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THE PRINCIPLES AND PRACTICE OF STATICS AND DYNAMICS,

EMBRACING A CLEAR DEVELOPMENT OF

HYDROSTATICS, HYDRODYNAMICS, AND PNEUMATICS:

WITH

CENTRAL FORCES AND SUPER-ELEVATION OF EXTERIOR RAIL.

FOR THE USE OF SCHOOLS AND PRIVATE STUDENTS.

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PREFACE.

In the following work, a great part of which is necessarily a compilation from the numerous Authors that have preceded me, will be found numerous new rules and formulæ adapted to the practical purposes of the present engineering age, many of which are not found in any other work of this kind.

All the fundamental principles of this work are rigidly demonstrated on the most elementary principles, and chiefly after the manner of the most approved authors of English works, excepting for the rotation of bodies, where *D'Alembert's* simple and elegant principle is adopted.

When the great quantity of matter, the numerous engravings, and small price of this work, are considered, I trust that no apology will be necessary for adding it to the great number of similar works that have preceded it.

A part of the engravings used in this work, are taken from *Tomlinson's Rudimentary Mechanics* (the present Series), for the use of beginners, which I would recommend the student to read carefully previous to studying this work, as that work contains a clear and popular exposition of a great many of the leading subjects, of which I have here treated in a more strictly scientific manner.

T. Baker.

CONTENTS.

PRINCIPLES AND PRACTICE

OF

STATICS AND DYNAMICS.

DEFINITIONS.

(1.) *Statics* treats of the laws of equilibrium of solid bodies.

(2.) *Dynamics* investigates the laws of motion of solid bodies.

(3.) *Hydrostatics* has for its object the laws of equilibrium of fluid bodies.

(4.) *Hydrodynamics* treats of the laws of motion of fluid bodies.

(5.) *Pneumatics* is a branch of Hydrostatics, and relates to properties and equilibrium of elastic fluids, such as common air and the gases.

1. *Motion* is a continual change of the place of a body.

NOTE.—If a body moves through equal spaces in equal times, it is called *equable motion*. If its motion continually increases or decreases, it is respectively called *accelerated* or *retarded motion*.

2. *Rest* is a permanency of a body in the same place.

3. *Matter* is the substance that affects our senses.

NOTE.—*Bodies* are certain portions of matter limited in magnitude. *Mass* is the quantity of matter of which a body is composed. An *elementary particle* is a body indefinitely small. The space occupied by a body is called its *volume* or *solid content*.

4. *Density* of a body is the proportional quantity of matter contained in it, to the quantity of matter contained in another body of the same magnitude; and it is called *uniform* when equal quantities of matter are contained in equal magnitudes.

5. *Force* is a power that tends to impress or destroy motion.

NOTE 1.—There are no means of estimating *force* except by its effects It is differently measured in Statics and Dynamics: in Statics, it is measured by the *pressure*, which it causes a body at rest to exert against another body with which it is in contact, or with which it is connected. The pressures exerted by means of cords pulled by any forces are called *tensions*. In Dynamics, force is measured by the velocity uniformly generated in a given time. See *Definitions* in *Dynamics*.

NOTE 2.—It is usual to represent forces or pressures by lines, the direction of the line coinciding with the direction of the force, and the length of the line expressing the amount or magnitude of the given force or pressure.

6. *Gravity* is the force by which bodies tend to descend in the direction of the centre of the earth: thus, gravity urges the fall of a stone, when left unsupported.

7. *Power* and *weight*, when opposed to one another, signify the body that moves and the body to be moved; *i. e.* the body that gives the motion is called the power, and that which receives the motion is called the weight.

8. *Velocity* is the swiftness or slowness of the motion of a body, and is *measured* by the space *uniformly* described in a unit of time, as for instance, in one second of time.

9. The *momentum* of a body is the product of its velocity and quantity of matter.

NOTE.—The remainder of the definitions, adapted to this work, will be found under the head of Dynamics, Part II., as placing them here would only tend to perplex the student.

