB, EF, i G, H:
therefore the angle AGH is equal to the alternate angle GHF. (I. 29
Again, because GHK cuts the parallel straight lines EF, CD, H, K;
therefore the exterior angle GHF is equal to the interior angle HK
and it was shewn that the angle AGH is equal to the angle GHF;
therefore the angle AGH is equal to the angle GKD;
and these are alternate angles;
therefore AB is parallel to CD. (I. 27.)
Wherefore, straight lines which are parallel, &c. Q. E. D.

PROPOSITION XXXI. PROBLEM.

To draw a straight line through a given point parallel to a given straigh line.

Let A be the given point, and BC the given straight line.
It is required to draw, through the point A, a straight line parall to the straight line BC.

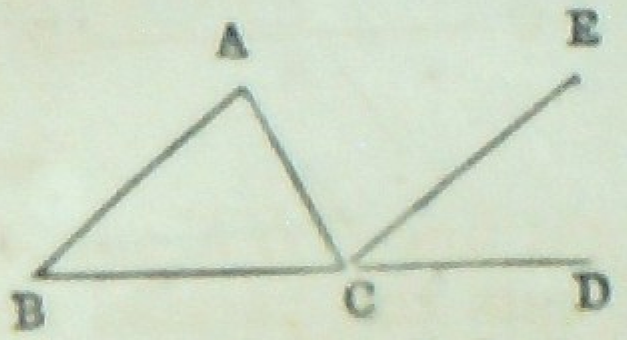


In the line BC take any point D, and join AD;
at the point A in the straight line AD,
make the angle DAE equal to the angle ADC, (I. 23.) on the opp site side of AD;
and produce the straight line EA to F.
Then EF shall be parallel to BC.
Because the straight line AD meets the two straight lines EF, BC,
and makes the alternate angles EAD, ADC, equal to one another,
therefore EF is parallel to BC. (I. 27.)
Wherefore, through the given point A, has been drawn a straight line EAF parallel to the given straight line BC. Q.E.F.

PROPOSITION XXXII. THEOREM.

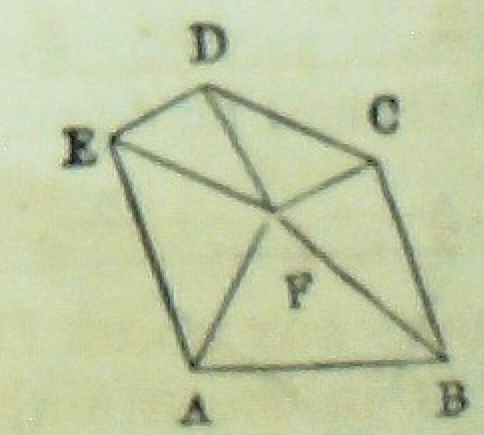
If a side of any triangle be produced, the exterior angle is equal to the two interior and opposite angles; and the three interior angles of every triangle are together equal to two right angles.

Let ABC be a triangle, and let one of its sides BC be produced to D.
Then the exterior angle ACD shall be equal to the two interior and opposite angles CAB, ABC:
and the three interior angles ABC, BCA, CAB shall be equal to two right angles.



Through the point C draw CE parallel to the side BA. (I. 31.)
Then because CE is parallel to BA, and AC meets them,
therefore the angle ACE is equal to the alternate angle BAC. (I. 29.)
Again, because CE is parallel to AB, and BD falls upon them,
therefore the exterior angle ECD is equal to the interior and opposite angle ABC; (I. 29.)
but the angle ACE was shewn to be equal to the angle BAC;
therefore the whole exterior angle ACD is equal to the two interior and opposite angles CAB, ABC. (ax. 2.)
Again, because the angle ACD is equal to the two angles ABC, BAC,
to each of these equals add the angle ACB,
therefore the angles ACD and ACB are equal to the three angles ABC, BAC, and ACB. (ax. 2.)
but the angles ACD, ACB are equal to two right angles, (I. 13.)
therefore also the angles ABC, BAC, ACB are equal to two right angles. (ax. 1.)
Wherefore, if a side of any triangle be produced, &c. Q.E.D.

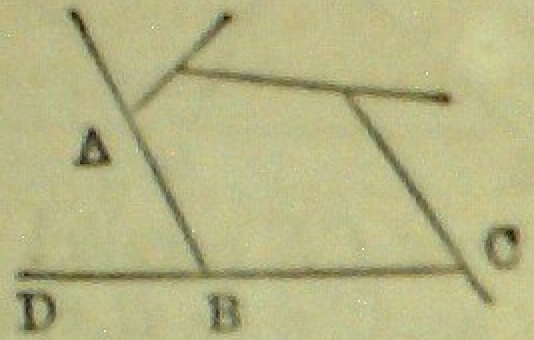
COR. 1. All the interior angles of any rectilineal figure together with four right angles, are equal to twice as many right angles as the figure has sides.



For any rectilineal figure $ABCDE$ can be divided into as many triangles as the figure has sides, by drawing straight lines from a point F within the figure to each of its angles.
Then, because the three interior angles of a triangle are equal to two right angles, and there are as many triangles as the figure has sides,
therefore all the angles of these triangles are equal to twice as many right angles as the figure has sides;
but the same angles of these triangles are equal to the interior angles of the figure together with the angles at the point F:

and the angles at the point F, which is the common vertex of
the triangles, are equal to four right angles, (I. 15. Cor. 2.)
therefore the same angles of these triangles are equal to the ang
of the figure together with four right angles;
but it has been proved that the angles of the triangles are equal to
twice as many right angles as the figure has sides;
therefore all the angles of the figure together with four right angl
are equal to twice as many right angles as the figure has sides.

COR. 2. All the exterior angles of any rectilineal figure, made b producing the sides successively in the same direction, are toget equal to four right angles.

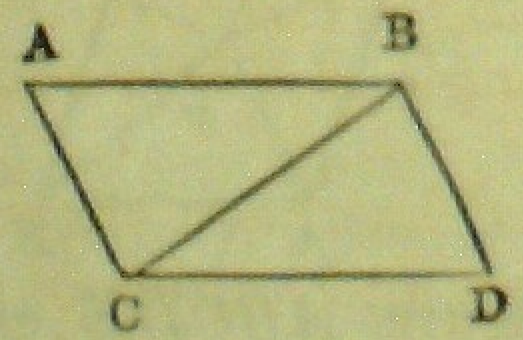


Since every interior angle ABC with its adjacent exterior ang ABD, is equal to two right angles, (I. 13.)
therefore all the interior angles, together with all the exterior angl are equal to twice as many right angles as the figure has sides;
but it has been proved by the foregoing corollary, that all the in terior angles together with four right angles are equal to twice as man right angles as the figure has sides;
therefore all the interior angles together with all the exterior angle are equal to all the interior angles and four right angles, (ax. 1.)
take from these equals all the interior angles,
therefore all the exterior angles of the figure are equal to four rig angles. (ax. 3.)

PROPOSITION XXXIII. THEOREM.

The straight lines which join the extremities of two equal and parall straight lines towards the same parts, are also themselves equal and parall

Let AB, CD be equal and parallel straight lines,
and joined towards the same parts by the straight lines AC, BD.
Then AC, BD shall be equal and parallel.



Join BC.
Then because AB is parallel to CD, and BC meets them,
therefore the angle ABC is equal to the alternate angle BCD; (I.
and because AB is equal to CD, and BC common to the two tria
ABC, DCB; the two sides AB, BC, are equal to the two DC, CB,
to each, and the angle ABC was proved to be equal to the angle
therefore the base AC is equal to the base BD, (I. 4.)
and the triangle ABC to the triangle BCD,

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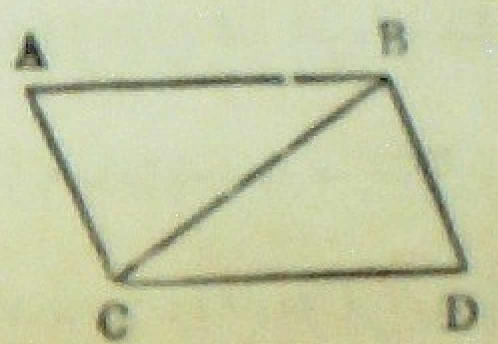
and the other angles to the other angles, each to each, to which the
equal sides are opposite;
therefore the angle ACB is equal to the angle CBD.
And because the straight line BC meets the two straight lines AC, BD, and makes the alternate angles ACB, CBD equal to one another;
therefore AC is parallel to BD; (I. 27.)
and AC was shewn to be equal to BD.
Therefore, straight lines which, &c. Q. E. D.

PROPOSITION XXXIV. THEOREM.

The opposite sides and angles of a parallelogram are equal to one another, and the diameter bisects it, that is, divides it into two equal parts.

Let $ACDB$ be a parallelogram, of which BC is a diameter.
Then the opposite sides and angles of the figure shall be equal to one another; and the diameter BC shall bisect it.



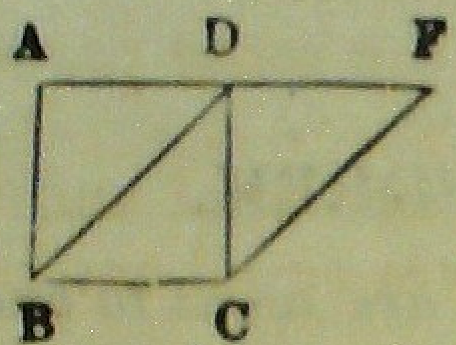
Because AB is parallel to CD, and BC meets them,
therefore the angle ABC is equal to the alternate angle BCD. (I. 29.)
And because AC is parallel to BD, and BC meets them,
therefore the angle ACB is equal to the alternate angle CBD. (I. 29.)
Hence in the two triangles ABC, CBD,
because the two angles ABC, BCA in the one, are equal to the two
angles BCD, CBD in the other, each to each;
and one side BC, which is adjacent to their equal angles, common to
the two triangles;
therefore their other sides are equal, each to each, and the third angle
of the one to the third angle of the other, (I. 26.)
namely, the side AB to the side CD, and AC to BD, and the angle
BAC to the angle BDC.
And because the angle ABC is equal to the angle BCD,
and the angle CBD to the angle ACB,
therefore the whole angle ABD is equal to the whole angle ACD; (ax. 2.)
and the angle BAC has been shewn to be equal to BDC;
therefore the opposite sides and angles of a parallelogram are equal to
one another.
Also the diameter BC bisects it.
For since AB is equal to CD, and BC common, the two sides AB,
BC, are equal to the two DC, CB, each to each,
and the angle ABC has been proved to be equal to the angle BCD;
therefore the triangle ABC is equal to the triangle BCD; (I. 4.) and
the diameter BC divides the parallelogram $ACDB$ into two equal parts.
Q. E. D.

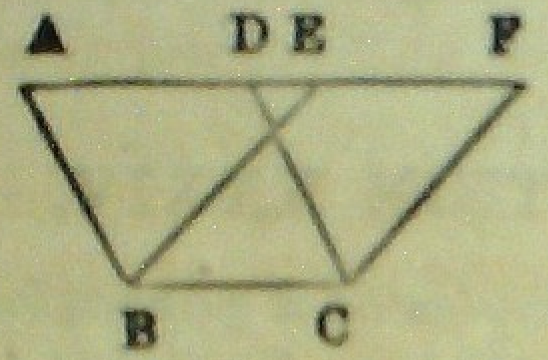
PROPOSITION XXXV. THEOREM.

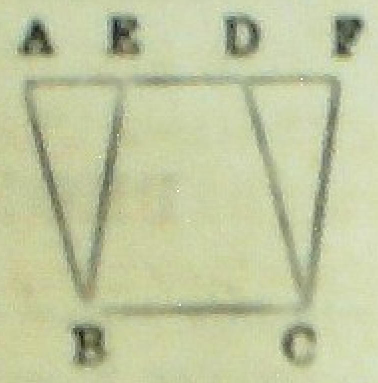
Parallelograms upon the same base, and between the same parallels, a equal to one another.

Let the parallelograms $ABCD$, $EBCF$ be upon the same base BC and between the same parallels AF, BC.

Then the parallelogram $ABCD$ shall be equal to the parallelogram $EBCF$.







If the sides AD, DF of the parallelograms $ABCD$, $DBCF$, opposite to the base BC, be terminated in the same point D;
then it is plain that each of the parallelograms is double of the triangle BDC; (I. 34.)
and therefore the parallelogram $ABCD$ is equal to the parallelogram $DBCF$. (ax. 6.)

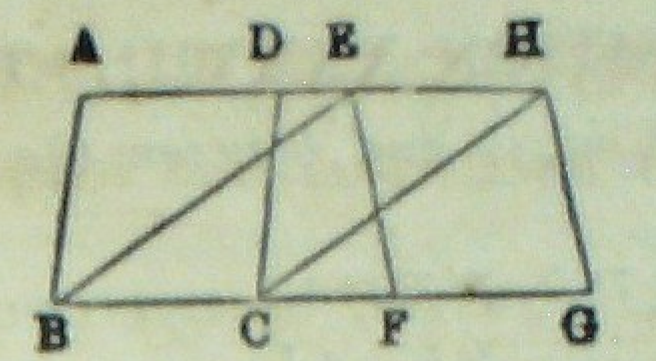
But if the sides AD, EF, opposite to the base BC, be not terminated in the same point;
Then, because $ABCD$ is a parallelogram,
therefore AD is equal to BC; (I. 34.)
and for a similar reason, EF is equal to BC;
wherefore AD is equal to EF; (ax. 1.)
and DE is common;
therefore the whole, or the remainder AE, is equal to the whole, or the remainder DF; (ax. 2 or 3.)
and AB is equal to DC; (I. 34.)
hence in the triangles EAB, FDC,
because FD is equal to EA, and DC to AB,
and the exterior angle FDC is equal to the interior and opposite angle EAB; (I. 29.)
therefore the base FC is equal to the base EB, (I. 4.)
and the triangle FDC is equal to the triangle EAB.
From the trapezium $ABCF$ take the triangle FDC,
and from the same trapezium take the triangle EAB,
and the remainders are equal, (ax. 3.)
therefore the parallelogram $ABCD$ is equal to the parallelogram $EBCF$.
Therefore, parallelograms upon the same, &c. Q. E. D.

PROPOSITION XXXVI. THEOREM.

Parallelograms upon equal bases and between the same parallels, are equal to one another.

Let $ABCD$, $EFGH$ be parallelograms upon equal bases BC, FG, and between the same parallels AH, BG.

Then the parallelogram $ABCD$ shall be equal to the parallelogram $EFGH$.

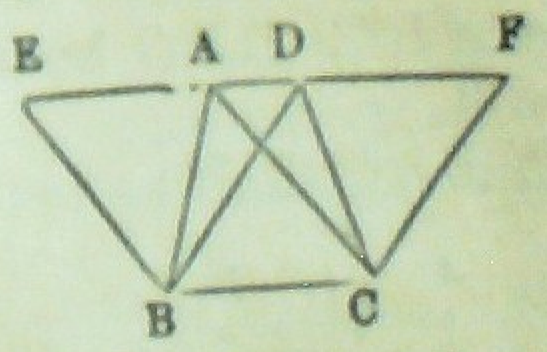


Join BE, CH.
Then because BC is equal to FG, (hyp.) and FG to EH, (I. 34.)
therefore BC is equal to EH; (ax. 1.)
and these lines are parallels, and joined towards the same parts by the straight lines BE, CH;
but straight lines which join the extremities of equal and parallel straight lines towards the same parts, are themselves equal and parallel; (I. 33.)
therefore BE, CH are both equal and parallel;
wherefore $EBCH$ is a parallelogram. (def. A.)
And because the parallelograms $ABCD$, $EBCH$, are upon the same base BC, and between the same parallels BC, AH;
therefore the parallelogram $ABCD$ is equal to the parallelogram $EBCH$. (I. 35.)
For the same reason, the parallelogram $EFGH$ is equal to the parallelogram $EBCH$;
therefore the parallelogram $ABCD$ is equal to the parallelogram $EFGH$. (ax. 1.)
Therefore, parallelograms upon equal, &c. Q. E. D.

PROPOSITION XXXVII. THEOREM.

Triangles upon the same base and between the same parallels, are equal to one another.

Let the triangles ABC, DBC be upon the same base BC, and between the same parallels AD, BC.
Then the triangle ABC shall be equal to the triangle DBC.



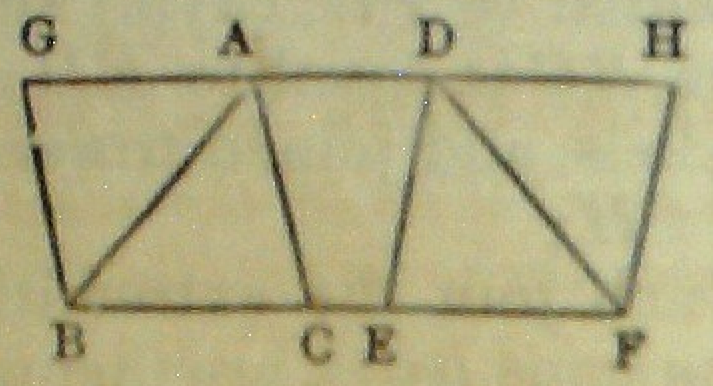
Produce AD both ways to the points E, F;
through B draw BE parallel to CA, (I. 31.)
and through C draw CF parallel to BD.
Then each of the figures $EBCA$, $DBCF$ is a parallelogram;
and $EBCA$ is equal to $DBCF$, (I. 35.) because they are upon the same base BC, and between the same parallels BC, EF.
And because the diameter AB bisects the parallelogram $EBCA$,
therefore the triangle ABC is half of the parallelogram $EBCA$; (I. 34.)
also because the diameter DC bisects the parallelogram $DBCF$,
therefore the triangle DBC is half of the parallelogram $DBCF$,
but the halves of equal things are equal; (ax. 7.)
therefore the triangle ABC is equal to the triangle DBC.
Wherefore, triangles, &c. Q. E. D.

PROPOSITION XXXVIII. THEOREM.

Triangles upon equal bases and between the same parallels, are equal to one another.

Let the triangles ABC, DEF be upon equal bases BC, EF, and between the same parallels BF, AD.

Then the triangle ABC shall be equal to the triangle DEF.



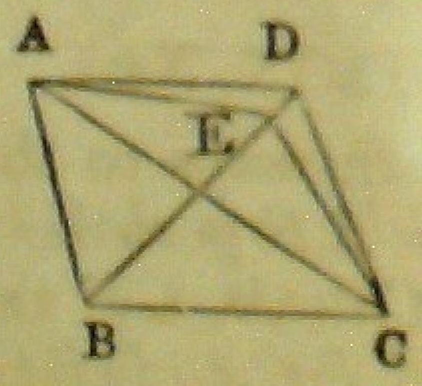
Produce AD both ways to the points G, H;
through B draw BG parallel to CA, (I. 31.)
and through F draw FH parallel to ED.
Then each of the figures $GBCA$, $DEFH$ is a parallelogram;
and they are equal to one another, (I. 36.)
because they are upon equal bases BC, EF,
and between the same parallels BF, GH.
And because the diameter AB bisects the parallelogram $GBCA$,
therefore the triangle ABC is the half of the parallelogram $GBCA$; (I. 34.)
also, because the diameter DF bisects the parallelogram $DEFH$,
therefore the triangle DEF is the half of the parallelogram $DEFH$;
but the halves of equal things are equal; (ax. 7.)
therefore the triangle ABC is equal to the triangle DEF.
Wherefore, triangles upon equal bases, &c. Q. E. D.

PROPOSITION XXXIX. THEOREM.

Equal triangles upon the same base and upon the same side of it, are between the same parallels.

Let the equal triangles ABC, DBC be upon the same base BC and upon the same side of it.

Then the triangles ABC, DBC shall be between the same parallels.



Join AD; then AD shall be parallel to BC.
For if AD be not parallel to BC,
if possible, through the point A, draw AE parallel to BC, (I. 31.)
meeting BD, or BD produced, in E, and join EC.
Then the triangle ABC is equal to the triangle EBC, (I. 37.)
because they are upon the same base BC,
and between the same parallels BC, AE:
but the triangle ABC is equal to the triangle DBC; (hyp.)
therefore the triangle DBC is equal to the triangle EBC,

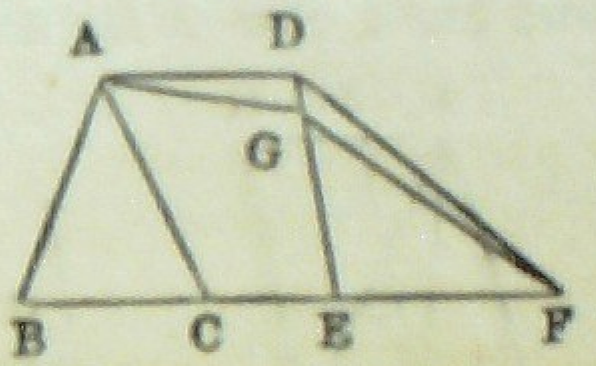
the greater triangle equal to the less, which is impossible:
therefore AE is not parallel to BC.
In the same manner it can be demonstrated,
that no other line drawn from A but AD is parallel to BC;
AD is therefore parallel to BC.
Wherefore, equal triangles upon, &c. Q. E. D.

PROPOSITION XL. THEOREM.

Equal triangles upon equal bases in the same straight line, and towards the same parts, are between the same parallels.

Let the equal triangles ABC, DEF be upon equal bases BC, EF, in the same straight line BF, and towards the same parts.

Then they shall be between the same parallels.



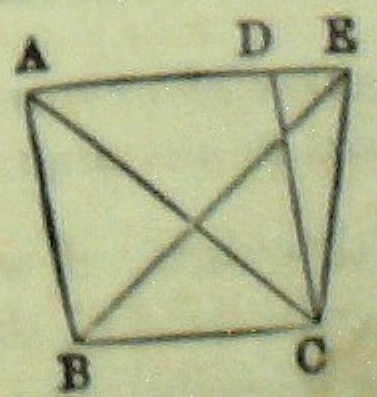
Join AD; then AD shall be parallel to BF.
For if AD be not parallel to BF,
if possible, through A draw AG parallel to BF, (I. 31.)
meeting ED, or ED produced in G, and join GF.
Then the triangle ABC is equal to the triangle GEF, (I. 38.)
because they are upon equal bases BC, EF,
and between the same parallels BF, AG;
but the triangle ABC is equal to the triangle DEF; (hyp.)
therefore the triangle DEF is equal to the triangle GEF, (ax. 1.)
the greater triangle equal to the less, which is impossible:
therefore AG is not parallel to BF.
And in the same manner it can be demonstrated,
that there is no other line drawn from A parallel to it but AD;
AD is therefore parallel to BF.
Wherefore, equal triangles upon, &c. Q. E. D.

PROPOSITION XLI. THEOREM.

If a parallelogram and a triangle be upon the same base, and between the same parallels; the parallelogram shall be double of the triangle.

Let the parallelogram $ABCD$, and the triangle EBC be upon the same base BC, and between the same parallels BC, AE.

Then the parallelogram $ABCD$ shall be double of the triangle EBC



Join AC.
Then the triangle ABC is equal to the triangle EBC, (I. 37.)

because they are upon the same base BC, and between the same parallels BC, AE.

But the parallelogram $ABCD$ is double of the triangle ABC, because the diameter AC bisects it; (I. 34.)

wherefore $ABCD$ is also double of the triangle EBC.

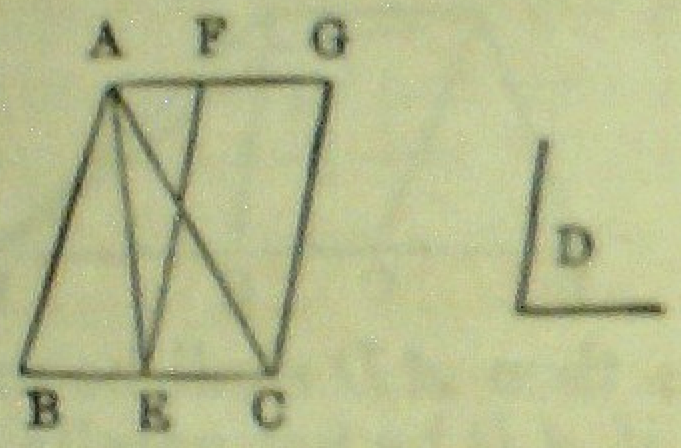
Therefore, if a parallelogram and a triangle, &c. Q. E. D.

PROPOSITION XLII. PROBLEM.

To describe a parallelogram that shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

Let ABC be the given triangle, and D the given rectilineal angle.

It is required to describe a parallelogram that shall be equal to the given triangle ABC, and have one of its angles equal to D.



Bisect BC in E, (I. 10.) and join AE;

at the point E in the straight line EC, make the angle CEF equal to the angle D; (I. 23.)

through C draw CG parallel to EF, and through A draw AFG parallel to BC, (I. 31.) meeting EF in F, and CG in G.

Then the figure $CEFG$ is a parallelogram. (def. A.)

And because the triangles ABE, AEC are on the equal bases BE, EC, and between the same parallels BC, AG; they are therefore equal to one another; (I. 38.)

and the triangle ABC is double of the triangle AEC;

but the parallelogram $FECG$ is double of the triangle AEC, (I. 41.) because they are upon the same base EC, and between the same parallels EC, AG;

therefore the parallelogram $FECG$ is equal to the triangle ABC, (ax. 6.)

and it has one of its angles CEF equal to the given angle D.

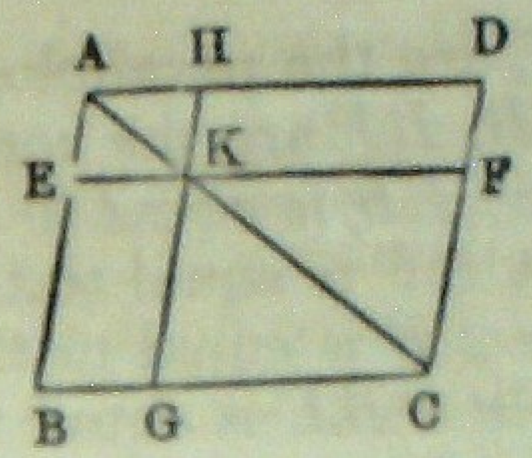
Wherefore, a parallelogram $FECG$ has been described equal to the given triangle ABC, and having one of its angles CEF equal to the given angle D. Q. E. F.

PROPOSITION XLIII. THEOREM.

The complements of the parallelograms, which are about the diameter of any parallelogram, are equal to one another.

Let $ABCD$ be a parallelogram, of which the diameter is AC: and EH, GF the parallelograms about AC, *that is, through which AC passes*: also BK, KD the other parallelograms which make up the whole figure $ABCD$, which are therefore called the complements.

Then the complement BK shall be equal to the complement KD.



Because $ABCD$ is a parallelogram, and AC its diameter, therefore the triangle ABC is equal to the triangle ADC. (I. 34.)

Again, because $EKHA$ is a parallelogram, and AK its diameter, therefore the triangle AEK is equal to the triangle AHK; (I. 34.)

and for the same reason, the triangle KGC is equal to the triangle KFC.

Wherefore the two triangles AEK, KGC are equal to the two triangles AHK, KFC, (ax. 2.)

but the whole triangle ABC is equal to the whole triangle ADC;

therefore the remaining complement BK is equal to the remaining complement KD. (ax. 3.)

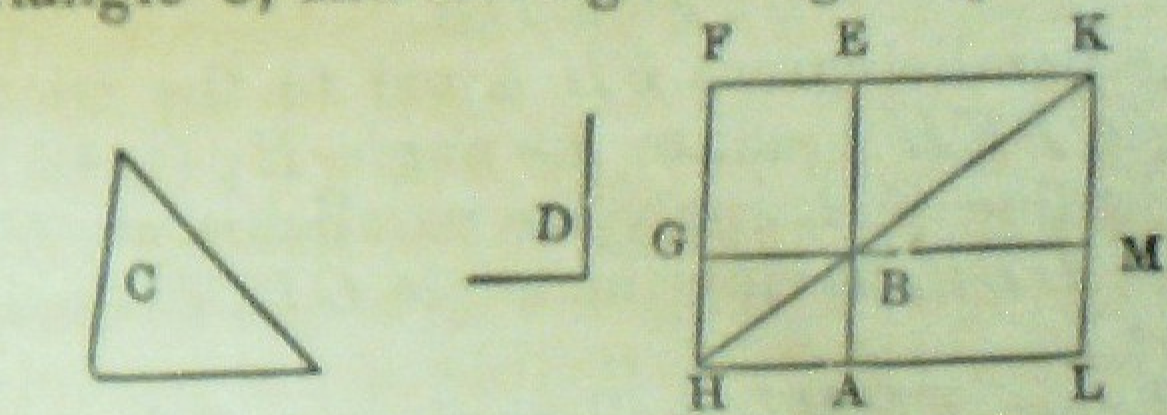
Wherefore the complements, &c. Q. E. D.

PROPOSITION XLIV. PROBLEM.

To a given straight line to apply a parallelogram, which shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

Let AB be the given straight line, and C the given triangle, and D the given rectilineal angle.

It is required to apply to the straight line AB, a parallelogram equal to the triangle C, and having an angle equal to the angle D.



Make the parallelogram $BEFG$ equal to the triangle C, and having the angle EBG equal to the angle D, (I. 42.) so that BE be in the same straight line with AB;

produce FG to H,

through A draw AH parallel to BG or EF, (I. 31.) and join HB.

Then because the straight line HF falls upon the parallels AH, EF, therefore the angles AHF, HFE are together equal to two right angles; (I. 29.)

wherefore the angles BHF, HFE are less than two right angles:

but straight lines which with another straight line, make the two interior angles upon the same side less than two right angles, do meet if produced far enough: (ax. 12.)

therefore HB, FE shall meet if produced;

let them be produced and meet in K,

through K draw KL parallel to EA or FH,

and produce HA, GB to meet KL in the points L, M.

Then $HLKF$ is a parallelogram, of which the diameter is HK;

and AG, ME, are the parallelograms about HK;
also LB, BF are the complements;
therefore the complement LB is equal to the complement BF; (I. 43.)
but the complement BF is equal to the triangle C; (constr.)
wherefore LB is equal to the triangle C.

And because the angle GBE is equal to the angle ABM, (I. 15.)
and likewise to the angle D; (constr.)
therefore the angle ABM is equal to the angle D. (ax. 1.)

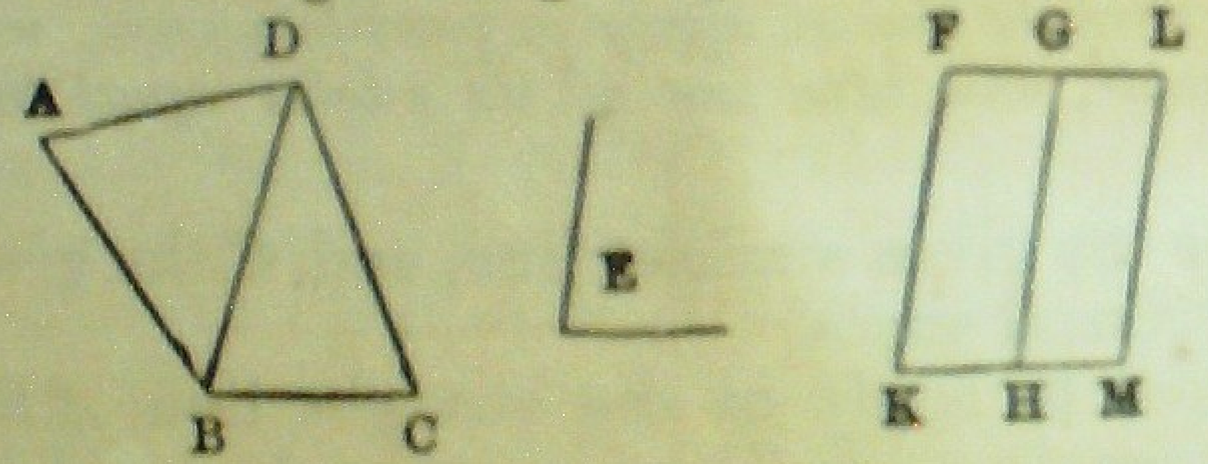
Therefore to the given straight line AB, the parallelogram LB has been applied, equal to the triangle C, and having the angle ABM equal to the given angle D. Q.E.F.

PROPOSITION XLV. PROBLEM.

To describe a parallelogram equal to a given rectilineal figure, and having an angle equal to a given rectilineal angle.

Let $ABCD$ be the given rectilineal figure, and E the given rectilineal angle.

It is required to describe a parallelogram that shall be equal to the figure $ABCD$, and having an angle equal to the given angle E.



Join DB.
Describe the parallelogram FH equal to the triangle ADB, and
having the angle FKH equal to the angle E; (I. 42.)
to the straight line GH, apply the parallelogram GM equal to the
triangle DBC, having the angle GHM equal to the angle E.
(I. 44.)
Then the figure $FKML$ shall be the parallelogram required.
Because each of the angles FKH, GHM, is equal to the angle E,
therefore the angle FKH is equal to the angle GHM;
add to each of these equals the angle KHG;
therefore the angles FKH, KHG are equal to the angles KHG, GHM;
but FKH, KHG are equal to two right angles; (I. 29.)
therefore also KHG, GHM are equal to two right angles;
and because at the point H, in the straight line GH, the two
straight lines KH, HM, upon the opposite sides of it, make the adjacent angles KHG, GHM equal to two right angles,
therefore HK is in the same straight line with HM. (I. 14.)
And because the line HG meets the parallels KM, FG,
therefore the angle MHG is equal to the alternate angle HGF; (I. 29.)
add to each of these equals the angle HGL;
therefore the angles MHG, HGL are equal to the angles HGF, HGL;
but the angles MHG, HGL are equal to two right angles; (I. 29.)
therefore also the angles HGF, HGL are equal to two right angles,
and therefore FG is in the same straight line with GL. (I. 14.)

And because KF is parallel to HG, and HG to ML,
therefore KF is parallel to ML; (I. 30.)
and FL has been proved parallel to KM,
wherefore the figure $FKML$ is a parallelogram;
and since the parallelogram HF is equal to the triangle ABD,
and the parallelogram GM to the triangle BDC;
therefore the whole parallelogram $KFLM$ is equal to the whole rectilineal figure $ABCD$.

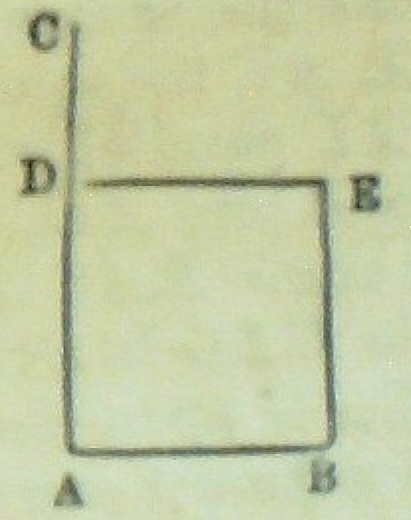
Therefore the parallelogram $KFLM$ has been described equal to the given rectilineal figure $ABCD$, having the angle FKM equal to the given angle E. Q.E.F.

COR. From this it is manifest how, to a given straight line, to apply a parallelogram, which shall have an angle equal to a given rectilineal angle, and shall be equal to a given rectilineal figure; viz. by applying to the given straight line a parallelogram equal to the first triangle ABD, (I. 44.) and having an angle equal to the given angle.

PROPOSITION XLVI. PROBLEM.

To describe a square upon a given straight line.

Let AB be the given straight line.



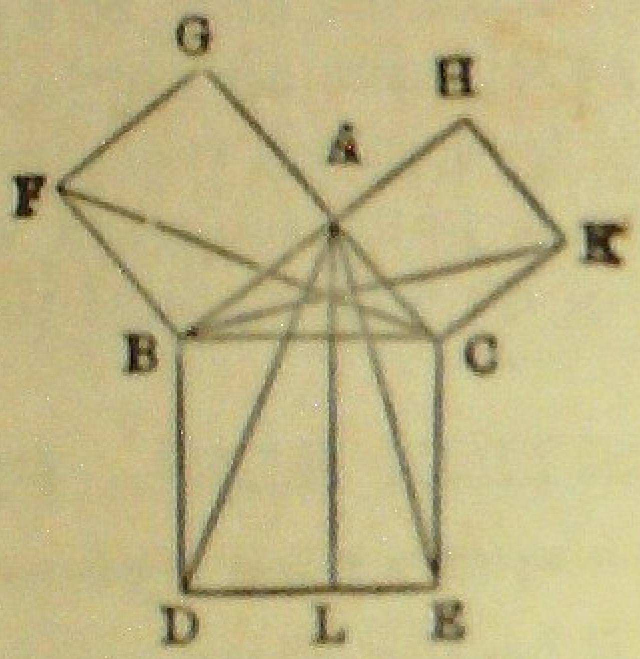
It is required to describe a square upon AB.
From the point A draw AC at right angles to AB; (I. 11.)
make AD equal to AB; (I. 3.)
through the point D draw DE parallel to AB; (I. 31.)
and through B, draw BE parallel to AD, meeting DE in E;
therefore $ABED$ is a parallelogram;
whence AB is equal to DE, and AD to BE; (I. 34.)
but AD is equal to AB,
therefore the four lines AB, BE, ED, DA are equal to one another,
and the parallelogram $ABED$ is equilateral.
It has likewise all its angles right angles;
since AD meets the parallels AB, DE,
therefore the angles BAD, ADE are equal to two right angles; (I. 29.)
but BAD is a right angle; (constr.)
therefore also ADE is a right angle.
But the opposite angles of parallelograms are equal; (I. 34.)
therefore each of the opposite angles ABE, BED is a right angle;
wherefore the figure $ABED$ is rectangular,
and it has been proved to be equilateral;
therefore the figure $ABED$ is a square. (def. 30.)
and it is described upon the given straight line AB. Q.E.F.

Cor. Hence, every parallelogram that has one of its angles a right angle, has all its angles right angles.

PROPOSITION XLVII. THEOREM.

In any right-angled triangle, the square which is described upon the side subtending the right angle, is equal to the squares described upon the sides which contain the right angle.

Let ABC be a right-angled triangle, having the right angle BAC.
Then the square described upon the side BC, shall be equal to the squares described upon BA, AC.



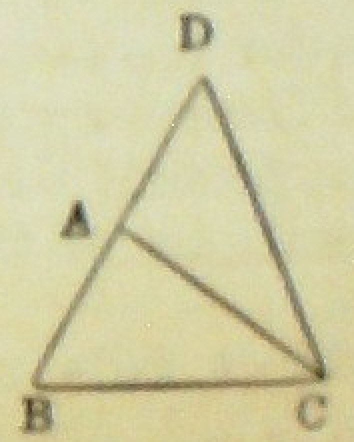
On BC describe the square $BDEC$, (I. 46.)
and on BA, AC the squares GB HC;
through A draw AL parallel to BD or CE; (I. 31.)
and join AD, FC.
Then because the angle BAC is a right angle, (hyp.)
and that the angle BAG is a right angle, (def. 30.)
the two straight lines AC, AG upon the opposite sides of AB, make with it at the point A, the adjacent angles equal to two right angles;
therefore CA is in the same straight line with AG. (I. 14.)
For the same reason, BA and AH are in the same straight line.
And because the angle DBC is equal to the angle FBA,
each of them being a right angle,
add to each of these equals the angle ABC,
therefore the whole angle ABD is equal to the whole angle FBC. (ax. 2.)
And because the two sides AB, BD, are equal to the two sides FB, BC, each to each, and the included angle ABD is equal to the included angle FBC,
therefore the base AD is equal to the base FC, (I. 4.)
and the triangle ABD to the triangle FBC.
Now the parallelogram BL is double of the triangle ABD, (I. 41.)
because they are upon the same base BD, and between the same parallels BD, AL;
also the square GB is double of the triangle FBC,
because these also are upon the same base FB, and between the same parallels FB, GC.
But the doubles of equals are equal to one another; (ax. 6.)
therefore the parallelogram BL is equal to the square GB.
Similarly, by joining AE, BK, it can be proved,
that the parallelogram CL is equal to the square HC.

Therefore the whole square $BDEC$ is equal to the two squares GB, HC; (ax. 2.)
and the square $BDEC$ is described upon the straight line BC,
and the squares GB, HC, upon AB, AC:
therefore the square upon the side BC, is equal to the squares upon the sides AB, AC.
Therefore, in any right-angled triangle, &c. Q.E.D.

PROPOSITION XLVIII. THEOREM.

If the square described upon one of the sides of a triangle, be equal to the squares described upon the other two sides of it; the angle contained by these two sides is a right angle.

Let the square described upon BC, one of the sides of the triangle ABC, be equal to the squares upon the other two sides, AB, AC.
Then the angle BAC shall be a right angle.



From the point A draw AD at right angles to AC, (I. 11.)
make AD equal to AB, and join DC.
Then, because AD is equal to AB,
the square on AD is equal to the square on AB;
to each of these equals add the square on AC;
therefore the squares on AD, AC are equal to the squares on AB, AC:
but the squares on AD, AC are equal to the square on DC, (I. 47.)
because the angle DAC is a right angle;
and the square on BC, by hypothesis, is equal to the squares on BA, AC;
therefore the square on DC is equal to the square on BC;
and therefore the side DC is equal to the side BC.
And because the side AD is equal to the side AB,
and AC is common to the two triangles DAC, BAC;
the two sides DA, AC, are equal to the two BA, AC, each to each:
and the base DC has been proved to be equal to the base BC;
therefore the angle DAC is equal to the angle BAC; (I. 8.)
but DAC is a right angle;
therefore also BAC is a right angle.
Therefore, if the square described upon, &c. Q.E.D.

NOTES TO BOOK I.

ON THE DEFINITIONS.

Geometry is one of the most perfect of the deductive Sciences, and seems to rest on the simplest inductions from experience and observation.

The first principles of Geometry are therefore in this view consistent hypotheses founded on facts cognizable by the senses, and it is a subject of primary importance to draw a distinction between the conception of things and the things themselves. These hypotheses do not involve any property contrary to the real nature of the things, and consequently cannot be regarded as arbitrary, but in certain respects, agree with the conceptions which the things themselves suggest to the mind through the medium of the senses. The essential definitions of Geometry therefore being inductions from observation and experience, rest ultimately on the evidence of the senses.

It is by experience we become acquainted with the existence of individual forms of magnitudes; but by the mental process of abstraction, which begins with a particular instance, and proceeds to the general idea of all objects of the same kind, we attain to the general conception of those forms which come under the same general idea.

The essential definitions of Geometry express generalized conceptions of real existences in their most perfect ideal forms: the laws and appearances of nature, and the operations of the human intellect being supposed uniform and consistent.

But in cases where the subject falls under the class of simple ideas, the terms of the definitions so called, are no more than merely equivalent expressions. The simple idea described by a proper term or terms, does not in fact admit of definition properly so called. The definitions in Euclid's Elements may be divided into two classes, those which merely explain the meaning of the terms employed, and those, which, besides explaining the meaning of the terms, suppose the existence of the things described in the definitions.

Definitions in Geometry cannot be of such a form as to explain the nature and properties of the figures defined: it is sufficient that they give marks whereby the thing defined may be distinguished from every other of the same kind. It will at once be obvious, that the definitions of Geometry, one of the pure sciences, being abstractions of space, are not like the definitions in any one of the physical sciences. The discovery of any new physical facts may render necessary some alteration or modification in the definitions of the latter.

Def. I. Simson has adopted Theon's definition of a point. Euclid's definition is, σημεῖον ἐστιν οὗ μέρος οὐδέν, "A point is that, of which there is no part," or which cannot be parted or divided, as it is explained by Proclus. The Greek term σημεῖον, literally means, a visible *sign* or mark on a surface, in other words, *a physical point*. The English term means the sharp end of any thing, or a mark made by it. The *point* comes from the Latin *punctum*, through the French word Neither of these terms, in its literal sense, appears to give a very notion of what is to be understood by a point in Geometry. Eu definition of a point merely expresses a negative property, which exc the proper and literal meaning of the Greek term, as applied to den physical point, or a mark which is visible to the senses.

Pythagoras defined a point to be μονὰς θέσιν ἔχουσα, "a monad h position." By uniting the positive idea of position, with the neg idea of defect of magnitude, the conception of a point in Geometry

be rendered perhaps more intelligible. A point is defined to be that which has no magnitude, but position only.

Def. II. Every visible line has both length and breadth, and it is impossible to draw any line whatever which shall have no breadth. The definition requires the conception of the length only of the line to be considered, abstracted from, and independently of, all idea of its breadth.

Def. III. This definition renders more intelligible the exact meaning of the definition of a point: and we may add, that, in the Elements, Euclid supposes that the intersection of two lines is a point, and that two lines can intersect each other in one point only.

Def. IV. The straight line or right line is a term so clear and intelligible as to be incapable of becoming more so by formal definition. Euclid's definition is Εὐθεῖα γραμμή ἐστιν, ἥτις ἐξ ἴσου τοῖς ἐφ' ἑαυτῆς σημείοις κεῖται, wherein he states it to lie *evenly*, or *equally*, or *upon an equality* (ἐξ ἴσου) between its extremities, and which Proclus explains as being stretched between its extremities, ἡ ἐπ' ἄκρων τεταμένη.

If the line be conceived to be drawn on a plane surface, the words ἐξ ἴσου may mean, that no part of the line which is called a straight line deviates either from one side or the other of the direction which is fixed by the extremities of the line; and thus it may be distinguished from a curved line, which does not lie, in this sense, evenly between its extreme points. If the line be conceived to be drawn in space, the words ἐξ ἴσου, must be understood to apply to every direction on every side of the line between its extremities.

Every straight line situated in a plane, is considered to have two sides; and when the direction of a line is known, the line is said to be given in position; also, when the length is known or can be found, it is said to be given in magnitude.

From the definition of a straight line, it follows, that two points fix a straight line in position, which is the foundation of the first and second postulates. Hence straight lines which are proved to coincide in two or more points, are called, "one and the same straight line," Prop. 14, Book I, or, which is the same thing, that "two straight lines cannot have a common segment," as Simson shews in his Corollary to Prop. 11, Book I.

The following definition of straight lines has also been proposed. "Straight lines are those which, if they coincide in any two points, coincide as far as they are produced." But this is rather a criterion of straight lines, and analogous to the eleventh axiom, which states that, "all right angles are equal to one another," and suggests that all straight lines may be made to coincide wholly, if the lines be equal; or partially, if the lines be of unequal lengths. A definition should properly be restricted to the description of the thing defined, as it exists, independently of any comparison of its properties or of tacitly assuming the existence of axioms.

Def. VII. Euclid's definition of a plane surface is Ἐπίπεδος ἐπιφάνειά ἐστιν ἥτις ἐξ ἴσου ταῖς ἐφ' ἑαυτῆς εὐθείαις κεῖται, "A plane surface is that which lies evenly or equally with the straight lines in it;" instead of which Simson has given the definition which was originally proposed by Hero the Elder. A plane superficies may be supposed to be situated in any position, and to be continued in every direction to any extent.

Def. VIII. Simson remarks that this definition seems to include the angles formed by two curved lines, or a curve and a straight line, as well as that formed by two straight lines.

Angles made by straight lines only, are treated of in Elementary Geometry.

ON THE AXIOMS.

Axioms are usually defined to be self-evident truths, which cannot be rendered more evident by demonstration; in other words, the axioms of Geometry are theorems, the truth of which is admitted without proof. It is by experience we first become acquainted with the different forms of geometrical magnitudes, and the axioms, or the fundamental ideas of their equality or inequality appear to rest on the same basis. The conception of the truth of the axioms does not appear to be more removed from experience than the conception of the definitions.

These axioms, or first principles of demonstration, are such theorems as cannot be resolved into simpler theorems, and no theorem ought to be admitted as a first principle of reasoning which is capable of being demonstrated. An axiom, and (when it is convertible) its converse, should both be of such a nature as that neither of them should require a formal demonstration.

The first and most simple idea, derived from experience is, that every magnitude fills a certain space, and that several magnitudes may successively fill the same space.

All the knowledge we have of magnitude is purely relative, and the most simple relations are those of equality and inequality. In the comparison of magnitudes, some are considered as given or known, and the unknown are compared with the known, and conclusions are synthetically deduced with respect to the equality or inequality of the magnitudes under consideration. In this manner we form our idea of equality, which is thus formally stated in the eighth axiom: "Magnitudes which coincide with one another, that is, which exactly fill the same space, are equal to one another."

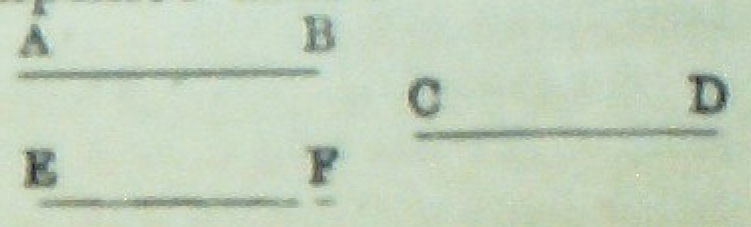
Every specific definition is referred to this universal principle. With regard to a few more general definitions which do not furnish an equality, it will be found that some hypothesis is always made reducing them to that principle, before any theory is built upon them. As for example, the definition of a straight line is to be referred to the tenth axiom; the definition of a right angle to the eleventh axiom; and the definition of parallel straight lines to the twelfth axiom.

The eighth axiom is called the principle of superposition, or, the mental process by which one Geometrical magnitude may be conceived to be placed on another, so as exactly to coincide with it, in the parts which are made the subject of comparison. Thus, if one straight line be conceived to be placed upon another, so that their extremities are coincident, the two straight lines are equal. If the directions of two lines which include one angle, coincide with the directions of the two lines which contain another angle, where the points, from which the angles diverge, coincide, then the two angles are equal: the lengths of the lines not affecting in any way the magnitudes of the angles. When one plane figure is conceived to be placed upon another, so that the boundaries of one exactly coincide with the boundaries of the other, then the two plane figures are equal. It may also be remarked, that the converse of this proposition is not universally true, namely, that when two magnitudes are equal, they coincide with one another: since two magnitudes may be equal in area, as two parallelograms or two triangles, Euc. I. 35, 37; but their boundaries may not be equal: and, consequently, by superposition, the figures could not exactly coincide: all such figures, however, having equal areas, by a different arrangement of their parts, may be made to coincide exactly.

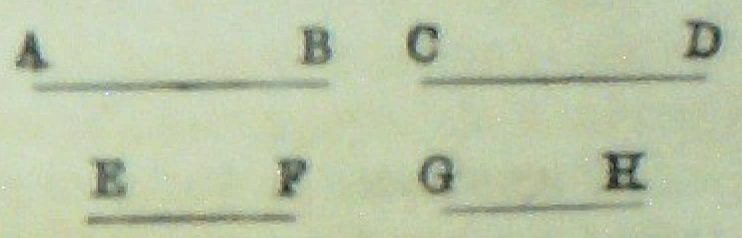
This axiom is the criterion of Geometrical equality, and is essentially different from the criterion of Arithmetical equality. Two geometrical magnitudes are equal, when they coincide or may be made to coincide: two abstract numbers are equal, when they contain the same aggregate of units; and two concrete numbers are equal, when they contain the same number of units of the same kind of magnitude. It is at once obvious, that Arithmetical representations of Geometrical magnitudes are not admissible in Euclid's criterion of Geometrical Equality, as he has not fixed the unit of magnitude of either the straight line, the angle, or the superficies. Perhaps Euclid intended that the first seven axioms should be applicable to numbers as well as to Geometrical magnitudes, and this is in accordance with the words of Proclus, who calls the axioms, *common notions*, not peculiar to the subject of Geometry.

Several of the axioms may be generally exemplified thus:

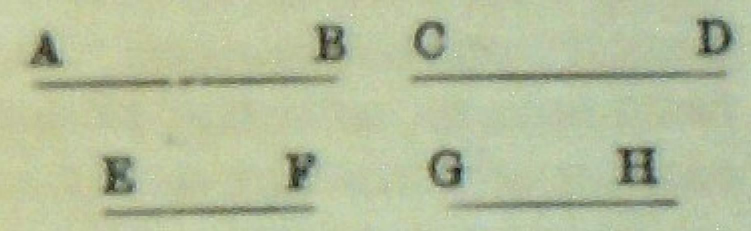
Axiom I. If the straight line AB be equal to the straight line CD; and if the straight line EF be also equal to the straight line CD; then the straight line AB is equal to the straight line EF.



Axiom II. If the line AB be equal to the line CD; and if the line EF be also equal to the line GH: then the sum of the lines AB and EF is equal to the sum of the lines CD and GH.

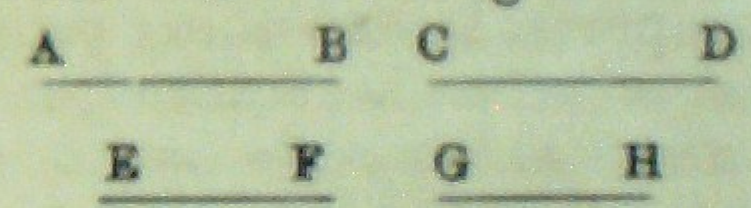


Axiom III. If the line AB be equal to the line CD; and if the line EF be also equal to the line GH; then the difference of AB and EF, is equal to the difference of CD and GH.

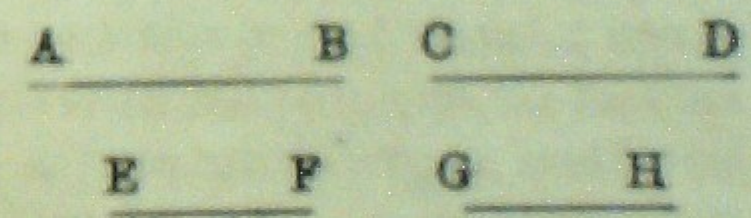


Axiom IV. admits of being exemplified under the two following forms:

1. If the line AB be equal to the line CD; and if the line EF *be greater than* the line GH; then the sum of the lines AB and EF *is greater than* the sum of the lines CD and GH.

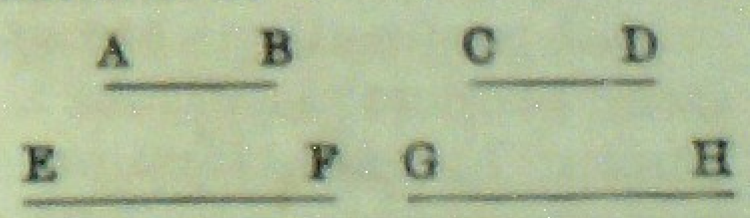


2. If the line AB be equal to the line CD; and if the line EF *be less than* the line GH; then the sum of the lines AB and EF *is less than* the sum of the lines CD and GH.

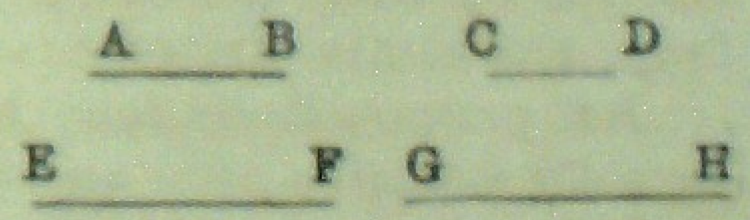


Axiom V. also admits of two forms of exemplification.

1. If the line AB be equal to the line CD; and if the line EF *be greater than* the line GH; then the difference of the lines AB and EF *is greater than* the difference of CD and GH.



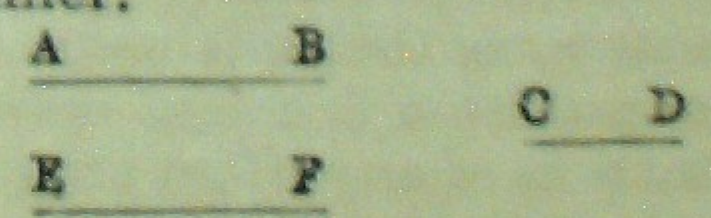
2. If the line AB be equal to the line CD; and if the line EF *be less than* the line GH; then the difference of the lines AB and EF *is less than* the difference of the lines CD and GH.



The axiom, "If unequals be taken from equals, the remainders are unequal," may be exemplified in the same manner.

Axiom VI. If the line AB be double of the line CD; and if the line EF be also double of the line CD;

then the line AB is equal to the line EF.



Axiom VII. If the line AB be the half of the line CD; and if the line EF be also the half of the line CD;

then the line AB is equal to the line EF.

It may be observed that when equal magnitudes are tak
equal magnitudes, the greater remainder exceeds the less re
as much as the greater of the unequal magnitudes exceeds the

If unequals be taken from unequals, the remainders are n
unequal; they may be equal: also if unequals be added to une
wholes are not always unequal, they may also be equal.

Axiom IX. The whole is greater than its part, and conver
part is less than the whole. This axiom appears to assert the
of the eighth axiom, namely, that two magnitudes, of whic
greater than the other, cannot be made to coincide with one ano

Axiom X. The property of straight lines expressed by t
axiom, namely, "that two straight lines cannot enclose a space,
viously implied in the definition of straight lines; for if they en
space, they could not coincide between their extreme points, w
two lines are equal.

Axiom XI. This axiom has been asserted to be a demonstra
rem. As an angle is a species of magnitude, this axiom is only
cular application of the eighth axiom to right angles.

Axiom XII. See the notes on Prop. XXIX. Book I.

ON THE PROPOSITIONS.

Whenever a judgment is formally expressed, there must b
thing respecting which the judgment is expressed, and someth
which constitutes the judgment. The former is called the *subje*
proposition, and the latter, the *predicate*, which may be anything
can be affirmed or denied respecting the *subject*.

The propositions in Euclid's Elements of Geometry may be
into two classes, *problems* and *theorems*. A proposition, as th
imports, is something proposed; it is a *problem*, when some Geo
construction is required to be *effected:* and it is a *theorem* when so
metrical *property* is to be *demonstrated*. Every proposition i
rally divided into two parts; a problem consists of the *data*, o
given; and the *quæsita*, or *things required:* a theorem, consists
subject or *hypothesis*, and the *conclusion*, or *predicate*. Hence the dis
between a problem and a theorem is this, that a problem consist
data and the quæsita, and requires solution: and a theorem co
the hypothesis and the predicate, and requires demonstration.

All propositions are *affirmative* or *negative;* that is, they eithe
some property, as Euc. I. 4, or deny the existence of some pro
Euc. I. 7; and every proposition which is affirmatively stated h
tradictory corresponding proposition. If the affirmative be pro
true, the contradictory is false.

All propositions may be viewed as (1) *universally affirmative,*
versally negative; (2) as *particularly affirmative*, or *particularly ne*

The connected course of reasoning by which any Geometrica
established is called a demonstration It is called a *direct* dem
when the predicate of the proposition is inferred directly
misses, as the conclusion of a series of successive deduc
monstration is called *indirect*, when the conclusion shows
duction of any other supposition contrary to the hypothesi
proposition, necessarily leads to an absurdity.

It has been remarked by Pascal, that "Geometry is almo
subject as to which we find truths wherein all men agree; and
of this is, that Geometers alone regard the true laws of demonst

These are enumerated by him as eight in number. "1. To define nothing which cannot be expressed in clearer terms than those in which it is already expressed. 2. To leave no obscure or equivocal terms undefined. 3. To employ in the definition no terms not already known. 4. To omit nothing in the principles from which we argue, unless we are sure it is granted. 5. To lay down no axiom which is not perfectly evident. 6. To demonstrate nothing which is as clear already as we can make it. 7. To prove every thing in the least doubtful by means of self-evident axioms, or of propositions already demonstrated. 8. To substitute mentally the definition instead of the thing defined." Of these rules, he says, "the first, fourth and sixth are not absolutely necessary to avoid error, but the other five are indispensable; and though they may be found in books of logic, none but the Geometers have paid any regard to them."

The course pursued in the demonstrations of the propositions in Euclid's Elements of Geometry, is always to refer directly to some expressed principle, to leave nothing to be inferred from vague expressions, and to make every step of the demonstrations the object of the understanding.

It has been maintained by some philosophers, that a genuine definition contains some property or properties which can form a basis for demonstration, and that the science of Geometry is deduced from the definitions, and that on them alone the demonstrations depend. Others have maintained that a definition explains only the meaning of a term, and does not embrace the nature and properties of the thing defined.

If the propositions usually called postulates and axioms are either tacitly assumed or expressly stated in the definitions; in this view, demonstrations may be said to be legitimately founded on definitions. If, on the other hand, a definition is simply an explanation of the meaning of a term, whether abstract or concrete, by such marks as may prevent a misconception of the thing defined; it will be at once obvious that some constructive and theoretic principles must be assumed, besides the definitions to form the ground of legitimate demonstration. These principles we conceive to be the postulates and axioms. The postulates describe constructions which may be admitted as possible by direct appeal to our experience; and the axioms assert general theoretic truths so simple and self-evident as to require no proof, but to be admitted as the assumed first principles of demonstration. Under this view all Geometrical reasonings proceed upon the admission of the hypotheses assumed in the definitions, and the unquestioned possibility of the postulates, and the truth of the axioms.

Deductive reasoning is generally delivered in the form of an enthymeme, or an argument wherein one enunciation is not expressed, but is readily supplied by the reader: and it may be observed, that although this is the ordinary mode of speaking and writing, it is not in the strictly syllogistic form; as either the *major* or the *minor* premiss only is formally stated before the conclusion: Thus in Euc. I. 1.

Because the point A is the center of the circle BCD;
therefore the straight line AB is equal to the straight line AC.

The premiss here omitted, is: all straight lines drawn from the center of a circle to the circumference are equal.

In a similar way may be supplied the reserved premiss in every enthymeme. The conclusion of two enthymemes may form the major and minor premiss of a third syllogism, and so on, and thus any process of reasoning is reduced to the strictly syllogistic form. And in this way it is shewn that the general theorems of Geometry are demonstrated by means syllogisms founded on the axioms and definitions.

Every syllogism consists of three propositions, of which, two are cal the premisses, and the third, the conclusion. These propositions cont three terms, the subject and predicate of the conclusion, and the midc term which connects the predicate and the conclusion together. T subject of the conclusion is called *the minor*, and the predicate of the cc clusion is called *the major* term, of the syllogism. The major term appe in one premiss, and the minor term in the other, with the middle te which is in both premisses. That premiss which contains the mi term and the major term, is called the *major premiss*; and that w contains the middle term and the minor term, is called the *minor pre* of the syllogism. As an example, we may take the syllogism in the dem stration of Prop. 1, Book 1, wherein it will be seen that the middle terr the subject of the major premiss and the predicate of the minor.

Major premiss: because the straight line AB is equal to the straight line
Minor premiss: and, because the straight line BC is equal to the strai line AB;
Conclusion: therefore the straight line BC is equal to the straight line

Here, BC is the subject, and AC the predicate of the conclusion.
BC is the subject, and AB the predicate of the minor premi
AB is the subject, and AC the predicate of the major premi

Also, AC is the major term, BC the minor term, and AB the middle t of the syllogism.

In this syllogism, it may be remarked that the definition of a stra line is assumed, and the definition of the Geometrical equality of straight lines; also that a general theoretic truth, or axiom, forms ground of the conclusion. And further, though it be impossible to m any point, mark or sign (σημεῖον) which has not both length and brea and any line which has not both length and breadth; the demonstrat in Geometry do not on this account become invalid. For they are pur on the hypothesis that the point has no parts, but position only: and line has length only, but no breadth or thickness: also that the su has length and breadth only, but no thickness: and all the conclu at which we arrive are independent of every other consideration.

The truth of the conclusion in the syllogism depends upon the of the premisses. If the premisses, or only one of them be not tru conclusion is false. The conclusion is said *to follow from* the prem whereas, in truth, it is *contained in* the premisses. The expression be understood of the mind apprehending in succession, the t the premisses, and subsequent to that, the truth of the co so that the conclusion *follows from* the premisses in order as far as reference is made to the mind's apprehension of the argument.

Every proposition, when complete, may be divided into six par Proclus has pointed out in his commentary.

1. *The proposition*, or *general enunciation*, which states in general the conditions of the problem or theorem.

2. *The exposition*, or *particular enunciation*, which exhibits the of the proposition in particular terms as a fact, and refers it t diagram described.

3. *The determination* contains the *predicate* in particular ter is pointed out in the diagram, and directs attention to the demon by pronouncing the thing sought.

4. *The construction* applies the postulates to prepare the diagram for the demonstration.

5. *The demonstration* is the connexion of syllogisms, which prove the truth or falsehood of the theorem, the possibility or impossibility of the problem, in that particular case exhibited in the diagram.

6. *The conclusion* is merely the repetition of the general enunciation, wherein the predicate is asserted as a demonstrated truth.

Prop. I. In the first two Books, the circle is employed as a mechanical instrument, in the same manner as the straight line, and the use made of it rests entirely on the third postulate. No properties of the circle are discussed in these books beyond the definition and the third postulate. When two circles are described, one of which has its center in the circumference of the other, the two circles being each of them partly within and partly without the other, their circumferences must intersect each other in two points; and it is obvious from the two circles cutting each other, in two points, one on each side of the given line, that two equilateral triangles may be formed on the given line.

Prop. II. When the given point is neither in the line, nor in the line produced, this problem admits of eight different lines being drawn from the given point in different directions, every one of which is a solution of the problem. For, 1. The given line has two extremities, to each of which a line may be drawn from the given point. 2. The equilateral triangle may be described on either side of this line. 3. And the side BD of the equilateral triangle ABD may be produced either way.

But when the given point lies either in the line or in the line produced, the distinction which arises from joining the two ends of the line with the given point, no longer exists, and there are only four cases of the problem.

The construction of this problem assumes a neater form, by first describing the circle CGH with center B and radius BC, and producing DB the side of the equilateral triangle DBA to meet the circumference in G: next, with center D and radius DG, describing the circle GKL, and then producing DA to meet the circumference in L.

By a similar construction the less of two given straight lines may be produced, so that the less together with the part produced may be equal to the greater.

Prop. III. This problem admits of two solutions, and it is left undetermined from which end of the greater line the part is to be cut off.

By means of this problem, a straight line may be found equal to the sum or the difference of two given lines.

Prop. IV. This forms the first case of equal triangles, two other cases are proved in Prop. VIII. and Prop. XXVI.

The term *base* is obviously taken from the idea of a building, and the same may be said of the term *altitude*. In Geometry, however, these terms are not restricted to one particular position of a figure, as in the case of a building, but may be in any position whatever.

Prop. V. Proclus has given, in his commentary, a proof for the equality of the angles at the base, without producing the equal sides. The construction follows the same order, taking in AB one side of the isosceles triangle ABC, a point D and cutting off from AC a part AE equal to AD, and then joining CD and BE.

A corollary is a theorem which results from the demonstration of a proposition.

Prop. VI. is the converse of one part of Prop. V. One proposition is defined to be the *converse* of another when the hypothesis former becomes the predicate of the latter; and vice versa.

There is besides this, another kind of conversion, when a th has several hypotheses and one predicate; by assuming the pre and one, or more than one of the hypotheses, some one of the hyp may be inferred as the predicate of the converse. In this n Prop. VIII. is the converse of Prop. IV. It may here be ob that converse theorems are not universally true: as for instanc following direct proposition is universally true; "If two triangle their three sides respectively equal, the three angles of each sh respectively equal." But the converse is not universally true; na "If two triangles have the three angles in each respectively the three sides are respectively equal." Converse theorems in some instances, the consideration of other conditions than which enter into the proof of the direct theorem. *Converse* and *con* propositions are by no means to be confounded; the *contrary* propo denies what is asserted, or asserts what is denied, in the *direct* position, but the subject and predicate in each are the same. A *co* *proposition* is a *completely contradictory proposition*, and the distin consists in this—that *two contrary propositions* may both be false of *two contradictory propositions*, one of them must be true, an other false. It may here be remarked, that one of the most com intellectual mistakes of learners, is to imagine that the denial proposition is a legitimate ground for affirming the contrary as whereas the rules of sound reasoning allow that the affirmation proposition as true, only affords a ground for the denial of the con as false.

Prop. VI. is the first instance of indirect demonstrations, and are more suited for the proof of converse propositions. All those positions which are demonstrated ex absurdo, are properly anal demonstrations, according to the Greek notion of analysis, which supposed the thing required, to be done, or to be true, and then sh the consistency or inconsistency of this construction or hyp with truths admitted or already demonstrated.

In indirect demonstrations, where hypotheses are made which not true and contrary to the truth stated in the proposition, it s desirable that a form of expression should be employed different that in which the hypotheses are true. In all cases therefore, wh noted by Euclid or not, the words *if possible* have been introd or some such qualifying expression, as in Euc. I. 6, so as not to upon the mind of the learner, the impression that the hypot which contradicts the proposition, is really true.

Prop. VIII. When the three sides of one triangle are shew coincide with the three sides of any other, the equality of the tri is at once obvious. This, however, is not stated at the conclus Prop. VIII. or of Prop. XXVI. For the equality of the areas coincident triangles, reference is always made by Euclid to Prop

A direct demonstration may be given of this proposition, a VII. may be dispensed with altogether.

Let the triangles *ABC*, *DEF* be so placed that the bas coincide with the base *EF*, and the vertices *A*, *D* may be c sides of *EF*. Join *AD*. Then because *EAD* is an isosceles the angle *EAD* is equal to the angle *EDA*; and because *CD* isosceles triangle, the angle *CAD* is equal to the angle *CDA*.

the angle *EAF* is equal to the angle *EDF*, (ax. 2 or 3): or the angle *BDC* is equal to the angle *EDF*.

Prop. IX. If *BA*, *AC* be in the same straight line. This problem then becomes the same as Prob. XI, which may be regarded as drawing a line which bisects an angle equal to two right angles.

If *FA* be produced in the fig. Prop. 9, it bisects the angle which is the defect of the angle *BAC from four right angles*.

By means of this problem, any angle may be divided into four, eight, sixteen, &c. equal angles.

Prop. X. A finite straight line may, by this problem, be divided into four, eight, sixteen, &c. equal parts

Prop. XI. When the point is at the extremity of the line; by the second postulate the line may be produced, and then the construction applies. See note on Euc. III. 31.

The distance between two points is the straight line which joins the points; but the distance between a point and a straight line, is the *shortest line* which can be drawn from the point to the line.

From this Prop. it follows that only one perpendicular can be drawn from a given point to a given line; and this perpendicular may be shewn to be less than any other line which can be drawn from the given point to the given line: and of the rest, the line which is nearer to the perpendicular is less than one more remote from it: also only two equal straight lines can be drawn from the same point to the line, one on each side of the perpendicular or the least. This property is analogous to Euc. III. 7, 8.

The corollary to this proposition is not in the Greek text, but was added by Simson, who states that it "is necessary to Prop. I, Book XI., and otherwise."

Prop. XII. The third postulate requires that the line *CD* should be drawn before the circle can be described with the center *C*, and radius *CD*.

Prop. XIV. is the converse of Prop. XIII. "Upon the opposite sides of it." If these words were omitted, it is possible for two lines to make with a third, two angles, which together are equal to two right angles, in such a manner that the two lines shall not be in the same straight line.

The line *BE* may be supposed to fall above, as in Euclid's figure, or below the line *BD*, and the demonstration is the same in form.

Prop. XV. is the development of the definition of an angle. If the lines at the angular point be produced, the produced lines have the same inclination to one another as the original lines, but in a different position.

The converse of this Proposition is not proved by Euclid, namely:— If the vertical angles made by four straight lines at the same point be respectively equal to each other, each pair of opposite lines shall be in the same straight line.

Prop. XVII. appears to be only a corollary to the preceding proposition, and it seems to be introduced to explain Axiom XII, of which it is the converse The exact truth respecting the angles of a triangle is proved in Prop. XXXII.

Prop. XVIII. It may here be remarked, for the purpose of guarding the student against a very common mistake, that in this proposition the *hypothesis* is stated before the *predicate*, and in the converse of it, the *predicate* is stated before the *hypothesis*. It may be remarked,

Prop. XIX. is the converse of Prop. XVIII. It may be remarked, that Prop. XIX. bears the same relation to Prop. XVIII., as Prop. VI does to Prop. V.

Prop. xx. The following corollary arises from this proposition:

A straight line is the shortest distance between two points. For the straight line *BC* is always less than *BA* and *AC*, however near the point *A* may be to the line *BC*.

It may be easily shewn from this proposition, that the difference of any two sides of a triangle is less than the third side.

Prop. xxii. When the sum of two of the lines is equal to, and when it is less than, the third line; let the diagrams be described, and they will exhibit the impossibility implied by the restriction laid down in the Proposition.

The same remark may be made here, as was made under the first Proposition, namely:—if one circle lies partly within and partly without another circle, the circumferences of the circles intersect each other in two points.

Prop. xxiii. *CD* might be taken equal to *CE*, and the construction effected by means of an isosceles triangle. It would, however, be less general than Euclid's, but is more convenient in practice.

Prop. xxiv. Simson makes the angle *EDG* at *D* in the line *ED*, the side which is not the greater of the two *ED*, *DF*; otherwise, three different cases would arise, as may be seen by forming the different figures. The point *G* might fall below or upon the base *EF* produced as well as above it. Prop. xxiv. and Prop. xxv. bear to each other the same relation as Prop. iv. and Prop. viii.

Prop. xxvi. This forms the third case of the equality of two triangles. Every triangle has three sides and three angles, and when any three of one triangle are given equal to any three of another, the triangles may be proved to be equal to one another, whenever the three magnitudes given in the hypothesis are independent of one another. Prop. iv. contains the first case, when the hypothesis consists of two sides and the included angle of each triangle. Prop. viii. contains the second, when the hypothesis consists of the three sides of each triangle. Prop. xxvi. contains the third, when the hypothesis consists of two angles, and one side either adjacent to the equal angles, or opposite to one of the equal angles in each triangle. There is another case, not proved by Euclid, when the hypothesis consists of two sides and one angle in each triangle, but these not the angles included by the two given sides in each triangle. This case however is only true under a certain restriction, thus:

If two triangles have two sides of one of them equal to two sides of the other, each to each, and have also the angles opposite to one of the equal sides in each triangle, equal to one another, and if the angles opposite to the other equal sides be both acute, or both obtuse angles; then shall the third sides be equal in each triangle, as also the remaining angles of the one to the remaining angles of the other.

Let *ABC*, *DEF* be two triangles which have the sides *AB*, *AC* equal to the two sides *DE*, *DF*, each to each, and the angle *ABC* equal to the angle *DEF*: then, if the angles *ACB*, *DEF*, be *both acute*, or *both obtuse angles*, the third side *BC* shall be equal to the third side *EF*, and also the angle *BCA* to the angle *EFD*, and the angle *BAC* to the angle *EDF*.

First. Let the angles *ACB*, *DFE* opposite to the equal sides *AB*, *DE*, be *both acute angles*.

If *BC* be not equal to *EF*, let *BC* be the greater, and from *BC* cut off *BG* equal to *EF*, and join *AG*.

Then in the triangles *ABG*, *DEF*, Euc. I. 4. *AG* is equal to *DF*,

and the angle *AGB* to *DFE*. But since *AC* is equal to *DF*, *AG* is equal to *AC*: and therefore the angle *ACG* is equal to the angle *AGC*, which is also an acute angle. But because *AGC*, *AGB* are together equal to two right angles, and that *AGC* is an acute angle, *AGB* must be an obtuse angle; which is absurd. Wherefore, *BC* is not unequal to *EF*, that is, *BC* is equal to *EF*, and also the remaining angles of one triangle to the remaining angles of the other.

Secondly. Let the angles *ACB*, *DFE*, be both *obtuse angles*. By proceeding in a similar way, it may be shewn that *BC* cannot be otherwise than equal to *EF*.

If *ACB*, *DFE* be *both right angles*: the case falls under Euc. I. 26.

Prop. xxvii. Alternate angles are defined to be the two angles which two straight lines make with another at its extremities, but upon opposite sides of it.

When a straight line intersects two other straight lines, two pairs of alternate angles are formed by the lines at their intersections, as in the figure, *BEF*, *EFC* are alternate angles as well as the angles *AEF*, *EFD*.

Prop. xxviii. One angle is called "the exterior angle," and another "the interior and opposite angle," when they are formed on the same side of a straight line which falls upon or intersects two other straight lines. It is also obvious that on each side of the line, there will be two exterior and two interior and opposite angles. The exterior angle *EGB* has the angle *GHD* for its corresponding interior and opposite angle: also the exterior angle *FHD* has the angle *HGB* for its interior and opposite angle.

Prop. xxix is the converse of Prop. xxvii and Prop. xxviii.

As the definition of parallel straight lines simply describes them by a statement of the negative property, that they never meet; it is necessary that some positive property of parallel lines should be assumed as an axiom, on which reasonings on such lines may be founded.

Euclid has assumed the statement in the twelfth axiom, which has been objected to, as not being self-evident. A stronger objection appears to be, that the converse of it forms Euc. I. 17; for both the assumed axiom and its converse, should be so obvious as not to require formal demonstration.

Simson has attempted to overcome the objection, not by any improved definition and axiom respecting parallel lines; but, by considering Euclid's twelfth axiom to be a theorem, and for its proof, assuming two definitions and one axiom, and then demonstrating five subsidiary Propositions.

Instead of Euclid's twelfth axiom, the following has been proposed as a more simple property for the foundation of reasonings on parallel lines; namely, "If a straight line fall on two parallel straight lines, the alternate angles are equal to one another." In whatever this may exceed Euclid's definition in simplicity, it is liable to a similar objection, being the converse of Euc. I. 27.

Professor Playfair has adopted in his Elements of Geometry, that "Two straight lines which intersect one another cannot be both parallel to the same straight line." This apparently more simple axiom follows as a direct inference from Euc. I. 30.

But one of the least objectionable of all the definitions which have been proposed on this subject, appears to be that which simply expresses the conception of equidistance. It may be formally stated thus: "Parallel lines are such as lie in the same plane, and which neither recede from, nor approach to, each other." This includes the conception stated by Euclid, that parallel lines never meet. Dr. Wallis observes on this subject, "Parallelismus et æquidistantia vel idem sunt, vel certe se mutuo comitantur."

As an additional reason for this definition being preferred, it may be remarked that the meaning of the terms γρυμμαὶ παράλληλοι, suggests the exact idea of such lines.

An account of thirty methods which have been proposed at different times for avoiding the difficulty in the twelfth axiom, will be found in the appendix to Colonel Thompson's "Geometry without Axioms."

Prop. xxx. In the diagram, the two lines *AB* and *CD* are placed one on each side of the line *EF*: the proposition may also be proved when both *AB* and *CD* are on the same side of *EF*.

Prop. xxxii. From this proposition, it is obvious that if one angle of a triangle be equal to the sum of the other two angles, that angle is a right angle, as is shewn in Euc. iii. 31, and that each of the angles of an equilateral triangle, is equal to two thirds of a right angle, as it is shewn in Euc. iv. 15. Also, if one angle of an isosceles triangle be a right angle, then each of the equal angles is half a right angle, as in Euc. ii. 9.

The three angles of a triangle may be shewn to be equal to two right angles without producing a side of the triangle, by drawing through any angle of the triangle a line parallel to the opposite side, as Proclus has remarked in his Commentary on this proposition. It is manifest from this proposition, that the third angle of a triangle is not independent of the sum of the other two; but is known if the sum of any two is known. Cor. 1 may be also proved by drawing lines from any one of the angles of the figure to the other angles. If any of the sides of the figure bend inwards and form what are called re-entering angles, the enunciation of these two corollaries will require some modification. As Euclid gives no definition of re-entering angles, it may fairly be concluded, he did not intend to enter into the proofs of the properties of figures which contain such angles.

Prop. xxxiii. The words "towards the same parts" are a necessary restriction: for if they were omitted, it would be doubtful whether the extremities *A*, *C*, and *B*, *D* were to be joined by the lines *AC* and *BD*; or the extremities *A*, *D*, and *B*, *C*, by the lines *AD* and *BC*.

Prop. xxxiv. If the other diameter be drawn, it may be shewn that the diameters of a parallelogram bisect each other, as well as bisect the area of the parallelogram. If the parallelogram be right angled, the diagonals are equal; if the parallelogram be a square or a rhombus, the diagonals bisect each other at right angles. The converse of this Prop., namely, "If the opposite sides or opposite angles of a quadrilateral figure be equal, the opposite sides shall also be parallel; that is, the figure shall be a parallelogram," is not proved by Euclid.

Prop. xxxv. The latter part of the demonstration is not expressed very intelligibly. Simson, who altered the demonstration, seems in fact to consider two trapeziums of the same form and magnitude, and from one of them, to take the triangle *ABE*; and from the other, the triangle *DCF*; and then the remainders are equal by the third axiom: that is, the parallelogram *ABCD* is equal to the parallelogram *EBCF*. Otherwise, the triangle, whose base is *DE*, (fig. 2.) is taken twice from the trapezium, which would appear to be impossible, if the sense in which Euclid applies the third axiom, is to be retained here.

It may be observed, that the two parallelograms exhibited in fig. 2 partially lie on one another, and that the triangle whose base is *BC* is a common part of them, but that the triangle whose base is *DE* is entirely without both the parallelograms. After having proved the triangle *ABE* equal to the triangle *DCF*, if we take from these equals (fig. 2.) the triangle whose base is *DE*, and to each of the remainders add the triangle whose base is *BC*, then the parallelogram *ABCD* is equal to the parallelogram *EBCF*. In fig. 3, the equality of the parallelograms *ABCD*, *EBCF*, is shewn by adding the figure *EBCD* to each of the triangles *ABE*, *DCF*.

In this proposition, the word *equal* assumes a new meaning, and is no longer restricted to mean coincidence in all the parts of two figures.

Prop. xxxviii. In this proposition, it is to be understood that the bases of the two triangles are in the same straight line. If in the diagram the point *E* coincide with *C*, and *D* with *A*, then the angle of one triangle is supplemental to the other. Hence the following property:—If two triangles have two sides of the one respectively equal to two sides of the other, and the contained angles supplemental, the two triangles are equal.

A distinction ought to be made between *equal* triangles and *equivalent* triangles, the former including those whose sides and angles mutually coincide, the latter those whose areas only are equivalent.

Prop. xxxix. If the vertices of all the equal triangles which can be described upon the same base, or upon the equal bases as in Prop. 40, be joined, the line thus formed will be a straight line, and is called the locus of the vertices of equal triangles upon the same base, or upon equal bases.

A locus in plane Geometry is a straight line or a plane curve, every point of which and none else satisfies a certain condition. With the exception of the straight line and the circle, the two most simple loci; all other loci, perhaps including also the Conic Sections, may be more readily and effectually investigated algebraically by means of their rectangular or polar equations.

Prop. xli. The converse of this proposition is not proved by Euclid; viz. If a parallelogram is double of a triangle, and they have the same base, or equal bases upon the same straight line, and towards the same parts, they shall be between the same parallels. Also, it may easily be shewn that if two equal triangles are between the same parallels; they are either upon the same base, or upon equal bases.

Prop. xliv. A parallelogram described on a straight line is said to be *applied* to that line.

Prop. xlv. The problem is solved only for a rectilineal figure of four sides. If the given rectilineal figure have more than four sides, it may be divided into triangles by drawing straight lines from any angle of the figure to the opposite angles, and then a parallelogram equal to the third triangle can be applied to *LM*, and having an angle equal to *E*: and so on for all the triangles of which the rectilineal figure is composed.

Prop. xlvi. The square being considered as an equilateral rectangle, its area or surface may be expressed numerically if the number of lineal units in a side of the square be given, as is shewn in the note on Prop. 1., Book ii.

The student will not fail to remark the analogy which exists between the *area of a square* and *the product of two equal numbers*; and between the *side of a square* and *the square root of a number*. There is, however, this distinction to be observed; it is always possible to find the product of two equal numbers, (or *to find the square of a number*, as it is usually called,) and to describe a square on a given line; but conversely, though the side of a given square is known from the figure itself, the exact number of units in the side of a square of given area, can only be found exactly, in such cases where the given number is a square number. For example, if the area of a square contain 9 square units, then the square root of 9 or 3, indicates the number of lineal units in the side of that square. Again, if the area of a square contain 12 square units, the side of the square is greater than 3, but less than 4 lineal units, and there is no number which will exactly express the side of that square: an approximation to the true length, however, may be obtained to any assigned degree of accuracy.

Prop. XLVII. In a right-angled triangle, the side opposite to the right angle is called the hypotenuse, and the other two sides, the base and perpendicular, according to their position.

In the diagram the three squares are described on the *outer* sides of the triangle *ABC*. The Proposition may also be demonstrated (1) when the three squares are described upon the *inner* sides of the triangle: (2) when one square is described on the outer side and the other two squares on the inner sides of the triangle: (3) when one square is described on the inner side and the other two squares on the outer sides of the triangle.

As one instance of the third case. If the square *BE* on the hypotenuse be described on the inner side of *BC* and the squares *BG*, *HC* on the outer sides of *AB*, *AC*; the point *D* falls on the side *FG* (Euclid's fig.) of the square *BG*, and *KH* produced meets *CE* in *E*. Let *LA* meet *BC* in *M*. Join *DA*; then the square *GB* and the oblong *LB* are each double of the triangle *DAB*, (Euc. I. 41.); and similarly by joining *EA*, the square *HC* and oblong *LC* are each double of the triangle *EAC*. Whence it follows that the squares on the sides *AB*, *AC* are together equal to the square on the hypotenuse *BC*.

By this proposition may be found a square equal to the sum of any given squares, or equal to any multiple of a given square: or equal to the difference of two given squares.

The truth of this proposition may be exhibited to the eye in some particular instances. As in the case of that right-angled triangle whose three sides are 3, 4, and 5 units respectively. If through the points of division of two contiguous sides of each of the squares upon the sides, lines be drawn parallel to the sides (see the notes on Book II.), it will be obvious, that the squares will be divided into 9, 16 and 25 small squares, each of the same magnitude; and that the number of the small squares into which the squares on the perpendicular and base are divided is equal to the number into which the square on the hypotenuse is divided.

Prop. XLVIII is the converse of Prop. XLVII. In this Prop. is assumed the Corollary that "the squares described upon two equal lines are equal," and the converse, which properly ought to have been appended to Prop. XLVI.

The First Book of Euclid's Elements, it has been seen, is conversant with the construction and properties of rectilineal figures. It first lays down the definitions which limit the subjects of discussion in the First Book, next the three postulates, which restrict the instruments by which the constructions in Plane Geometry are effected; and thirdly, the twelve axioms, which express the principles by which a comparison is made between the ideas of the things defined.

This Book may be divided into three parts. The first part treats of the origin and properties of triangles, both with respect to their sides and angles; and the comparison of these mutually, both with regard to equality and inequality. The second part treats of the properties of parallel lines and of parallelograms. The third part exhibits the connection of the properties of triangles and parallelograms, and the equality of the squares on the base and perpendicular of a right-angled triangle to the square on the hypotenuse.

When the propositions of the First Book have been read with the notes, the student is recommended to use different letters in the diagrams, and where it is possible, diagrams of a form somewhat different from those exhibited in the text, for the purpose of testing the accuracy of his knowledge of the demonstrations. And further, when he has become sufficiently familiar with the method of geometrical reasoning, he may dispense with the aid of letters altogether, and acquire the power of expressing in general terms the process of reasoning in the demonstration of any proposition. Also, he is advised to answer the following questions before he attempts to apply the principles of the First Book to the solution of Problems and the demonstration of Theorems.

QUESTIONS ON BOOK I.

1. What is the name of the Science of which Euclid gives the Elements? What is meant by *Solid Geometry?* Is there any distinction between *Plane Geometry*, and the *Geometry of Planes?*

2. Define the term *magnitude*, and specify the different kinds of magnitude considered in Geometry. What dimensions of space belong to figures treated of in the first six Books of Euclid?

3. Give Euclid's definition of a "straight line." What does he really use as his test of rectilinearity, and where does he first employ it? What objections have been made to it, and what substitute has been proposed as an available definition? How many points are necessary to fix the position of a straight line in a plane? When is one straight line said to *cut*, and when to *meet* another?

4. What positive property has a Geometrical point? From the definition of a straight line, shew that the intersection of two lines is a point.

5. Give Euclid's definition of a plane rectilineal angle. What are the limits of the angles considered in Geometry? Does Euclid consider angles greater than two right angles?

6. When is a straight line said to be drawn at *right angles*, and when *perpendicular*, to a given straight line?

7. Define a *triangle*; shew how many kinds of triangles there are according to the variation both of the *angles*, and of the *sides*.

8. What is Euclid's definition of a circle? Point out the assumption involved in your definition. Is any axiom implied in it? Shew that in this as in all other definitions, some geometrical fact is assumed as somehow previously known.

9. Define the quadrilateral figures mentioned by Euclid.

10. Describe briefly the use and foundation of definitions, axioms, and postulates: give illustrations by an instance of each.

11. What objection may be made to the method and order in which Euclid has laid down the elementary abstractions of the Science of Geometry? What other method has been suggested?

12. What distinctions may be made between definitions in the Science of Geometry and in the Physical Sciences?

13. What is necessary to constitute an exact definition? Are definitions propositions? Are they arbitrary? Are they convertible? Does a Mathematical definition admit of proof on the principles of the Science to which it relates?

14. Enumerate the principles of construction assumed by Euclid.

15. Of what instruments may the use be considered to meet approximately the demands of Euclid's postulates? Why only *approximately?*

16. "A circle may be described from any center, with any straight line as radius." How does this postulate differ from Euclid's, and which of his problems is assumed in it?

17. What principles in the Physical Sciences correspond to axioms in Geometry?

18. Enumerate Euclid's twelve axioms and point out those which have special reference to Geometry. State the converse of those which admit of being so expressed.

19. What two tests of equality are assumed by Euclid? Is the assumption of the principle of superposition (ax. 8.), essential to all Geometrical reasoning? Is it correct to say, that it is "an appeal, though of the most familiar sort, to external observation"?

20. Could any, and if any, which of the axioms of Euclid be turned into definitions; and with what advantages or disadvantages?

21. Define the terms, Problem, Postulate, Axiom and Theorem. Are any of Euclid's axioms improperly so called?

22. Of what two parts does the enunciation of a Problem, and of a Theorem consist? Distinguish them in Euc. I. 4, 5, 18, 19.

23. When is a problem said to be indeterminate? Give an example.

24. When is one proposition said to be the converse or reciprocal of another? Give examples. Are converse propositions universally true? If not, under what circumstances are they necessarily true? Why is it necessary to demonstrate converse propositions? How are they proved?

25. Explain the meaning of the word *proposition.* Distinguish between *converse* and *contrary* propositions, and give examples.

26. State the grounds as to whether Geometrical reasonings depend for their conclusiveness upon axioms or definitions.

27. Explain the meaning of *enthymeme* and *syllogism.* How is the enthymeme made to assume the form of the syllogism? Give examples.

28. What constitutes a demonstration? State the laws of demonstration.

29. What are the principal parts, in the entire process of establishing a proposition?

30. Distinguish between a *direct* and *indirect* demonstration.

31. What is meant by the term *synthesis,* and what, by the term, *analysis?* Which of these modes of reasoning does Euclid adopt in his Elements of Geometry?

32. In what sense is it true that the conclusions of Geometry are necessary truths?

33. Enunciate those Geometrical definitions which are used in the proof of the propositions of the First Book.

34. If in Euclid I. 1, an equal triangle be described on the other side of the given line, what figure will the two triangles form?

35. In the diagram, Euclid I. 2, if DB a side of the equilateral triangle DAB be produced both ways and cut the circle whose center is B and radius BC in two points G and H; shew that either of the distances DG, DH may be taken as the radius of the second circle; and give the proof in each case.

36. Explain how the propositions Euc. I. 2, 3, are rendered necessary by the restriction imposed by the third postulate. Is it necessary for the proof, that the triangle described in Euc. I. 2, should be equilateral? Could we, at this stage of the subject, describe an isosceles triangle on a given base?

37. State how Euc. I. 2, may be extended to the following problem: "From a given point to draw a straight line *in a given direction* equal to a given straight line."

38. How would you cut off from a straight line unlimited in both directions, a length equal to a given straight line?

39. In the proof of Euclid I. 4, how much depends upon Definition, how much upon Axiom?

40. Draw the figure for the third case of Euc. I. 7, and state why it *needs no demonstration.*

41. In the construction Euclid I. 9, is it indifferent in all cases on which side of the joining line the equilateral triangle is described?

42. Shew how a given straight line may be bisected by Euc. I. 1.

43. In what cases do the lines which bisect the interior angles of plane triangles, also bisect one, or more than one of the corresponding opposite sides of the triangles?

44. "Two straight lines cannot have a common segment." Has this corollary been tacitly assumed in any preceding proposition?

45. In Euc. I. 12, must the given line necessarily be "of unlimited length"?

46. Shew that (fig. Euc. I. 11) every point without the perpendicular drawn from the middle point of every straight line DE, is at unequal distances from the extremities D, E of that line.

47. From what proposition may it be inferred that a straight line is the shortest distance between two points?

48. Enunciate the propositions you employ in the proof of Euc. I. 16.

49. Is it essential to the truth of Euc. I. 21, that the two straight lines be drawn from the extremities of the base?

50. In the diagram, Euc. I. 21, by how much does the greater angle BDC exceed the less BAC?

51. To form a triangle with three straight lines, any two of them must be greater than the third: is a similar limitation necessary with respect to the three angles?

52. Is it possible to form a triangle with three lines whose lengths are 1, 2, 3 units: or one with three lines whose lengths are $1, \sqrt{2}, \sqrt{3}$?

53. Is it possible to construct a triangle whose angles shall be as the numbers 1, 2, 3? Prove or disprove your answer.

54. What is the reason of the limitation in the construction of Euc. I. 24. viz. "that DE is that side which is not greater than the other?"

55. Quote the first proposition in which the equality of two areas which cannot be superposed on each other is considered.

56. Is the following proposition universally true? "If two plane triangles have three elements of the one respectively equal to three elements of the other, the triangles are equal in every respect." Enumerate all the cases in which this equality is proved in the First Book. What case is omitted?

57. What parts of a triangle must be given in order that the triangle may be described?

58. State the converse of the second case of Euc. I. 26? Under what limitations is it true? Prove the proposition so limited?

59. Shew that the angle contained between the perpendiculars drawn to two given straight lines which meet each other, is equal to the angle contained by the lines themselves.

60. Are two triangles necessarily equal in all respects, where a side and two angles of the one are equal to a side and two angles of the other, each to each?

61. Illustrate fully the difference between analytical and synthetical proofs. What propositions in Euclid are demonstrated analytically?

62. Can it be properly predicated of any two straight lines that they never meet if indefinitely produced either way, antecedently to our knowledge of some other property of such lines, which makes the property first predicated of them a necessary conclusion from it?

63. Enunciate Euclid's definition and axiom relating to parallel straight lines; and state in what Props. of Book I. they are used

64. What proposition is the converse to the twelfth axiom of the First Book? What other two propositions are complementary to these?

65. If lines being produced ever so far do not meet; can they be otherwise than parallel? If so, under what circumstances?

66. Define *adjacent angles*, *opposite angles*, *vertical angles*, and *alternate angles;* and give examples from the First Book of Euclid.

67. Can you suggest anything to justify the assumption in the twelfth axiom upon which the proof of Euc. I. 29, depends?

68. What objections have been urged against the definition and the doctrine of parallel straight lines as laid down by Euclid? Where does the difficulty originate? What other assumptions have been suggested and for what reasons?

69. Assuming as an axiom that two straight lines which cut one another cannot both be parallel to the same straight line; deduce Euclid's twelfth axiom as a corollary of Euc. I. 29.

70. From Euc. I. 27, shew that the distance between two parallel straight lines is constant?

71. If two straight lines be not parallel, shew that all straight lines falling on them, make alternate angles, which differ by the same angle.

72. Taking as the definition of parallel straight lines that they are equally inclined to the same straight line towards the same parts; prove that "being produced ever so far both ways they do not meet?" Prove also Euclid's axiom 12, by means of the same definition.

73. What is meant by *exterior* and *interior* angles? Point out examples.

74. Can the three angles of a triangle be proved equal to two right angles without producing a side of the triangle?

75. Shew how the corners of a triangular piece of paper may be turned down, so as to exhibit to the eye that the three angles of a triangle are equal to two right angles.

76. Explain the meaning of the term *corollary.* Enunciate the two corollaries appended to Euc. I. 32, and give another proof of the first. What other corollaries may be deduced from this proposition?

77. Shew that the two lines which bisect the exterior and interior angles of a triangle, as well as those which bisect any two interior angles of a parallelogram, contain a right angle.

78. The opposite sides and angles of a parallelogram are equal to one another, and the diameters bisect it. State and prove the converse of this proposition. Also shew that a quadrilateral figure, is a parallelogram, when its diagonals bisect each other: and when its diagonals divide it into four triangles, which are equal, two and two, viz. those which have the same vertical angles.

79. If two straight lines join the extremities of two parallel straight lines, but *not* towards the same parts, when are the joining lines equal, and when are they unequal?

80. If either diameter of a four-sided figure divide it into two equal triangles, is the figure necessarily a parallelogram? Prove your answer.

81. Shew how to divide one of the parallelograms in Euc. I. 35, by straight lines so that the parts when properly arranged shall make up the other parallelogram.

82. Distinguish between *equal* triangles and *equivalent* triangles, and give examples from the First Book of Euclid.

83. What is meant by the locus of a point? Adduce instances of loci from the first Book of Euclid.

84. How is it shewn that equal triangles upon the same base or equal bases have equal altitudes, whether they are situated on the same or opposite sides of the same straight line?

85. In Euc. I. 37, 38, if the triangles are not towards the same parts, shew that the straight line joining the vertices of the triangles is bisected by the line containing the bases.

86. If the complements (fig. Euc. I. 43) be squares, determine their relation to the whole parallelogram.

87. What is meant by a parallelogram being applied to a straight line?

88. Is the proof of Euc. I. 45, perfectly general?

89. Define a square without including superfluous conditions, and explain the mode of constructing a square upon a given straight line in conformity with such a definition.

90. The sum of the angles of a square is equal to four right angles. Is the converse true? If not, why?

91. Conceiving a square to be a figure bounded by four equal straight lines not necessarily in the same plane, what condition respecting the angles is necessary to complete the definition?

92. In Euclid I. 47, why is it necessary to prove that one side of each square described upon each of the sides containing the right angle, should be in the same straight line with the other side of the triangle?

93. On what assumption is an analogy shewn to exist between the product of two equal numbers and the surface of a square?

94. Is the triangle whose sides are 3, 4, 5 right-angled, or not?

95. Can the side and diagonal of a square be represented simultaneously by any finite numbers?

96. By means of Euc. I. 47, the square roots of the natural numbers, 1, 2, 3, 4, &c. may be represented by straight lines.

97. If the square on the hypotenuse in the fig. Euc. I. 47, be described on the other side of it: shew from the diagram how the squares on the two sides of the triangle may be made to cover exactly the square on the hypotenuse.

98. If Euclid II. 2, be assumed, enunciate the form in which Euc. I. 47 may be expressed.

99. Classify all the properties of *triangles* and *parallelograms*, proved in the First Book of Euclid.

100. Mention any propositions in Book I. which are included in more general ones which follow.

ON THE ANCIENT GEOMETRICAL ANALYSIS.

SYNTHESIS, or the method of composition, is a mode of reasoning which begins with something given, and ends with something required, either to be done or to be proved. This may be termed a *direct process*, as it leads from principles to consequences.

Analysis, or the method of resolution, is the reverse of synthesis, and thus it may be considered an *indirect process*, a method of reasoning from consequences to principles.

The synthetic method is pursued by Euclid in his Elements of Geometry. He commences with certain assumed principles, and proceeds to the solution of problems and the demonstration of theorems by undeniable and successive inferences from them.

The Geometrical Analysis was a process employed by the ancient Geometers, both for the discovery of the solution of problems and for the investigation of the truth of theorems. In the analysis of a *problem*, the quæsita, or what is required to be done, is supposed to have been effected, and the consequences are traced by a series of geometrical constructions and reasonings, till at length they terminate in the data of the problem, or in some previously demonstrated or admitted truth, whence the direct solution of the problem is deduced.

In the Synthesis of a *problem*, however, the last consequence of the analysis is assumed as the first step of the process, and by proceeding in a contrary order through the several steps of the analysis until the process terminate in the quæsita, the solution of the problem is effected.

But if, in the analysis, we arrive at a consequence which contradicts any truth demonstrated in the Elements, or which is inconsistent with the data of the problem, the problem must be impossible: and further, if in certain relations of the given magnitudes the construction be possible, while in other relations it is impossible, the discovery of these relations will become a necessary part of the solution of the problem.

In the analysis of a *theorem*, the question to be determined, is, whether by the application of the geometrical truths proved in the Elements, the predicate is consistent with the hypothesis. This point is ascertained by assuming the predicate to be true, and by deducing the successive consequences of this assumption combined with proved geometrical truths, till they terminate in the hypothesis of the theorem or some demonstrated truth. The theorem will be proved synthetically by retracing, in order, the steps of the investigation pursued in the analysis, till they terminate in the predicate, which was assumed in the analysis. This process will constitute the demonstration of the theorem.

If the assumption of the truth of the predicate in the analysis lead to some consequence which is inconsistent with any demonstrated truth, the false conclusion thus arrived at, indicates the falsehood of the predicate; and by reversing the process of the analysis, it may be demonstrated, that the theorem cannot be true.

It may here be remarked, that the geometrical analysis is more extensively useful in discovering the solution of problems than for investigating the demonstration of theorems.

From the nature of the subject, it must be at once obvious, that no general rules can be prescribed, which will be found applicable in all cases, and infallibly lead to the solution of every problem. The conditions of problems must suggest what constructions may be possible; and the consequences which follow from these constructions and the assumed solution, will shew the possibility or impossibility of arriving at some known property consistent with the data of the problem.

Though the data of a problem may be given in magnitude and position, certain ambiguities will arise, if they are not properly restricted. Two points may be considered as situated on the same side, or one on each side of a given line; and there may be two lines drawn from a given point making equal angles with a line given in position; and to avoid ambiguity, it must be stated on which side of the line the angle is to be formed.

A problem is said to be *determinate* when, with the prescribed conditions, it admits of one definite solution; the same construction which may be made on the other side of any given line, not being considered a different solution: and a problem is said to be *indeterminate* when it admits of more than one definite solution. This latter circumstance arises from the data not *absolutely fixing*, but *merely restricting* the quæsita, leaving certain points or lines not fixed in one position only. The number of given conditions may be insufficient for a single determinate solution; or relations may subsist among some of the given conditions from which one or more of the remaining given conditions may be deduced.

If the base of a right-angled triangle be given, and also the difference of the squares on the hypotenuse and perpendicular, the triangle is indeterminate. For though apparently here are three things given, the right angle, the base, and the difference of the squares on the hypotenuse and perpendicular, it is obvious that these three apparent conditions are in fact reducible to two: for since in a right-angled triangle, the sum of the squares on the base and on the perpendicular, is equal to the square on the hypotenuse, it follows that the difference of the squares on the hypotenuse and perpendicular, is equal to the square on the base of the triangle, and therefore the base is known from the difference of the squares on the hypotenuse and perpendicular being known. The conditions therefore are insufficient to determine a right-angled triangle; an indefinite number of triangles may be found with the prescribed conditions, whose vertices will lie in the line which is perpendicular to the base.

If a problem relate to the determination of *a single point*, and the data be sufficient to determine the position of that point, the problem is *determinate:* but if one or more of the conditions be omitted, the data which remain may be sufficient for the determination of more than one point, each of which satisfies the conditions of the problem; in that case, the problem is *indeterminate:* and in general, such points are found to be situated in some line, and hence such line is called the locus of the point which satisfies the conditions of the problem.

If any two given points A and B (fig. Euc. IV. 5.) be joined by a straight line AB, and this line be bisected in D, then if a perpendicular be drawn from the point of bisection, it is manifest that a circle described with *any* point in the perpendicular as a center, and a radius equal to its distance from one of the given points, will pass through the other point, and the perpendicular will be the locus of all the circles which can be described passing through the two given points.

Again, if a third point *C* be taken, but not in the same straight line with the other two, and this point be joined with the first point *A*; then the perpendicular drawn from the bisection *E* of this line will be the locus of the centers of all circles which pass through the first and third points *A* and *C*. But the perpendicular at the bisection of the first and second points *A* and *B* is the locus of the centers of circles which pass through these two points. Hence the intersection *F* of these two perpendiculars, will be the center of a circle which passes through the three points and is called the intersection of the two loci. Sometimes this method of solving geometrical problems may be pursued with advantage, by constructing the locus of every two points separately, which are given in the conditions of the problem. In the Geometrical Exercises which follow, only those local problems are given where the locus is either a straight line or a circle.

Whenever the quæsitum is a point, the problem on being rendered indeterminate, becomes a locus, whether the deficient datum be of the essential or of the accidental kind. When the quæsitum is a straight line or a circle, (which were the only two loci admitted into the ancient Elementary Geometry) the problem *may* admit of an *accidentally indeterminate* case; but will not *invariably* or even very frequently do so. This will be the case, when the line or circle shall be so far arbitrary in its position, as depends upon the deficiency of a *single* condition to fix it perfectly;—that is, (for instance) one point in the line, or two points in the circle, may be determined from the given conditions, but the remaining one is indeterminate from the accidental relations among the data of the problem.

Determinate Problems become indeterminate by the merging of some one datum in the results of the remaining ones. This may arise in three different ways; first, from the coincidence of two points; secondly, from that of two straight lines; and thirdly, from that of two circles. These, further, are the only three ways in which this accidental coincidence of data can produce this indeterminateness; that is, in other words, convert the problem into a Porism.

In the original Greek of Euclid's Elements, the corollaries to the propositions are called porisms (πορισματα); but this scarcely explains the nature of *porisms*, as it is manifest that they are different from simple deductions from the demonstrations of propositions. Some analogy, however, we may suppose them to have to the porisms or corollaries in the Elements. Pappus (Coll. Math. Lib. VII. pref.) informs us that Euclid wrote three books on Porisms. He defines "a porism to be something between a problem and a theorem, or that in which something is proposed to be investigated." Dr. Simson, to whom is due the merit of having restored the porisms of Euclid, gives the following definition of that class of propositions: "Porisma est propositio in qua proponitur demonstrare rem aliquam, vel plures datas esse, cui, vel quibus, ut et cuilibet ex rebus innumeris, non quidem, datis, sed quæ ad ea quæ data sunt eandem habent relationem, convenire ostendendum est affectionem quandam communem in propositione descriptam." That is, "A Porism is a proposition in which it is proposed to demonstrate that some one thing, or more things than one, are given, to which, as also to each of innumerable other things, not given indeed, but which have the same relation to those which are given, it is to be shewn that there belongs some common affection described in the proposition." Professor Dugald Stewart defines a porism to be "A proposition affirming the possibility of finding one or more of the conditions of an indeterminate theorem." Professor Playfair in a paper (from which the following account is taken) on Porisms, printed in the Transactions of the Royal Society of Edinburgh, for the year 1792, defines a porism to be "A proposition affirming the possibility of finding such conditions as will render a certain problem indeterminate or capable of innumerable solutions."

It may without much difficulty be perceived that this definition represents a porism as almost the same as an indeterminate problem. There is a large class of indeterminate problems which are, in general, loci, and satisfy certain defined conditions. Every indeterminate problem containing a locus may be made to assume the form of a porism, but not the converse. Porisms are of a more general nature than indeterminate problems which involve a locus.

The ancient geometers appear to have undertaken the solution of problems with a scrupulous and minute attention, which would scarcely allow any of the collateral truths to escape their observation. They never considered a problem as solved till they had distinguished all its varieties, and evolved separately every different case that could occur, carefully distinguishing whatever change might arise in the construction from any change that was supposed to take place among the magnitudes which were given. This cautious method of proceeding soon led them to see that there were circumstances in which the solution of a problem would cease to be possible; and this always happened when one of the conditions of the data was inconsistent with the rest. Such instances would occur in the simplest problems; but in the analysis of more complex problems, they must have remarked that their constructions failed, for a reason directly contrary to that assigned. Instances would be found where the lines, which, by their intersection, were to determine the thing sought, instead of intersecting one another, as they did in general, or of not meeting at all, would coincide with one another entirely, and consequently leave the question unresolved. The confusion thus arising would soon be cleared up, by observing, that a problem before determined by the intersection of two lines, would now become capable of an indefinite number of solutions. This was soon perceived to arise from one of the conditions of the problem involving another, or from two parts of the data becoming one, so that there was not left a sufficient number of independent conditions to confine the problem to a single solution, or any determinate number of solutions. It was not difficult afterwards to perceive, that these cases of problems formed very curious propositions, of an indeterminate nature between problems and theorems, and that they admitted of being enunciated separately. It was to such propositions so enunciated that the ancient geometers gave the name of *Porisms*.

Besides, it will be found, that some problems are possible within certain limits, and that certain magnitudes increase while others decrease within those limits; and after having reached a certain value, the former begin to decrease, while the latter increase. This circumstance gives rise to questions of *maxima* and *minima*, or the greatest and least values which certain magnitudes may admit of in indeterminate problems.

In the following collection of problems and theorems, most will be found to be of so simple a character, (being almost obvious deductions from propositions in the Elements) as scarcely to admit of the principle of the Geometrical Analysis being applied, in their solution.

It must however be recollected that a clear and exact knowledge of the first principles of Geometry must necessarily precede any intelligent application of them. Indistinctness or defectiveness of understanding with respect to these, will be a perpetual source of error and confusion. The learner is therefore recommended to understand the principles of the Science, and their connexion, fully, before he attempt any applications of them. The following directions may assist him in his proceedings.

ANALYSIS OF THEOREMS.

1. Assume that the Theorem is true.
2. Proceed to examine any consequences that result from this admission, by the aid of other truths respecting the diagram, which have been already proved.
3. Examine whether any of these consequences are already known to be *true*, or to be *false*.
4. If any one of them be false, we have arrived at a *reductio ad absurdum*, which proves that the theorem itself is false, as in Euc. I. 25.
5. If none of the consequences so deduced be *known* to be either true or false, proceed to deduce other consequences from all or any of these, as in (2).
6. Examine these results, and proceed as in (3) and (4); and if still without any conclusive indications of the truth or falsehood of the alleged theorem, proceed still further, until such are obtained.

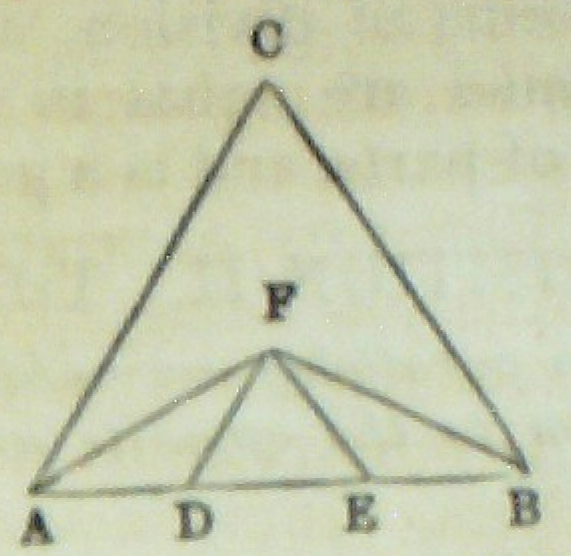
ANALYSIS OF PROBLEMS.

1. In general, any given problem will be found to depend on several problems and theorems, and these ultimately on some problem or theorem in Euclid.
2. Describe the diagram as directed in the enunciation, and suppose the solution of the problem effected.
3. Examine the relations of the lines, angles, triangles, &c. in the diagram, and find the dependence of the assumed solution on some theorem or problem in the Elements.
4. If such cannot be found, draw other lines parallel or perpendicular as the case may require, join given points, or points assumed in the solution, and describe circles if need be: and then proceed to trace the dependence of the assumed solution on some theorem or problem in Euclid.
5. Let not the first unsuccessful attempts at the solution of a Problem be considered as of no value; such attempts have been found to lead to the discovery of other theorems and problems.

PROPOSITION I. PROBLEM.

To trisect a given straight line.

ANALYSIS. Let AB be the given straight line, and suppose it divided into three equal parts in the points D, E.



On DE describe an equilateral triangle DEF,
then DF is equal to AD, and FE to EB.
On AB describe an equilateral triangle ABC,
and join AF, FB.
Then because AD is equal to DF,
therefore the angle AFD is equal to the angle DAF,
and the two angles DAF, DFA are double of one of them DAF.
But the angle FDE is equal to the angles DAF, DFA,
and the angle FDE is equal to DAC, each being an angle of an equilateral triangle;
therefore the angle DAC is double the angle DAF;
wherefore the angle DAC is bisected by AF.
Also because the angle FAC is equal to the angle FAD,
and the angle FAD to DFA;
therefore the angle CAF is equal to the alternate angle AFD:
and consequently FD is parallel to AC.

Synthesis. Upon AB describe an equilateral triangle ABC,
bisect the angles at A and B by the straight lines AF, BF, meeting in F;
through F draw FD parallel to AC, and FE parallel to BC.
Then AB is trisected in the points D, E.
For since AC is parallel to FD and FA meets them,
therefore the alternate angles FAC, AFD are equal;
but the angle FAD is equal to the angle FAC,
hence the angle DAF is equal to the angle AFD,
and therefore DF is equal to DA.
But the angle FDE is equal to the angle CAB,
and FED to CBA; (I. 29.)
therefore the remaining angle DFE is equal to the remaining angle ACB.
Hence the three sides of the triangle DFE are equal to one another,
and DF has been shewn to be equal to DA,
therefore AD DE, EB are equal to one another.

Hence the following theorem.

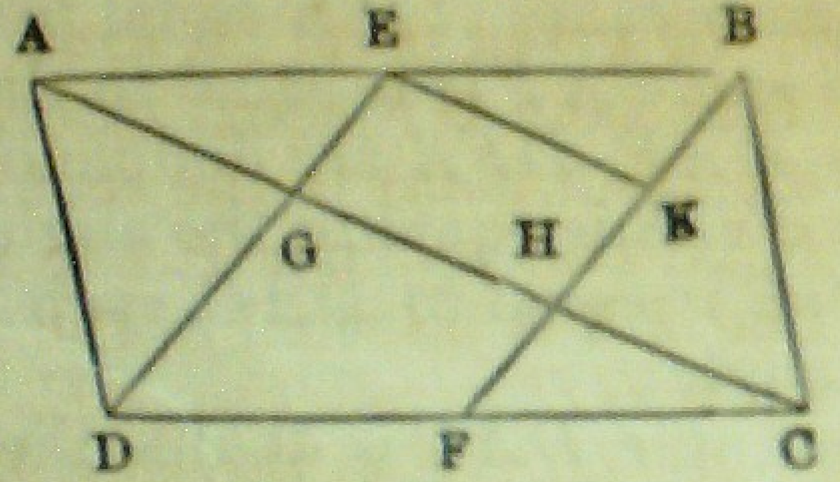
If the angles at the base of an equilateral triangle be bisected by two lines which meet at a point within the triangle; the two lines drawn from this point parallel to the sides of the triangle, divide the base into three equal parts.

Note. There is another method whereby a line may be divided into three equal parts:—by drawing from one extremity of the given line, another making an acute angle with it, and taking three equal distances from the extremity, then joining the extremities, and through the other two points of division, drawing lines parallel to this line through the other two points of division, and to the given line; the three triangles thus formed are equal in all respects. This may be extended for any number of parts, and is a particular case of Euc. VI. 10.

PROPOSITION II. THEOREM.

If two opposite sides of a parallelogram be bisected, and two lines be drawn from the points of bisection to the opposite angles, these two lines trisect the diagonal.

Let $ABCD$ be a parallelogram of which the diagonal is AC.
Let AB be bisected in E, and DC in F,
also let DE, FB be joined cutting the diagonal in G, H.
Then AC is trisected in the points G, H.



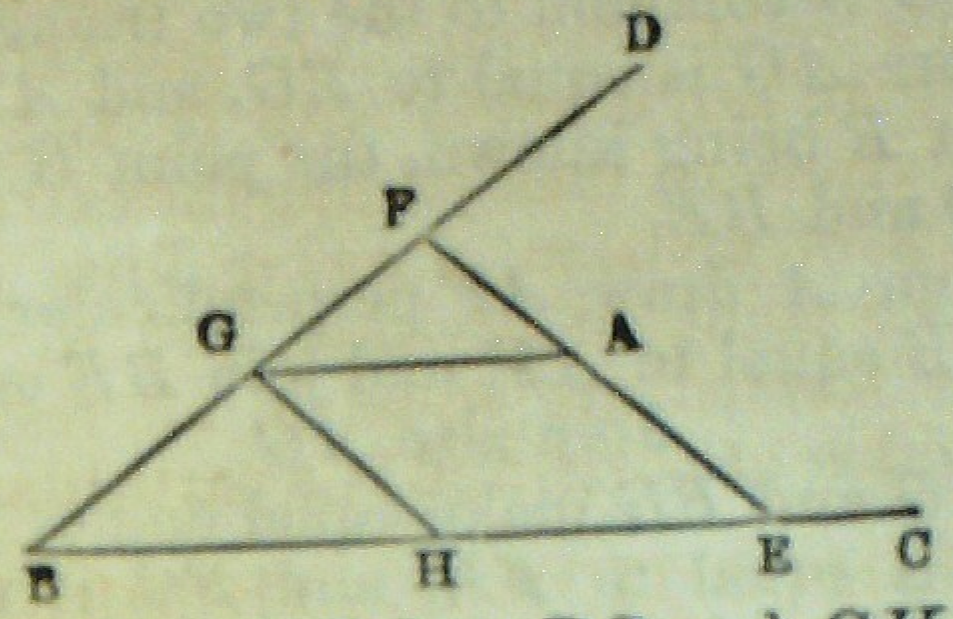
Through E draw EK parallel to AC and meeting FB in K.
Then because EB is the half of AB, and DF the half of DC;
therefore EB is equal to DF;
and these equal and parallel straight lines are joined towards the same parts by DE and FB;
therefore DE and FB are equal and parallel. (I. 33.)
And because AEB meets the parallels EK, AC,
therefore the exterior angle BEK is equal to the interior angle EAG.
For a similar reason, the angle EBK is equal to the angle AEG.
Hence in the triangles AEG, EBK, there are the two angles GAE, AEG in the one, equal to the two angles KEB, EBK in the other, and one side adjacent to the equal angles in each triangle, namely AE equal to EB;
therefore AG is equal to EK, (I. 26.)
but EK is equal to GH, (I. 34.) therefore AG is equal to GH.
By a similar process, it may be shewn that GH is equal to HC.
Hence AG, GH, HC are equal to one another,
and therefore AC is trisected in the points G, H.
It may also be proved that BF is trisected in H and K.

PROPOSITION III. PROBLEM.

Draw through a given point, between two straight lines not parallel, a straight line which shall be bisected in that point.

Analysis. Let BC BD be the two lines meeting in B, and let A be the given point between them.

Suppose the line EAF drawn through A, so that EA is equal to AF;



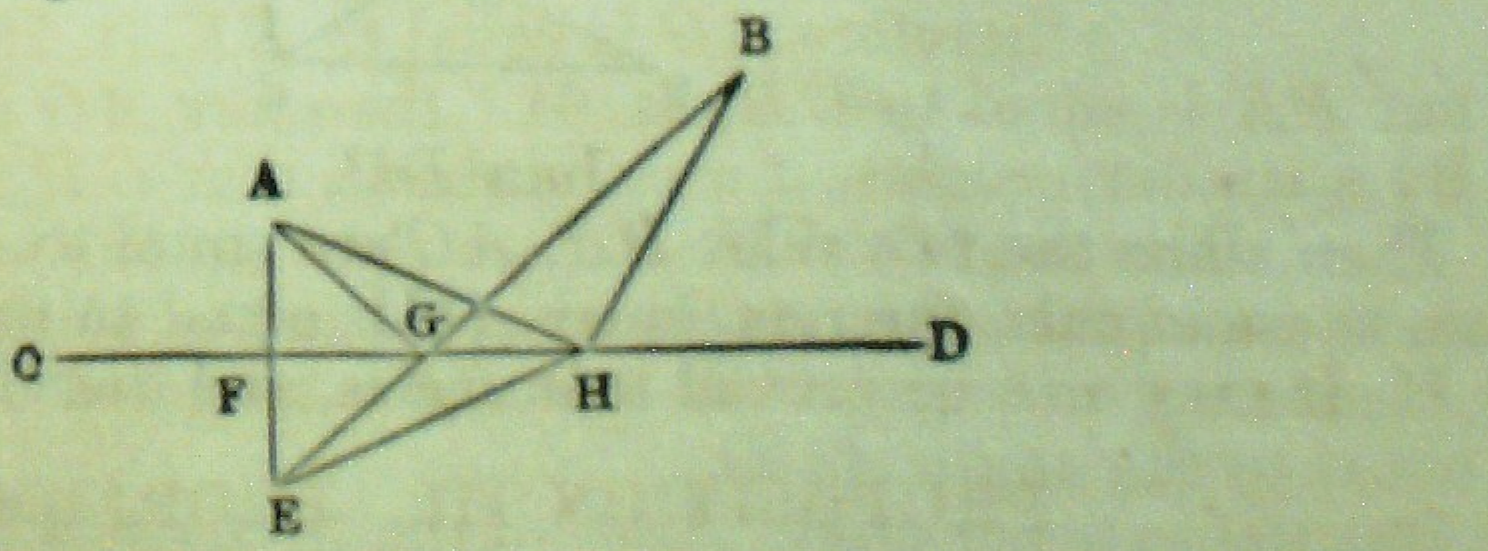
through A draw AG parallel to BC, and GH parallel to EF.
Then $AGHE$ is a parallelogram, wherefore AE is equal to GH,
but EA is equal to AF by hypothesis; therefore GH is equal to AF.
Hence in the triangles BHG, GAF,
the angles HBG, AGF are equal, as also BGH, GFA, (I. 29.)
also the side GH is equal to AF;
whence the other parts of the triangles are equal, (I. 26.)
therefore BG is equal to GF.

Synthesis. Through the given point A, draw AG parallel to BC;
on GD, take GF equal to GB;
then F is a second point in the required line:
join the points F, A, and produce FA to meet BC in E;
then the line FE is bisected in the point A;
draw GH parallel to AE.
Then in the triangles BGH, GFA, the side BG is equal to GF,
and the angles GBH, BGH are respectively equal to FGA, GFA,
wherefore GH is equal to AF, (I. 26.)
but GH is equal to AE, (I. 34.)
therefore AE is equal to AF, or EF is bisected in A.

PROPOSITION IV. PROBLEM.

From two given points on the same side of a straight line given in position, draw two straight lines which shall meet in that line, and make equal angles with it; also prove, that the sum of these two lines is less than the sum of any other two lines drawn to any other point in the line.

Analysis. Let A, B be the two given points, and CD the given line.
Suppose G the required point in the line, such that AG and BG being joined, the angle AGC is equal to the angle BGD.



Draw AF perpendicular to CD and meeting BG produced in E.
Then, because the angle BGD is equal to AGF, (hyp.)
and also to the vertical angle FGE, (I. 15.)
therefore the angle AGF is equal to the angle EGF;

also the right angle AFG is equal to the right angle
and the side FG is common to the two triangles AFG,
therefore AG is equal to EG, and AF to FE.

Hence the point E being known, the point G is determi
intersection of CD and BE.

Synthesis. From A draw AF perpendicular to CD an
it to E, making FE equal to AF, and join BE cutting CD in G.
Join also AG.

Then AG and BG make equal angles with CD.

For since AF is equal to FE, and FG is common to the two
triangles AGF, EGF, and the included angles AFG, EFG are equal;
therefore the base AG is equal to the base EG,
and the angle AGF to the angle EGF;
but the angle EGF is equal to the vertical angle BGD,
therefore the angle AGF is equal to the angle BGD;
that is, the straight lines AG and BG make equal angles with
the straight line CD.

Also the sum of the lines AG, GB is a minimum.

For take any other point H in CD, and join EH, HB, AH.
Then since any two sides of a triangle are greater than the third side,
therefore EH, HB are greater than EB in the triangle EHB.
But EG is equal to AG, and EH to AH;
therefore AH, HB are greater than AG, GB.

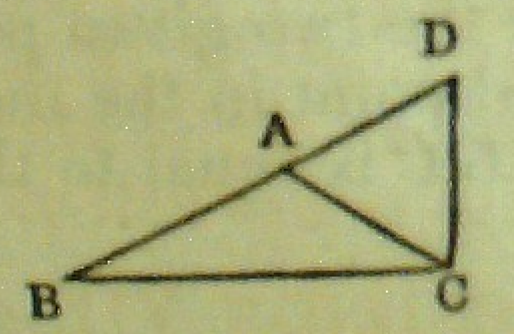
That is, AG, GB are less than any other two lines which can be drawn from A, B, to any other point H in the line CD.

By means of this Proposition may be found the shortest path from one given point to another, subject to the condition, that it shall meet two given lines.

PROPOSITION V. PROBLEM.

Given one angle, a side opposite to it, and the sum of the other two sides, construct the triangle.

Analysis. Suppose BAC the triangle required, having BC equal to the given side, BAC equal to the given angle opposite to BC, also BD equal to the sum of the other two sides.



Join DC.

Then since the two sides BA, AC are equal to BD, by taking BA
from these equals, the remainder AC is equal to the remainder AD.

Hence the triangle ACD is isosceles, and therefore the angle ADC
is equal to the angle ACD.

But the exterior angle BAC of the triangle ADC is equal to the
two interior and opposite angles ACD and ADC:

Wherefore the angle BAC is double the angle BDC, and BDC
the half of the angle BAC.

Hence the synthesis.

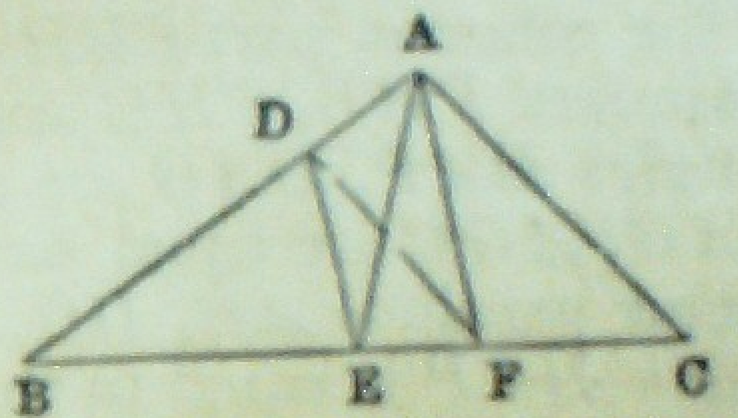
the point D in BD, make the angle BDC equal to half the angle,
d from B the other extremity of BD, draw BC equal to the given side, and meeting DC in C,
C in CD make the angle DCA equal to the angle CDA, so that CA may meet BD in the point A.

Then the triangle ABC shall have the required conditions.

PROPOSITION VI. PROBLEM.

To bisect a triangle by a line drawn from a given point in one of the sides.

Analysis. Let ABC be the given triangle, and D the given point in the side AB.



Suppose DF the line drawn from D which bisects the triangle;
therefore the triangle DBF is half of the triangle ABC.
Bisect BC in E, and join AE, DE, AF,
then the triangle ABE is half of the triangle ABC:
hence the triangle ABE is equal to the triangle DBF;
take away from these equals the triangle DBE,
therefore the remainder ADE is equal to the remainder DEF.

But ADE, DEF are equal triangles upon the same base DE, and on the same side of it.
they are therefore between the same parallels, (I. 39.)
that is, AF is parallel to DE,
therefore the point F is determined.

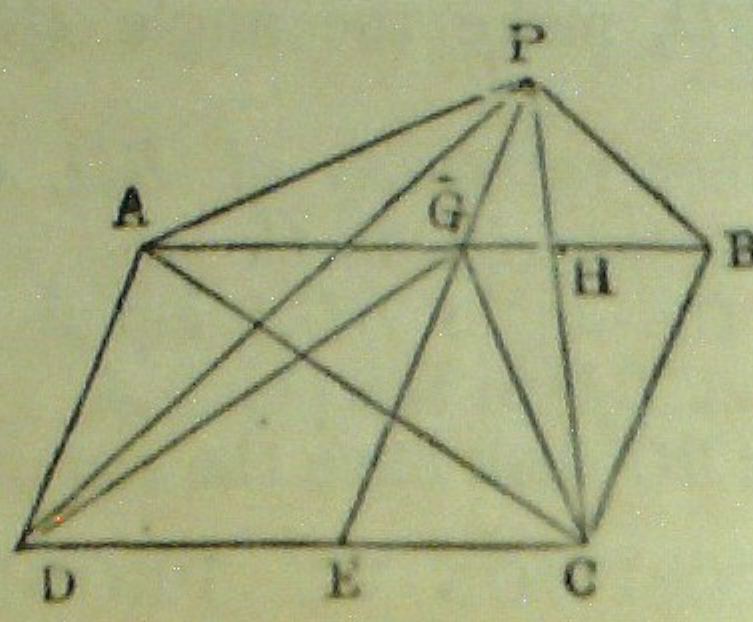
Synthesis. Bisect the base BC in E, join DE,
from A, draw AF parallel to DE, and join DF.
Then because DE is parallel to AF,
therefore the triangle ADE is equal to the triangle DEF; (I. 37.)
to each of these equals, add the triangle BDE,
therefore the whole triangle ABE is equal to the whole DBF,
but ABE is half of the whole triangle ABC;
therefore BDF is also half of the triangle ABC.

PROPOSITION VII. THEOREM.

If from a point without a parallelogram lines be drawn to the extremities of two adjacent sides, and of the diagonal which they include; of the triangles thus formed, that, whose base is the diagonal, is equal to the sum of the other two.

Let $ABCD$ be a parallelogram of which AC is one of the diagonals, and let P be any point without it: and let AP, PC, BP, PD be joined.

Then the triangles APD, APB are together equivalent to the triangle APC.



Draw PGE parallel to AD or BC, and meeting AB in G, an
in E; and join DG, GC.

Then the triangles CBP, CBG are equal: (I. 37.)
and taking the common part CBH from each,
the remainders PHB, CHG are equal.
Again, the triangles DAP, DAG are equal; (I. 37.)
also the triangles DAG, AGC are equal, being on the same
AG, and between the same parallels AG, DC:
therefore the triangle DAP is equal to the triangle AGC:
but the triangle PHB is equal to the triangle CHG,
wherefore the triangles PHB, DAP are equal to AGC, CHG or
ACH, add to these equals the triangle APH,
therefore the triangles APH, PHB, DAP are equal to APH, ACH,
that is, the triangles APB, DAP are together equal to the triangle
PAC.

If the point P be within the parallelogram, then the *difference* of the triangles APB, DAP may be proved to be equal to the triangle PAC.

I.

8. Describe an isosceles triangle upon a given base and having each of the sides double of the base, without using any proposition of the Elements subsequent to the first three. If the base and sides be given, what condition must be fulfilled with regard to the magnitude of each of the equal sides in order that an isosceles triangle may be constructed?

9. In the fig. Euc. I. 5. If FC and BG meet in H, then prove that AH bisects the angle BAC.

10. In the fig. Euc. I. 5. If the angle FBG be equal to the angle ABC, and BG, CF, intersect in O; the angle BOF is equal to twice the angle BAC.

11. From the extremities of the base of an isosceles triangle straight lines are drawn perpendicular to the sides, the angles made by them with the base are each equal to half the vertical angle.

12. A line drawn bisecting the angle contained by the two equal sides of an isosceles triangle, bisects the third side at right angles.

13. If a straight line drawn bisecting the vertical angle of a triangle also bisect the base, the triangle is isosceles

14. Given two points one on each side of a given straight line; find a point in the line such that the angle contained by two lines drawn to the given points may be bisected by the given line.

15. In the fig. Euc. I 5, let F and G be the points in the sides AB and AC produced, and let lines FH and GK be drawn perpendicular and equal to FC and GB respectively: also if BH, CK, or these lines produced meet in O; prove that BH is equal to CK and BO to CO.

16. From every point of a given straight line, the straight lines drawn to each of two given points on opposite sides of the line are equal: prove that the line joining the given points will cut the given line at right angles.

17. If A be the vertex of an isosceles triangle ABC, and BA be produced so that AD is equal to BA, and DC be drawn; shew that BCD is a right angle.

18. The straight line EDF, drawn at right angles to BC the base of an isosceles triangle ABC, cuts the side AB in D, and CA produced in E; shew that AED is an isosceles triangle.

19. In the fig. Euc. I. 1, if AB be produced both ways to meet the circles in D and E, and from C, CD and CE be drawn; the figure CDE is an isosceles triangle having each of the angles at the base, equal to one fourth of the angle at the vertex of the triangle.

20. From a given point, draw two straight lines making equal angles with two given straight lines intersecting one another.

21. From a given point to draw a straight line to a given straight line, that shall be bisected by another given straight line.

22. Place a straight line of given length between two given straight lines which meet, so that it shall be equally inclined to each of them.

23. To determine that point in a straight line from which the straight lines drawn to two other given points shall be equal, provided the line joining the two given points is not perpendicular to the given line.

24. In a given straight line to find a point equally distant from two given straight lines. In what case is this impossible?

25. If a line intercepted between the extremity of the base of an isosceles triangle, and the opposite side (produced if necessary) be equal to a side of the triangle, the angle formed by this line and the base produced, is equal to three times either of the equal angles of the triangle.

26. In the base BC of an isosceles triangle ABC, take a point D, and in CA take CE equal to CD, let ED produced meet AB produced in F; then $3.AEF = 2$ right angles $+ AFE$, or $= 4$ right angles $+ AFE$.

27. If from the base to the opposite sides of an isosceles triangle, three straight lines be drawn, making equal angles with the base, viz. one from its extremity, the other two from any other point in it, these two shall be together equal to the first.

28. A straight line is drawn, terminated by one of the sides of an isosceles triangle, and by the other side produced, and bisected by the base; prove that the straight lines, thus intercepted between the

vertex of the isosceles triangle, and this straight line, are equal to the two equal sides of the triangle.

29. In a triangle, if the lines bisecting the angles at the [illegible] equal, the triangle is isosceles, and the angle contained by the [illegible]ing lines is equal to an exterior angle at the base of the triangle.

30. In a triangle, if lines be equal when drawn from the [illegible]ties of the base, (1) perpendicular to the sides, (2) bisecting the [illegible] (3) making equal angles with the sides; the triangle is isosceles: and then these lines which respectively join the intersections of the sides, are parallel to the base.

II.

31. *ABC* is a triangle right-angled at *B*, and having the angle *A* double the angle *C*; shew that the side *BC* is less than double the side *AB*.

32. If one angle of a triangle be equal to the sum of the other two, the greatest side is double of the distance of its middle point from the opposite angle.

33. If from the right angle of a right-angled triangle, two straight lines be drawn, one perpendicular to the base, and the other bisecting it, they will contain an angle equal to the difference of the two acute angles of the triangle.

34. If the vertical angle *CAB* of a triangle *ABC* be bisected by *AD*, to which the perpendiculars *CE*, *BF* are drawn from the remaining angles: bisect the base *BC* in *G*, join *GE*, *GF*, and prove these lines equal to each other.

35. The difference of the angles at the base of any triangle, is double the angle contained by a line drawn from the vertex perpendicular to the base, and another bisecting the angle at the vertex.

36. If one angle at the base of a triangle be double of the other, the less side is equal to the sum or difference of the segments of the base made by the perpendicular from the vertex, according as the angle is greater or less than a right angle.

37. If two exterior angles of a triangle be bisected, and from the point of intersection of the bisecting lines, a line be drawn to the opposite angle of the triangle, it will bisect that angle.

38. From the vertex of a scalene triangle draw a right line to the base, which shall exceed the less side as much as it is exceeded by the greater.

39. Divide a right angle into three equal angles.

40. One of the acute angles of a right-angled triangle is three times as great as the other; trisect the smaller of these.

41. Prove that the sum of the distances of any point within a triangle from the three angles is greater than half the perimeter of the triangle.

42. The perimeter of an isosceles triangle is less than that of any other equal triangle upon the same base.

43. If from the angles of a triangle *ABC*, straight lines *ADE*, *BDF*, *CDG* be drawn through a point *D* to the opposite sides, prove that the sides of the triangle are together greater than the three

lines drawn to the point *D*, and less than twice the same, but greater than two-thirds of the lines drawn through the point to the opposite sides.

44. In a plane triangle an angle is right, acute or obtuse, according as the line joining the vertex of the angle with the middle point of the opposite side is equal to, greater or less than half of that side.

45. If the straight line *AD* bisect the angle *A* of the triangle *ABC*, and *BDE* be drawn perpendicular to *AD* and meeting *AC* or *AC* produced in *E*, shew that $BD = DE$.

46. The side *BC* of a triangle *ABC* is produced to a point *D*. The angle *ACB* is bisected by a line *CE* which meets *AB* in *E*. A line is drawn through *E* parallel to *BC* and meeting *AC* in *F*, and the line bisecting the exterior angle *ACD*, in *G*. Shew that *EF* is equal to *FG*.

47. The sides *AB*, *AC*, of a triangle are bisected in *D* and *E* respectively, and *BE*, *CD*, are produced until $EF = EB$, and $GD = DC$; shew that the line *GF* passes through *A*.

48. In a triangle *ABC*, *AD* being drawn perpendicular to the straight line *BD* which bisects the angle *B*, shew that a line drawn from *D* parallel to *BC* will bisect *AC*.

49. If the sides of a triangle be trisected and lines be drawn through the points of section adjacent to each angle so as to form another triangle, this shall be in all respects equal to the first triangle.

50. Between two given straight lines it is required to draw a straight line which shall be equal to one given straight line, and parallel to another.

51. If from the vertical angle of a triangle three straight lines be drawn, one bisecting the angle, another bisecting the base, and the third perpendicular to the base, the first is always intermediate in magnitude and position to the other two.

52. In the base of a triangle, find the point from which, lines drawn parallel to the sides of the triangle and limited by them, are equal.

53. In the base of a triangle, to find a point from which if two lines be drawn, (1) perpendicular, (2) parallel, to the two sides of the triangle, their sum shall be equal to a given line.

III.

54. In the figure of Euc. I. 1, the given line is produced to meet either of the circles in *P*; shew that *P* and the points of intersection of the circles, are the angular points of an equilateral triangle.

55. If each of the equal angles of an isosceles triangle be one-fourth of the third angle, and from one of them a line be drawn at right angles to the base meeting the opposite side produced; then will the part produced, the perpendicular, and the remaining side, form an equilateral triangle.

56. In the figure Euc. I. 1, if the sides *CA*, *CB* of the equilateral triangle *ABC* be produced to meet the circles in *F*, *G*, respectively, and if *C'* be the point in which the circles cut one another on th

other side of AB: prove the points F, C', G to be in the same straight line; and the figure CFG to be an equilateral triangle.

57. ABC is a triangle and the exterior angles at B and C are bisected by lines BD, CD respectively, meeting in D: shew that the angle BDC and half the angle BAC make up a right angle.

58. If the exterior angle of a triangle be bisected, and the angles of the triangle made by the bisectors be bisected, and so on, the triangles so formed will tend to become eventually equilateral.

59. If in the three sides AB, BC, CA of an equilateral triangle ABC, distances AE, BF, CG be taken, each equal to a third of one of the sides, and the points E, F, G be respectively joined (1) with each other, (2) with the opposite angles: shew that the two triangles so formed, are equilateral triangles.

IV.

60. Describe a right-angled triangle upon a given base, having given also the perpendicular from the right angle upon the hypotenuse.

61. Given one side of a right-angled triangle, and the difference between the hypotenuse and the sum of the other two sides, to construct the triangle.

62. Construct an isosceles right-angled triangle, having given (1) the sum of the hypotenuse and one side; (2) their difference.

63. Describe a right-angled triangle of which the hypotenuse and the difference between the other two sides are given.

64. Given the base of an isosceles triangle, and the sum or difference of a side and the perpendicular from the vertex on the base. Construct the triangle.

65. Make an isosceles triangle of given altitude whose sides shall pass through two given points and have its base on a given straight line.

66. Construct an equilateral triangle, having given the length of the perpendicular drawn from one of the angles on the opposite side.

67. Having given the straight lines which bisect the angles at the base of an equilateral triangle, determine a side of the triangle.

68. Having given two sides and an angle of a triangle, construct the triangle, distinguishing the different cases.

69. Having given the base of a triangle, the difference of the sides, and the difference of the angles at the base; to describe the triangle.

70. Given the perimeter and the angles of a triangle, to construct it.

71. Having given the base of a triangle, and half the sum and half the difference of the angles at the base; to construct the triangle.

72. Having given two lines, which are not parallel, and a point between them; describe a triangle having two of its angles in the respective lines, and the third at the given point; and such that the sides shall be equally inclined to the lines which they meet.

73. Construct a triangle, having given the three lines drawn from the angles to bisect the sides opposite.

74. Given one of the angles at the base of a triangle, the base itself, and the sum of the two remaining sides, to construct the triangle.

75. Given the base, an angle adjacent to the base, and the difference of the sides of a triangle, to construct it.

76. Given one angle, a side opposite to it, and the difference of the other two sides; to construct the triangle.

77. Given the base and the sum of the two other sides of a triangle, construct it so that the line which bisects the vertical angle shall be parallel to a given line.

V.

78. From a given point without a given straight line, to draw a line making an angle with the given line equal to a given rectilineal angle.

79. Through a given point A, draw a straight line ABC meeting two given parallel straight lines in B and C, such that BC may be equal to a given straight line.

80. If the line joining two parallel lines be bisected, all the lines drawn through the point of bisection and terminated by the parallel lines are also bisected in that point.

81. Three given straight lines issue from a point: draw another straight line cutting them so that the two segments of it intercepted between them may be equal to one another.

82. AB, AC are two straight lines, B and C given points in the same; BD is drawn perpendicular to AC, and DE perpendicular to AB; in like manner CF is drawn perpendicular to AB, and FG to AC. Shew that EG is parallel to BC.

83. ABC is a right-angled triangle, and the sides AC, AB are produced to D and F; bisect FBC and BCD by the lines BE, CE, and from E let fall the perpendiculars EF, ED. Prove (without assuming any properties of parallels) that $ADEF$ is a square.

84. Two pairs of equal straight lines being given, shew how to construct with them the greatest parallelogram.

85. On the sides AB, BC, CD of a parallelogram are described equilateral triangles ABE, CDF without, and BCG within the figure; prove that EG is equal to one, and FG the other diagonal.

86. Having given one of the diagonals of a parallelogram, the sum of the two adjacent sides and the angle between them, construct the parallelogram.

87. One of the diagonals of a parallelogram being given, and the angle which it makes with one of the sides, complete the parallelogram, so that the other diagonal may be parallel to a given line.

88. $ABCD$, $A'B'C'D'$ are two parallelograms whose corresponding sides are equal, but the angle A is greater than the angle A', prove that the diameter AC is less than $A'C'$, but BD greater than $B'D'$.

89. If in the diagonal of a parallelogram any two points equidistant from its extremities be joined with the opposite angles, a figure will be formed which is also a parallelogram.

90. From each angle of a parallelogram a line is drawn making

the same angle towards the same parts with an adjacent side, taken always in the same order; shew that these lines form another parallelogram *similar* to the original one.

91. Along the sides of a parallelogram taken in order, measure $AA' = BB' = CC' = DD'$: the figure $A'B'C'D'$ will be a parallelogram.

92. On the sides AB, BC, CD, DA, of a parallelogram, set off AE, BF, CG, DH, equal to each other, and join AF, BG, CH, DE: these lines form a parallelogram, and the difference of the angles AFB, BGC, equals the difference of any two proximate angles of the two parallelograms.

93. OB, OC are two straight lines at right angles to each other, through any point P any two straight lines are drawn intersecting OB, OC, in B, B', C, C', respectively. If D and D' be the middle points of BB' and CC', shew that the angle BPD' is equal to the angle DOD'.

94. $ABCD$ is a parallelogram of which the angle C is opposite to the angle A. If through A any straight line be drawn, then the distance of C is equal to the sum or difference of the distances of B and of D from that straight line, according as it lies without or within the parallelogram.

95. Upon stretching two chains AC, BD, across a field $ABCD$, I find that BD and AC make equal angles with DC, and that AC makes the same angle with AD that BD does with BC; hence prove that AB is parallel to CD.

96. To find a point in the side or side produced of any parallelogram, such that the angle it makes with the line joining the point and one extremity of the opposite side, may be bisected by the line joining it with the other extremity.

97. When the corner of the leaf of a book is turned down a second time, so that the lines of folding are parallel and equidistant, the space in the second fold is equal to three times that in the first.

VI.

98. If the points of bisection of the sides of a triangle be joined, the triangle so formed shall be one-fourth of the given triangle.

99. If in the triangle ABC, BC be bisected in D, AD joined and bisected in E, BE joined and bisected in F, and CF joined and bisected in G; then the triangle EFG will be equal to one-eighth of the triangle ABC.

100. Shew that the areas of the two equilateral triangles in Prob. 59, p. 78, are respectively, one-third and one-seventh of the area of the original triangle.

101. To describe a triangle equal to a given triangle, (1) when the base, (2) when the altitude of the required triangle is given.

102. To describe a triangle equal to the sum or difference of two given triangles.

103. Upon a given base describe an isosceles triangle equal to a given triangle.

104. Describe a right-angled triangle equal to a given triangle ABC.

105. To a given straight line apply a triangle which shall be equal to a given parallelogram and have one of its angles equal to a given rectilineal angle.

106. Transform a given rectilineal figure into a triangle whose vertex shall be in a given angle of the figure, and whose base shall be in one of the sides.

107. Divide a triangle by two straight lines into three parts which when properly arranged shall form a parallelogram whose angles are of a given magnitude.

108. Shew that a scalene triangle cannot be divided into two parts which will coincide.

109. If two sides of a triangle be given, the triangle will be greatest when they contain a right angle.

110. Of all triangles having the same vertical angle, and whose bases pass through a given point, the least is that whose base is bisected in the given point.

111. Of all triangles having the same base and the same perimeter, that is the greatest which has the two undetermined sides equal.

112. Divide a triangle into three equal parts, (1) by lines drawn from a point in one of the sides: (2) by lines drawn from the angles to a point within the triangle: (3) by lines drawn from a given point within the triangle. In how many ways can the third case be done?

113. Divide an equilateral triangle into nine equal parts.

114. Bisect a parallelogram, (1) by a line drawn from a point in one of its sides: (2) by a line drawn from a given point within or without it: (3) by a line perpendicular to one of the sides: (4) by a line drawn parallel to a given line.

115. From a given point in one side produced of a parallelogram, draw a straight line which shall divide the parallelogram into two equal parts.

116. To trisect a parallelogram by lines drawn (1) from a given point in one of its sides, (2) from one of its angular points.

VII.

117. To describe a rhombus which shall be equal to any given quadrilateral figure.

118. Describe a parallelogram which shall be equal in area and perimeter to a given triangle.

119. Find a point in the diagonal of a square produced, from which if a straight line be drawn parallel to any side of the square, and meeting another side produced, it will form together with the produced diagonal and produced side, a triangle equal to the square.

120. If from any point within a parallelogram, straight lines be drawn to the angles, the parallelogram shall be divided into four triangles, of which each two opposite are together equal to one-half of the parallelogram.

121. If $ABCD$ be a parallelogram, and E any point in the diagonal AC, or AC produced; shew that the triangles EBC, EDC, are equal, as also the triangles EBA and EBD.

122. $ABCD$ is a parallelogram, draw DFG meeting BC in F

and AB produced in G; join AF, CG; then will the triangles ABF, CFG be equal to one another.

123. $ABCD$ is a parallelogram, E the point of intersection of its diagonals, and K any point in AD. If KB, KC be joined, shew that the figure $BKEC$ is one-fourth of the parallelogram.

124. Let $ABCD$ be a parallelogram, and O any point within it, through O draw lines parallel to the sides of $ABCD$, and join OA, OC; prove that the difference of the parallelograms DO, BO is twice the triangle OAC.

125. The diagonals AC, BD of a parallelogram intersect in O and P is a point within the triangle AOB; prove that the difference of the triangles APB, CPD is equal to the sum of the triangles APC, BPD.

126. If K be the common angular point of the parallelograms about the diameter AC (fig. Euc. I. 43.) and BD be the other diameter, the difference of these parallelograms is equal to twice the triangle BKD.

127. The perimeter of a square is less than that of any other parallelogram of equal area.

128. Shew that of all equiangular parallelograms of equal perimeters, that which is equilateral is the greatest.

129. Prove that the perimeter of an isosceles triangle is greater than that of an equal right-angled parallelogram of the same altitude.

VIII.

130. If a quadrilateral figure is bisected by one diagonal, the second diagonal is bisected by the first.

131. If two opposite angles of a quadrilateral figure are equal, shew that the angles between opposite sides produced are equal.

132. Prove that the sides of any four-sided rectilinear figure are together greater than the two diagonals.

133. The sum of the diagonals of a trapezium is less than the sum of any four lines which can be drawn to the four angles, from any point within the figure, except their intersection.

134. The longest side of a given quadrilateral is opposite to the shortest; shew that the angles adjacent to the shortest side are together greater than the sum of the angles adjacent to the longest side.

135. Give any two points in the opposite sides of a trapezium, inscribe in it a parallelogram having two of its angles at these points.

136. Shew that in every quadrilateral plane figure, two parallelograms can be described upon two opposite sides as diagonals, such that the other two diagonals shall be in the same straight line and equal.

137. Describe a quadrilateral figure whose sides shall be equal to four given straight lines. What limitation is necessary?

138. If the sides of a quadrilateral figure be bisected and the points of bisection joined, the included figure is a parallelogram, and equal in area to half the original figure.

139. A trapezium is such, that the perpendiculars let fall on a diagonal from the opposite angles are equal. Divide the trapezium into four equal triangles, by straight lines drawn to the angles from a point within it.

140. If two opposite sides of a trapezium be parallel to one another, the straight line joining their bisections, bisects the trapezium.

141. If of the four triangles into which the diagonals divide a trapezium, any two opposite ones are equal, the trapezium has two of its opposite sides parallel.

142. If two sides of a quadrilateral are parallel but not equal, and the other two sides are equal but not parallel, the opposite angles of the quadrilateral are together equal to two right angles: and conversely.

143. If two sides of a quadrilateral be parallel, and the line joining the middle points of the diagonals be produced to meet the other sides; the line so produced will be equal to half the sum of the parallel sides, and the line between the points of bisection equal to half their difference.

144. To bisect a trapezium, (1) by a line drawn from one of its angular points: (2) by a line drawn from a given point in one side.

145. To divide a square into four equal portions by lines drawn from any point in one of its sides.

146. It is impossible to divide a quadrilateral figure (except it be a parallelogram) into equal triangles by lines drawn from a point within it to its four corners.

IX.

147. If the greater of the acute angles of a right-angled triangle, be double the other, the square on the greater side is three times the square on the other.

148. Upon a given straight line construct a right-angled triangle such that the square on the other side may be equal to seven times the square on the given line.

149. If from the vertex of a plane triangle, a perpendicular fall upon the base or the base produced, the difference of the squares on the sides is equal to the difference of the squares on the segments of the base.

150. If from the middle point of one of the sides of a right-angled triangle, a perpendicular be drawn to the hypotenuse, the difference of the squares on the segments into which it is divided, is equal to the square on the other side.

151. If a straight line be drawn from one of the acute angles of a right-angled triangle, bisecting the opposite side, the square upon that line is less than the square upon the hypotenuse by three times the square upon half the line bisected.

152. If the sum of the squares on the three sides of a triangle be equal to eight times the square on the line drawn from the vertex to the point of bisection of the base, then the vertical angle is a right angle.

153. If a line be drawn parallel to the hypotenuse of a right-angled triangle, and each of the acute angles be joined with the points where this line intersects the sides respectively opposite to them, the squares on the joining lines are together equal to the squares on the hypotenuse and on the line drawn parallel to it.

154. Let ACB, ADB be two right-angled triangles having a common hypotenuse AB, join CD, and on CD produced both ways draw perpendiculars AE, BF. Shew that $CE^2 + CF^2 = DE^2 + DF^2$.

155. If perpendiculars AD, BE, CF, drawn from the angles on the opposite sides of a triangle intersect in G, the difference of the squares on the sides AC, AB, is equal to the difference of the squares on the lines CG, BG.

156. If ABC be a triangle of which the angle A is a right angle; and BE, CF be drawn bisecting the opposite sides respectively: shew that four times the sum of the squares on BE and CF is equal to five times the square on BC.

157. If ABC be an isosceles triangle, and CD be drawn perpendicular to AB; the sum of the squares on the three sides is equal to

$$AD^2 + 2.\ BD^2 + 3.\ CD^2.$$

158. The sum of the squares described upon the sides of a rhombus is equal to the squares described on its diameters.

159. A point is taken within a square, and straight lines drawn from it to the angular points of the square, and perpendicular to the sides; the squares on the first are double the sum of the squares on the last. Shew that these sums are least when the point is in the center of the square.

160. In the figure Euc. I. 47,

(*a*) Shew that the diagonals FA, AK of the squares on AB, AC, lie in the same straight line.

(*b*) If DF, EK be joined, the sum of the angles at the bases of the triangles BFD, CEK is equal to one right angle.

(*c*) If BG and CH be joined, those lines will be parallel.

(*d*) If perpendiculars be let fall from F and K on BC produced, the parts produced will be equal; and the perpendiculars together will be equal to BC.

(*e*) Join GH, KE, FD, and prove that each of the triangles so formed, equals the given triangle ABC.

(*f*) The sum of the squares on GH, KE, and FD will be equal to six times the square on the hypotenuse.

(*g*) The difference of the squares on AB, AC, is equal to the difference of the squares on AD, AE.

161. The area of any two parallelograms described on the two sides of a triangle, is equal to that of a parallelogram on the base, whose side is equal and parallel to the line drawn from the vertex of the triangle, to the intersection of the two sides of the former parallelograms produced to meet.

162. If one angle of a triangle be a right angle, and another equal to two-thirds of a right angle, prove from the First Book of Euclid, that the equilateral triangle described on the hypotenuse, is equal to the sum of the equilateral triangles described upon the sides which contain the right angle.

GEOMETRICAL EXERCISES ON BOOK I.

HINTS, &c.

8. This is a particular case of Euc. I. 22. The triangle however may be described by means of Euc. I. 1. Let AB be the given base, produce AB both ways to meet the circles in D, E (fig. Euc. I. 1.); with center A, and radius AE, describe a circle, and with center B and radius BD, describe another circle cutting the former in G. Join GA, GB.

9. Apply Euc. I. 6, 8.

10. This is proved by Euc. I. 32, 13, 5.

11. Let fall also a perpendicular from the vertex on the base.

12. Apply Euc. I. 4.

13. Let CAB be the triangle (fig. Euc. I. 10.) CD the line bisecting the angle ACD and the base AB. Produce CD, and make DE equal to CD, and join AE. Then CB may be proved equal to AE, also AE to AC.

14. Let AB be the given line, and C, D the given points. From C draw CE perpendicular to AB, and produce it making EF equal to CE, join FD, and produce it to meet the given line in G, which will be the point required.

15. Make the construction as the enunciation directs, then by Euc. I. 4, BH is proved equal to CK: and by Euc. I. 13, 6, OB is shewn to be equal to OC.

16. This proposition requires for its proof the case of equal triangles omitted in Euclid:—namely, when two sides and one angle are given, but not the angle included by the given sides.

17. The angle BCD may be shewn to be equal to the sum of the angles ABC, ADC.

18. The angles ADE, AED may be each proved to be equal to the complements of the angles at the base of the triangle.

19. The angles CAB, CBA, being equal, the angles CAD, CBE are equal, Euc. I. 13. Then, by Euc. I. 4, CD is proved to be equal to CE. And by Euc. I. 5, 32, the angle at the vertex is shewn to be four times either of the angles at the base.

20. Let AB, CD be two straight lines intersecting each other in E, and let P be the given point, within the angle AED. Draw EF bisecting the angle AED, and through P draw PGH parallel to EF, and cutting ED, EB in G, H. Then EG is equal to EH. And by bisecting the angle DEB and drawing through P a line parallel to this line, another solution is obtained. It will be found that the two lines are at right angles to each other.

21. Let the two given straight lines meet in A, and let P be the given point. Let PQR be the line required, meeting the lines AQ, AR in Q and R, so that PQ is equal to QR. Through P draw PS parallel to AR and join RS. Then APSR is a parallelogram and AS, PR the diagonals. Hence the construction.

22. Let the two straight lines AB, AC meet in A. In AB take any point D, and from AC cut off AE equal to AD, and join DE. On DE, or DE produced, take DF equal to the given line, and through F draw FG parallel to AB meeting AC in G, and through G draw GH parallel to DE meeting AB in H. Then GH is the line required.

23. The two given points may be both on the same side, or one point may be on each side of the line. If the point required in the line be supposed to be found, and lines be drawn joining this point and the given points, an isosceles triangle is formed, and if a perpendicular be drawn on the base from the point in the line: the construction is obvious.

24. The problem is simply this—to find a point in one side of a triangle from which the perpendiculars drawn to the other two sides shall be equal. If all the positions of these lines be considered, it will readily be seen in what case the problem is impossible.

25. If the *isosceles triangle* be obtuse-angled, by Euc. I. 5, 32, the truth will be made evident. If the triangle be acute-angled, the enunciation of the proposition requires some modification.

26. Construct the figure and apply Euc. I. 5, 32, 15.

If the isosceles triangle have its vertical angle less than two-thirds of a right-angle, the line ED produced, meets AB produced towards the base, and then 3 . AEF = 4 right angles + AFE. If the vertical angle be greater than two-thirds of a right angle, ED produced meets AB produced towards the vertex, then 3 . AEF = 2 right angles + AFE.

27. Let ABC be an isosceles triangle, and from any point D in the base BC, and the extremity B, let three lines DE, DF, BG be drawn to the sides and making equal angles with the base. Produce ED and make DH equal to DF and join BH.

28. In the isosceles triangle ABC, let the line DFE which meets the side AC in D and AB produced in E, be bisected by the base in the point E. Then DC may be shewn to be equal to BE.

29. If two equal straight lines be drawn terminated by two lines which meet in a point, they will cut off triangles of equal area. Hence the two triangles have a common vertical angle and their areas and bases equal. By Euc. I. 32 it is shewn that the angle contained by the bisecting lines is equal to the exterior angle at the base.

30. There is an omission in this question. After the words "making equal angles with the sides," add, "and be equal to each other respectively." (1), (3) Apply Euc. I. 26, 4. (2) The equal lines which bisect the sides may be shewn to make equal angles with the sides.

31. At C make the angle BCD equal to the angle ACB, and produce AB to meet CD in D.

32. By bisecting the hypotenuse, and drawing a line from the vertex to the point of bisection, it may be shewn that this line forms with the shorter side and half the hypotenuse an isosceles triangle.

33. Let ABC be a triangle, having the right angle at A, and the angle at C greater than the angle at B, also let AD be perpendicular to the base, and AE be the line drawn to E the bisection of the base. Then AE may be proved equal to BE or EC independently of Euc. III. 31.

34. Produce EG, FG to meet the perpendiculars CE, BF, produced if necessary. The demonstration is obvious.

35. If the given triangle have both of the angles at the base acute angles; the difference of the angles at the base is at once obvious from Euc. I. 32. If one of the angles at the base be obtuse, does the property hold good?

36. Let ABC be a triangle having the angle ACB double of the angle ABC, and let the perpendicular AD be drawn to the base BC. Take DE equal to DC and join AE. Then AE may be proved to be equal to EB.

If ACB be an obtuse angle, then AC is equal to the sum of the segments of the base, made by the perpendicular from the vertex A.

37. Let the sides AB, AC of any triangle ABC be produced, the exterior angles bisected by two lines which meet in D, and let AD be joined, then AD bisects the angle BAC. For draw DE perpendicular on BC, also DF, DG perpendiculars on AB, AC produced, if necessary. Then DF may be proved equal to DG, and the squares on DF, DA are equal to the squares on FG, GA, of which the square on FD is equal to the square on DG; hence AF is equal to AG, and Euc. I. 8, the angle BAC is bisected by AD.

38. The line required will be found to be equal to half the sum of the two sides of the triangle.

39. Apply Euc. I. 1, 9.

40. The angle to be trisected is one-fourth of a right angle. If an equilateral triangle be described on one of the sides of a triangle which contains the given angle, and a line be drawn to bisect that angle of the equilateral triangle which is at the given angle, the angle contained between this line and the other side of the triangle will be one-twelfth of a right angle, or equal to one-third of the given angle.

It may be remarked, generally, that any angle which is the half, fourth, eighth, &c. part of a right angle, may be trisected by Plane Geometry.

41. Apply Euc. I. 20.

42. Let ABC, DBC be two equal triangles on the same base, of which ABC is isosceles, fig. Euc. I. 37. By producing AB and making AG equal to AB or AC, and joining GD, the perimeter of the triangle ABC may be shewn to be less than the perimeter of the triangle DBC.

43. Apply Euc. I. 20.

44. For the first case, see Theo. 32, p. 76: for the other two cases, apply Euc. I. 19.

45. This is obvious from Euc. I. 26.

46. By Euc. I. 29, 6, FC may be shewn equal to each of the lines EF, FG.

47. Join GA and AF, and prove GA and AF to be in the same straight line.

48. Let the straight line drawn through D parallel to BC meet the side AB in E, and AC in F. Then in the triangle EBD, EB is equal to ED, by Euc. I. 29, 6. Also, in the triangle EAD, the angle EAD may be shewn equal to the angle EDA, whence EA is equal to ED, and therefore AB is bisected in E. In a similar way it may be shewn, by bisecting the angle C, that AC is bisected in F. Or the bisection of AC in F may be proved when AB is shewn to be bisected in E.

49. The triangle formed will be found to have its sides respectively parallel to the sides of the original triangle.

50. If a line equal to the given line be drawn from the point where the two lines meet, and parallel to the other given line; a parallelogram may be formed, and the construction effected.

51. Let ABC be the triangle; AD perpendicular to BC, AE drawn to the bisection of BC, and AF bisecting the angle BAC. Produce AD and make DA′ equal to AD: join FA′, EA′.

52. If the point in the base be supposed to be determined, and lines drawn from it parallel to the sides, it will be found to be in the line which bisects the vertical angle of the triangle.

53. Let ABC be the triangle, at C draw CD perpendicular to CB and equal to the sum of the required lines, through D draw DE parallel to CB meeting AC in E, and draw EF parallel to DC, meeting BC in F. Then EF is equal to DC. Next produce CB, making CG equal to CE, and join EG cutting AB in H. From H draw HK perpendicular to EAC, and

HL perpendicular to BC. Then HK and HL together are equal to DC, The proof depends on Theorem 27, p. 75.

54. Let C′ be the intersection of the circles on the other side of the base, and join AC′, BC′. Then the angles CBA, C′BA being equal, the angles CBP, C′BP are also equal, Euc. I. 13: next by Euc. I. 4, CP, PC′ are proved equal; lastly prove CC′ to be equal to CP or PC′.

55. In the fig. Euc. I. 1, produce AB both ways to meet the circles in D and E, join CD, CE, then CDE is an isosceles triangle, having each of the angles at the base one-fourth of the angle at the vertex. At E draw EG perpendicular to DB and meeting DC produced in G. Then CEG is an equilateral triangle.

56. Join CC′, and shew that the angles CC′F, CC′G are equal to two right angles; also that the line FC′G is equal to the diameter.

57. Construct the figure and by Euc. I. 32. If the angle BAC be a right angle, then the angle BDC is half a right angle.

58. Let the lines which bisect the three exterior angles of the triangle ABC form a new triangle A′B′C′. Then each of the angles at A′, B′, C′ may be shewn to be equal to half of the angles at A and B, B and C, C and A respectively. And it will be found that half the sums of every two of three unequal numbers whose sum is constant, have less differences than the three numbers themselves.

59. The first case may be shewn by Euc. I. 4: and the second by Euc. I. 32, 6, 15.

60. At D any point in a line EF, draw DC perpendicular to EF and equal to the given perpendicular on the hypotenuse. With centre C and radius equal to the given base describe a circle cutting EF in B. At C draw CA perpendicular to CB and meeting EF in A. Then ABC is the triangle required.

61. Let ABC be the required triangle having the angle ACB a right angle. In BC produced, take CE equal to AC, and with center B and radius BA describe a circular arc cutting CE in D, and join AD. Then DE is the difference between the sum of the two sides AC, CB and the hypotenuse AB; also one side AC the perpendicular is given. Hence the construction. On any line EB take EC equal to the given side. ED equal to the given difference. At C, draw CA perpendicular to CB, and equal to EC, join AD, at A in AD make the angle DAB equal to ADB, and let AB meet EB in B. Then ABC is the triangle required.

62. (1) Let ABC be the triangle required, having ACB the right angle. Produce AB to D making AD equal to AC or CB: then BD is the sum of the sides. Join DC: then the angle ADC is one-fourth of a right angle, and DBC is one-half of a right angle. Hence to construct: at B in BD make the angle DBM equal to half a right angle, and at D the angle BDC equal to one-fourth of a right angle, and let DC meet BM in C. At C draw CA at right angles to BC meeting BD in A: and ABC is the triangle required.

(2) Let ABC be the triangle, C the right angle: from AB cut off AD equal to AC; then BD is the difference of the hypotenuse and one side. Join CD; then the angles ACD, ADC are equal, and each is half the supplement of DAC, which is half a right angle. Hence the construction.

63. Take any straight line terminated at A. Make AB equal to the difference of the sides, and AC equal to the hypotenuse. At B make the angle CBD equal to half a right angle, and with center A and radius AC describe a circle cutting BD in D: join AD, and draw DE perpendicular to AC. Then ADE is the required triangle.

64. Let BC the given base be bisected in D. At D draw DE at right angles to BC and equal to the sum of one side of the triangle and the perpendicular from the vertex on the base: join DB, and at B in BE make the angle EBA equal to the angle BED, and let BA meet DE in A: join AC, and ABC is the isosceles triangle.

65. This construction may be effected by means of Prob. 4, p. 71.

66. The perpendicular from the vertex on the base of an equilateral triangle bisects the angle at the vertex which is two-thirds of one right angle.

67. Let ABC be the equilateral triangle of which a side is required to be found, having given BD, CD the lines bisecting the angles at B, C. Since the angles DBC, DCB are equal, each being one-third of a right angle, the sides BD, DC are equal, and BDC is an isosceles triangle having the angle at the vertex the supplement of a third of two right angles. Hence the side BC may be found.

68. Let the given angle be taken, (1) as the *included angle* between the given sides; and (2) as the *opposite angle* to one of the given sides. In the latter case, an ambiguity will arise if the angle be an acute angle, and opposite to the less of the two given sides.

69. Let ABC be the required triangle, BC the given base, CD the given difference of the sides AB, AC: join BD, then DBC by Euc. I. 18, can be shewn to be half the difference of the angles at the base, and AB is equal to AD. Hence at B in the given base BC, make the angle CBD equal to half the difference of the angles at the base. On CB take CE equal to the difference of the sides, and with center C and radius CE, describe a circle cutting BD in D: join CD and produce it to A, making DA equal to DB. Then ABC is the triangle required.

70. On the line which is equal to the perimeter of the required triangle describe a triangle having its angles equal to the given angles. Then bisect the angles at the base; and from the point where these lines meet, draw lines parallel to the sides and meeting the base.

71. Let ABC be the required triangle, BC the given base, and the side AB greater than AC. Make AD equal to AC, and draw CD. Then the angle BCD may be shewn to be equal to half the difference, and the angle DCA equal to half the sum of the angles at the base. Hence ABC, ACB the angles at the base of the triangle are known.

72. Let the two given lines meet in A, and let B be the given point. If BC, BD be supposed to be drawn making equal angles with AC, and if AD and DC be joined, BCD is the triangle required, and the figure ACBD may be shewn to be a parallelogram. Whence the construction.

73. It can be shewn that lines drawn from the angles of a triangle to bisect the opposite sides, intersect each other at a point which is two-thirds of their lengths from the angular points from which they are drawn. Let ABC be the triangle required, AD, BE, CF the given lines from the angles drawn to the bisections of the opposite sides and intersecting in G. Produce GD, making DH equal to DG, and join BH, CH: the figure GBHC is a parallelogram. Hence the construction.

74. Let ABC (fig. to Euc. I. 20.) be the required triangle, having the base BC equal to the given base, the angle ABC equal to the given angle, and the two sides BA, AC together equal to the given line BD. Join DC, then since AD is equal to AC, the triangle ACD is isosceles, and therefore the angle ADC is equal to the angle ACD. Hence the construction.

75. Let ABC be the required triangle (fig. to Euc. I. 18), having the angle ACB equal to the given angle, and the base BC equal to the given

HL perpendicular to BC. Then HK and HL together are equal to DC. The proof depends on Theorem 27, p. 75.

54. Let C′ be the intersection of the circles on the other side of the base, and join AC′, BC′. Then the angles CBA, C′BA being equal, the angles CBP, C′BP are also equal, Euc. I. 13: next by Euc. I. 4, CP, PC′ are proved equal; lastly prove CC′ to be equal to CP or PC′.

55. In the fig. Euc. I. 1, produce AB both ways to meet the circles in D and E, join CD, CE, then CDE is an isosceles triangle, having each of the angles at the base one-fourth of the angle at the vertex. At E draw EG perpendicular to DB and meeting DC produced in G. Then CEG is an equilateral triangle.

56. Join CC′, and shew that the angles CC′F, CC′G are equal to two right angles; also that the line FC′G is equal to the diameter.

57. Construct the figure and by Euc. I. 32. If the angle BAC be a right angle, then the angle BDC is half a right angle.

58. Let the lines which bisect the three exterior angles of the triangle ABC form a new triangle A′B′C′. Then each of the angles at A′, B′, C′ may be shewn to be equal to half of the angles at A and B, B and C, C and A respectively. And it will be found that half the sums of every two of three unequal numbers whose sum is constant, have less differences than the three numbers themselves.

59. The first case may be shewn by Euc. I. 4: and the second by Euc. I. 32, 6, 15.

60. At D any point in a line EF, draw DC perpendicular to EF and equal to the given perpendicular on the hypotenuse. With centre C and radius equal to the given base describe a circle cutting EF in B. At C draw CA perpendicular to CB and meeting EF in A. Then ABC is the triangle required.

61. Let ABC be the required triangle having the angle ACB a right angle. In BC produced, take CE equal to AC, and with center B and radius BA describe a circular arc cutting CE in D, and join AD. Then DE is the difference between the sum of the two sides AC, CB and the hypotenuse AB; also one side AC the perpendicular is given. Hence the construction. On any line EB take EC equal to the given side, ED equal to the given difference. At C, draw CA perpendicular to CB, and equal to EC, join AD, at A in AD make the angle DAB equal to ADB, and let AB meet EB in B. Then ABC is the triangle required.

62. (1) Let ABC be the triangle required, having ACB the right angle. Produce AB to D making AD equal to AC or CB: then BD is the sum of the sides. Join DC: then the angle ADC is one-fourth of a right angle, and DBC is one-half of a right angle. Hence to construct: at B in BD make the angle DBM equal to half a right angle, and at D the angle BDC equal to one-fourth of a right angle, and let DC meet BM in C. At C draw CA at right angles to BC meeting BD in A: and ABC is the triangle required.

(2) Let ABC be the triangle, C the right angle: from AB cut off AD equal to AC; then BD is the difference of the hypotenuse and one side. Join CD; then the angles ACD, ADC are equal, and each is half the supplement of DAC, which is half a right angle. Hence the construction.

63. Take any straight line terminated at A. Make AB equal to the difference of the sides, and AC equal to the hypotenuse. At B make the angle CBD equal to half a right angle, and with center A and radius AC describe a circle cutting BD in D: join AD, and draw DE perpendicular to AC. Then ADE is the required triangle.

64. Let BC the given base be bisected in D. At D draw DE at right angles to BC and equal to the sum of one side of the triangle and the perpendicular from the vertex on the base: join DB, and at B in BE make the angle EBA equal to the angle BED, and let BA meet DE in A: join AC, and ABC is the isosceles triangle.

65. This construction may be effected by means of Prob. 4, p. 71.

66. The perpendicular from the vertex on the base of an equilateral triangle bisects the angle at the vertex which is two-thirds of one right angle.

67. Let ABC be the equilateral triangle of which a side is required to be found, having given BD, CD the lines bisecting the angles at B, C. Since the angles DBC, DCB are equal, each being one-third of a right angle, the sides BD, DC are equal, and BDC is an isosceles triangle having the angle at the vertex the supplement of a third of two right angles. Hence the side BC may be found.

68. Let the given angle be taken, (1) as the *included angle* between the given sides; and (2) as the *opposite angle* to one of the given sides. In the latter case, an ambiguity will arise if the angle be an acute angle, and opposite to the less of the two given sides.

69. Let ABC be the required triangle, BC the given base, CD the given difference of the sides AB, AC: join BD, then DBC by Euc. I. 18, can be shewn to be half the difference of the angles at the base, and AB is equal to AD. Hence at B in the given base BC, make the angle CBD equal to half the difference of the angles at the base. On CB take CE equal to the difference of the sides, and with center C and radius CE, describe a circle cutting BD in D: join CD and produce it to A, making DA equal to DB. Then ABC is the triangle required.

70. On the line which is equal to the perimeter of the required triangle describe a triangle having its angles equal to the given angles. Then bisect the angles at the base; and from the point where these lines meet, draw lines parallel to the sides and meeting the base.

71. Let ABC be the required triangle, BC the given base, and the side AB greater than AC. Make AD equal to AC, and draw CD. Then the angle BCD may be shewn to be equal to half the difference, and the angle DCA equal to half the sum of the angles at the base. Hence ABC, ACB the angles at the base of the triangle are known.

72. Let the two given lines meet in A, and let B be the given point. If BC, BD be supposed to be drawn making equal angles with AC, and if AB and DC be joined, BCD is the triangle required, and the figure ACBD may be shewn to be a parallelogram. Whence the construction.

73. It can be shewn that lines drawn from the angles of a triangle to bisect the opposite sides, intersect each other at a point which is two-thirds of their lengths from the angular points from which they are drawn. Let ABC be the triangle required, AD, BE, CF the given lines from the angles drawn to the bisections of the opposite sides and intersecting in G. Produce GD, making DH equal to DG, and join BH, CH: the figure GBHC is a parallelogram. Hence the construction.

74. Let ABC (fig. to Euc. I. 20.) be the required triangle, having the base BC equal to the given base, the angle ABC equal to the given angle, and the two sides BA, AC together equal to the given line BD. Join DC, then since AD is equal to AC, the triangle ACD is isosceles, and therefore the angle ADC is equal to the angle ACD. Hence the construction.

75. Let ABC be the required triangle (fig. to Euc. I. 18), having the angle ACB equal to the given angle, and the base BC equal to the given

line, also CD equal to the difference of the two sides AB, AC. If BD be joined, then ABD is an isosceles triangle. Hence the synthesis. Does this construction hold good in all cases?

76. Let ABC be the required triangle, (fig. Euc. I. 18), of which the side BC is given and the angle BAC, also CD the difference between the sides AB, AC. Join BD; then AB is equal to AD, because CD is their difference, and the triangle ABD is isosceles, whence the angle ABD is equal to the angle ADB; and since BAD and twice the angle ABD are equal to two right angles, it follows that ABD is half the supplement of the given angle BAC. Hence the construction of the triangle.

77. Let AB be the given base: at A draw the line AD to which the line bisecting the vertical angle is to be parallel. At B draw BE parallel to AD; from A draw AE equal to the given sum of the two sides to meet BE in E. At B make the angle EBC equal to the angle BEA, and draw CF parallel to AD. Then ACB is the triangle required.

78. Take any point in the given line, and apply Euc. I. 23, 31.

79. On one of the parallel lines take EF equal to the given line, and with center E and radius EF describe a circle cutting the other in G. Join EG, and through A draw ABC parallel to EG.

80. This will appear from Euc. I. 29, 15, 26.

81. Let AB, AC, AD, be the three lines. Take any point E in AC, and on EC make EF equal to EA. through F draw FG parallel to AB, join GE and produce it to meet AB in H. Then GE is equal to GH.

82. Apply Euc. I. 32, 29.

83. From E draw EG perpendicular on the base of the triangle, then ED and EF may each be proved equal to EG, and the figure shewn to be equilateral. Three of the angles of the figure are right angles.

84. The greatest parallelogram which can be constructed with given sides can be proved to be rectangular.

85. Let the lines EG and FG be drawn, as also the diagonals AC, BD, then the angle ABE is equal to the angle CBG, and when the angle ABG is added to each of these, the angles EBG, ABC, are equal in the triangles EBG, ABC, whence by Euc. I. 4, EG is proved equal to the diagonal AC. And similarly FG is proved equal to BD.

86. This problem is the same as the following; having given the base of a triangle, the vertical angle and the sum of the sides, to construct the triangle. This triangle is one half of the required parallelogram.

87. Draw a line AB equal to the given diagonal, and at the point A make an angle BAC equal to the given angle. Bisect AB in D, and through D draw a line parallel to the given line and meeting AC in C. This will be the position of the other diagonal. Through B draw BE parallel to CA, meeting CD produced in E; join AE, and BC. Then ACBE is the parallelogram required.

88. Construct the figures and by Euc. I. 24.
89. By Euc. I. 4, the opposite sides may be proved to be equal.
90. Let ABCD be the given parallelogram; construct the other
parallelogram A′B′C′D′ by drawing the lines required, also the dia-
gonals AC, A′C′, and shew that the triangles ABC, A′B′C′ are equi-
angular.
91. A′D′ and B′C′ may be proved to be parallel.
92. Apply Euc. I. 29, 32.
93. The points D, D′, are the intersections of the diagonals of two
rectangles: if the rectangles be completed, and the lines OD′, OD be
produced, they will be the other two diagonals.
94. Let the line drawn from A fall without the parallelogram, and

be the perpendiculars from C, B, D, on the line drawn
let CC′, BB′, DD′, be the perpendiculars from C, B, D, on the line drawn
from A; from B draw BE parallel to AC′, and the truth is manifest.
Next, let the line from A be drawn so as to fall within the parallelogram.
95. Let the diagonals intersect in E. In the triangles DCB, CDA,
two angles in each are respectively equal and one side DE: wherefore
the diagonals DB, AC are equal: also since DE, EC are equal, it follows
that EA, EB are equal. Hence DEC, AEB are two isosceles triangles
having their vertical angles equal, wherefore the angles at their bases
are equal respectively, and therefore the angle CDB is equal to DBA.
96. (1) By supposing the point P found in the side AB of the paral-
lelogram ABCD, such that the angle contained by AP, PC may be bisected
by the line PD; CP may be proved equal to CD; hence the solution.
(2) By supposing the point P found in the side AB produced, so that
PD may bisect the angle contained by ABP and PC; it may be shewn
that the side AB must be produced, so that BP is equal to BD.
97. This may be shewn by Euc. I. 35.
98. Let D, E, F be the bisections of the sides AB, BC, CA of the
triangle ABC: draw DE, EF, FD; the triangle DEF is one-fourth of the
triangle ABC. The triangles DBE, FBE are equal, each being one-fourth
of the triangle ABC: DF is therefore parallel to BE, and DBEF is a
parallelogram of which DE is a diagonal.
99. This may be proved by applying Euc I. 38.
100. Apply Euc. I. 37, 38.
101. On any side BC of the given triangle ABC, take BD equal to the
given *base*; join AD, through C draw CE parallel to AD, meeting BA pro-
duced if necessary in E, join ED; then BDE is the triangle required.
By a process somewhat similar the triangle may be formed when the *al-
titude* is given.
102. Apply the preceding problem (101) to make a triangle equal to
one of the given triangles and of the same altitude as the other given tri-
angle. Then the sum or difference can be readily found.
103. First construct a triangle on the given base equal to the given
triangle; next form an isosceles triangle on the same base equal to this
triangle.
104. Make an isosceles triangle equal to the given triangle, and
then this isosceles triangle into an equal equilateral triangle.
105. Make a triangle equal to the given parallelogram upon the
given line, and then a triangle equal to this triangle, having an angle
equal to the given angle.
106. If the figure ABCD be one of four sides; join the opposite
angles A, C of the figure, through D draw DE parallel to AC meeting
BC produced in E, join AE:—the triangle ABE is equal to the four-
sided figure ABCD.
If the figure ABCDE be one of five sides, produce the base both ways,
and the figure may be transformed into a triangle, by two constructions
similar to that employed for a figure of four sides. If the figure consists
of six, seven, or any number of sides, the same process must be repeated.
107. Draw two lines from the bisection of the base parallel to the
two sides of the triangle.
108. This may be shewn ex absurdo.
109. On the same base AB, and on the same side of it, let two triangles
ABC, ABD be constructed, having the side BD equal to BC, the angle
ABC a right angle, but the angle ABD not a right angle; then the triangle
ABC is greater than ABD, whether the angle ABD be acute or obtuse.
110. Let ABC be a triangle whose vertical angle is A, and whose

base BC is bisected in D; let any line EDG be drawn through D, meeting AC the greater side in G and AB produced in E, and forming a triangle AEG having the same vertical angle A. Draw BH parallel to AC, and the triangles BDH, GDC are equal. Euc. I. 26.

111. Let two triangles be constructed on the same base with equal perimeters, of which one is isosceles. Through the vertex of that which is not isosceles draw a line parallel to the base, and intersecting the perpendicular drawn from the vertex of the isosceles triangle upon the common base. Join this point of intersection and the extremities of the base.

112. (1) DF bisects the triangle ABC (fig. Prop. 6, p. 73.) On each side of the point F in the line BC, take FG, FH, each equal to one-third of BF, the lines DG, DH shall trisect the triangle. Or,

Let ABC be any triangle, D the given point in BC. Trisect BC in E, F. Join AD, and draw EG, FH parallel to AD. Join DG, DH; these lines trisect the triangle. Draw AE, AF and the proof is manifest.

(2) Let ABC be any triangle; trisect the base BC in D, E, and join AD, AE. From D, E, draw DP, EP parallel to AB, AC and meeting in P. Join AP, BP, CP; these three lines trisect the triangle.

(3) Let P be the given point within the triangle ABC. Trisect the base BC in D, E. From the vertex A draw AD, AE, AP. Join PD, draw AG parallel to PD and join PG. Then BGPA is one-third of the triangle. The problem may be solved by trisecting either of the other two sides and making a similar construction.

113. The base may be divided into nine equal parts, and lines may be drawn from the vertex to the points of division. Or, the sides of the triangle may be trisected, and the points of trisection joined.

114. It is proved, Euc. I. 34, that each of the diagonals of a parallelogram bisects the figure, and it may be shewn that they also bisect each other. It is hence manifest that any straight line, whatever may be its position, which bisects a parallelogram, *must* pass through the intersection of the diagonals.

115. See the remark on the preceding problem 114.

116. Trisect the side AB in E, F, and draw EG, FH parallel to AD or BC, meeting DC in G and H. If the given point P be in EF, the two lines drawn from P through the bisections of EG and FH will trisect the parallelogram. If P be in FB, a line from P through the bisection of FH will cut off one-third of the parallelogram, and the remaining trapezium is to be bisected by a line from P, one of its angles. If P coincide with E or F, the solution is obvious.

117. Construct a right-angled parallelogram by Euc. I. 44, equal to the given quadrilateral figure, and from one of the angles, draw a line to meet the opposite side and equal to the base of the rectangle, and a line from the adjacent angle parallel to this line will complete the rhombus.

118. Bisect BC in D, and through the vertex A, draw AE parallel to BC, with center D and radius equal to half the sum of AB, AC, describe a circle cutting AE in E.

119. Produce one side of the square till it becomes equal to the diagonal, the line drawn from the extremity of this produced side and parallel to the adjacent side of the square, and meeting the diagonal produced, determines the point required.

120. Let fall upon the diagonal perpendiculars from the opposite angles of the parallelogram. These perpendiculars are equal, and each pair of triangles is situated on different sides of the same base and has equal altitudes. If the point be not on the diagonal, draw through the given point, a line parallel to a side of the parallelogram.

121. One case is included in Theo. 120. The other case, when the point is in the diagonal produced, is obvious from the same principle.

122. The triangles DCF, ABF may be proved to be equal to half of the parallelogram by Euc. I. 41.

123. Apply Euc. I. 41, 38.

124. If a line be drawn parallel to AD through the point of intersection of the diagonal, and the line drawn through O parallel to AB; then by Euc. I. 43, 41, the truth of the theorem is manifest.

125. It may be remarked that parallelograms, are divided into pairs of equal triangles by the diagonals, and therefore by taking the triangle ABD equal to the triangle ABC, the property may be easily shewn.

126. The triangle ABD is one half of the parallelogram ABCD, Euc. I. 34. And the triangle DKC is one half of the parallelogram CDHG, Euc. I. 41, also for the same reason the triangle AKB is one half of the parallelogram AHGB: therefore the two triangles DKC, AKB are together one half of the whole parallelogram ABCD. Hence the two triangles DKC, AKB are equal to the triangle ABD: take from these equals the equal parts which are common, therefore the triangle CKF is equal to the triangles AHK, KBD: wherefore also taking AHK from these equals, then the difference of the triangles CKF, AHK is equal to the triangle KBC: and the doubles of these are equal, or the difference of the parallelograms CFKG, AHKE is equal to twice the triangle KBD.

127. First prove that the perimeter of a square is less than the perimeter of an equal rectangle: next, that the perimeter of the rectangle is less than the perimeter of any other equal parallelogram.

128. This may be proved by shewing that the area of the isosceles triangle is greater than the area of any other triangle which has the same vertical angle, and the sum of the sides containing that angle is equal to the sum of the equal sides of the isosceles triangle.

129. Let ABC be an isosceles triangle (fig. Euc. I. 42), AE perpendicular to the base BC, and AECG the equivalent rectangle. Then AC is greater than AE, &c.

130. Let the diagonal AC bisect the quadrilateral figure ABCD. Bisect AC in E, join BE, ED, and prove BE, ED in the same straight line and equal to one another.

131. Apply Euc. I. 15.

132. Apply Euc. I. 20.

133. This may be shewn by Euc. I. 20.

134. Let AB be the longest and CD the shortest side of the rectangular figure. Produce AD, BC to meet in E. Then by Euc. I. 32.

135. Let ABCD be the quadrilateral figure, and E, F, two points in the opposite sides AB, CD, join EF and bisect it in G; and through G draw a straight line HGK terminated by the sides AD, BC; and bisected in the point G. Then EF, HK are the diagonals of the required parallelogram.

136. After constructing the figure, the proof offers no difficulty.

137. If any line be assumed as a diagonal, if the four given lines taken two and two be always greater than this diagonal, a four-sided figure may be constructed having the assumed line as one of its diagonals: and it may be shewn that when the quadrilateral is possible, the sum of every three given sides is greater than the fourth.

138. Draw the two diagonals, then four triangles are formed, two on one side of each diagonal. Then two of the lines drawn through the points of bisection of two sides may be proved parallel to one diagonal, and two

parallel to the other diagonal, in the same way as Theo. 97, supra The other property is manifest from the relation of the areas of the triangles made by the lines drawn through the bisections of the sides.

139. It is sufficient to suggest, that triangles on equal bases, and of equal altitudes, are equal.

140. Let the side AB be parallel to CD, and let AB be bisected in E and CD in F, and let EF be drawn. Join AF, BF, then Euc. I. 38.

141. Let BCED be a trapezium of which DC, BE are the diagonals intersecting each other in G. If the triangle DBG be equal to the triangle EGC, the side DE may be proved parallel to the side BC, by Euc. I. 39.

142. Let ABCD be the quadrilateral figure having the sides AB, CD, parallel to one another, and AD, BC equal. Through B draw BE parallel to AD, then ABED is a parallelogram.

143. Let ABCD be the quadrilateral having the side AB parallel to CD. Let E, F be the points of bisection of the diagonals BD, AC, and join EF and produce it to meet the sides AD, BC in G and H. Through H draw LHK parallel to DA meeting DC in L and AB produced in K. Then BK is half the difference of DC and AB.

144. (1) Reduce the trapezium ABCD to a triangle BAE by Prob. 106, supra, and bisect the triangle BAE by a line AF from the vertex. If F fall without BC, through F draw FG parallel to AC or DE, and join AG.

Or thus. Draw the diagonals AC, BD: bisect BD in E, and join AE, EC. Draw FEG parallel to AC the other diagonal, meeting AD in F, and DC in G. AG being joined, bisects the trapezium.

(2) Let E be the given point in the side AD. Join EB. Bisect the quadrilateral EBCD by EF. Make the triangle EFG equal to the triangle EAB, on the same side of EF as the triangle AB. Bisect the triangle EFG by EH. EH bisects the figure.

145. If a straight line be drawn from the given point through the intersection of the diagonals and meeting the opposite side of the square; the problem is then reduced to the bisection of a trapezium by a line drawn from one of its angles.

146. If the four sides of the figure be of different lengths, the truth of the theorem may be shewn. If, however, two adjacent sides of the figure be equal to one another, as also the other two, the lines drawn from the angles to the bisection of the longer diagonal, will be found to divide the trapezium into four triangles which are equal in area to one another. Euc. I. 38.

147. Apply Euc. I. 47, observing that the shortest side is one half of the longest.

148. Find by Euc. I. 47, a line the square on which shall be seven times the square on the given line. Then the triangle which has these two lines containing the right angle shall be the triangle required.

149. Apply Euc. I. 47.

150. Let the base BC be bisected in D, and DE be drawn perpendicular to the hypotenuse AC. Join AD: then Euc. I. 47.

151. Construct the figure, and the truth is obvious from Euc. I. 47.

152. See Theo. 32, p. 76, and apply Euc. I. 47.

153. Draw the lines required and apply Euc. I. 47.

154. Apply Euc. I. 47.

155. Apply Euc. I. 47.

156. Apply Euc. I. 47, observing that the square on any line is four times the square on half the line.

157. Apply Euc. I. 47, to express the squares of the three sides in erms of the squares on the perpendiculars and on the segments of AB.

158. By Euc. I. 47. bearing in mind that the square described on any ne is four times the square described upon half the line.

159. The former part is at once manifest by Euc. I. 47. Let the diaonals of the square be drawn, and the given point be supposed to coincide ith the intersection of the diagonals, the minimum is obvious. Find its alue in terms of the side.

160. (a) This is obvious from Euc. I. 13.

(b) Apply Euc. I. 32, 29.

(c) Apply Euc. I. 5, 29.

(d) Let AL meet the base BC in P, and let the perpendiculars from C meet BC produced in M and N respectively; then the triangles B, FMB may be proved to be equal in all respects, as also APC, CKN.

) Let fall DQ perpendicular on FB produced. Then the triangle B may be proved equal to each of the triangles ABC, DBF; whence he triangle DBF is equal to the triangle ABC.

Perhaps however the better method is to prove at once that the trigles ABC, FBD are equal, by shewing that they have two sides equal n each triangle, and the included angles, one the supplement of the other.

(f) If DQ be drawn perpendicular on FB produced, FQ may be roved to be bisected in the point B, and DQ equal to AC. Then the quare on FD is found by the right-angled triangle FQD. Similarly, the quare on KE is found, and the sum of the squares on FD, EK, GH will be nd to be six times the square on the hypotenuse.

(g) Through A draw PAQ parallel to BC and meeting DB, EC duced in P, Q. Then by the right angled triangles.

161. Let any parallelograms be described on any two sides AB, AC a triangle ABC, and the sides parallel to AB, AC be produced to meet n a point P. Join PA. Then on either side of the base BC, let a parallelogram be described having two sides equal and parallel to AP. Produce AP and it will divide the parallelogram on BC into two parts respectively equal to the parallelograms on the sides. Euc. I. 35, 36.

162. Let the equilateral triangles ABD, BCE, CAF be described on AB, BC, CA, the sides of the triangle ABC having the right angle at A. Join DC, AK: then the triangles DBC. ABE are equal. Next draw DG perpendicular to AB and join CG: then the triangles BDG, DAG, DGC are equal to one another. Also draw AH, EK perpendicular to BC; the triangles EKH, EKA are equal. Whence may be shewn that he triangle ABD is equal to the triangle BHE, and in a similar way may e shewn that CAF is equal to CHE.

The restriction is unnecessary: it only brings AD, AE into the same ne.

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