PART I.

STATICS.

ON THE COMPOSITION AND RESOLUTION OF FORCES.

10. PROPOSITION.—Let A B, A C (see Note 2, Art. 5.) represent two forces acting on a point or particle A, then these forces will be proportional to the velocities communicated to the particle A in their respective directions, and consequently to the spaces which it would uniformly describe in a given time. Complete the parallelogram A B D C, then the motion in the direction A C, can neither accelerate nor repel the approach of

the body or particle to the line B D, which is parallel to A C; hence the body will arrive at B D in the same time that it would have done if no motion had been given to it in the direction A C. In the same manner, the motion in the direction A B can neither accelerate nor retard the approach of the body to the line C D; therefore, in consequence of the motion in the direction A C, it will arrive in the same time that it would have done if no motion had been given to it in the direction A B. It hence follows that, by the joint effect of the two motions, the body will be found both in B D and C D at the end of this time, and will therefore be found at D, the point of their intersection: consequently, by the simultaneous action of the two motions, the body will evidently describe the diagonal A D of the parallelogram. And since A B, A C, A D, represent the spaces uniformly moved over by the body A in the same time, they are proportional to the forces acting in these directions; that is, the forces A B, A C, acting at the same time, produce a force which is represented in magnitude and direction by A D.

11. COROLLARY 1.—Hence, if any two forces act from the same point, the force which is equivalent to these two is expressed in *direction* and *magnitude* by the diagonal of the parallelogram, the sides of which represent the direction and magnitude of the two forces.

12. COR. 2.—The force in the direction A D is called the *resultant* of the two forces in the directions A B, A C; and the forces in the directions A B, A C, are called the *components* of the force in A D.

13. COR. 3.—A force represented in magnitude and direction by A S, which is equal to and directly opposed to A D, will evidently just balance the forces A B, A C.

14. COR. 4.—If A B, taken from a scale of equal parts, represents the magnitude and direction of one of the *component* forces or weights in pounds, cwts, &c., and A C, taken from the same scale, represents the magnitude and direction of the other *component* force or weight in pounds, cwts, &c.; then, if on the two lines A B, A C, the parallelogram A B D C be constructed, the diagonal A D will be the *direction* and *magnitude* of the *resultant* force or weight, and its length, taken from the same scale, will give the pounds, cwts, &c., in the resultant force or weight.

15. Cor. 5.—Let the component A B = P pounds, the component A C = Q pounds, and the resultant A D = W pounds; also let the angle B A C = α, then, by trigonometry, $AD^2 = AC^2 + CD^2 (= AB^2) + 2 AC \times CD \cos BAC$*,

$$\text{that is } W^2 = P^2 + Q^2 + 2PQ \cos \alpha,$$

$$\text{or } W = \sqrt{(P^2 + Q^2 + 2PQ \cos \alpha)};$$

also, to find the angle B A D, we have

$$W : Q :: \sin \alpha : \sin BAD,$$

$$\text{or, } \sin BAD = \frac{Q \sin \alpha}{W}.$$

Example.—Two forces of 4 and 5 tons act in directions inclined to each other at an angle of 60°; it is required to find the weight of the resultant force, and its inclination to the greater of the component forces.

Let P = 5 tons, Q = 4 tons, and W = resultant force or weight, then $W = \sqrt{(P^2 + Q^2 + 2PQ \cos 60^\circ)} = \sqrt{(25 + 16 + 2 \times 4 \times 5 \times \frac{1}{2})} = \sqrt{61} = 7{\cdot}81$ tons = required force, and $\sin BAD = \dfrac{Q \sin \alpha}{W} = {\cdot}443 = \sin 26^\circ\ 20'$ = inclination to the greater force.

16. Cor. 6.—If three forces acting on a point, keep it at rest, each of these forces is proportional to the sine of the angle made by the other two. Let the forces or weights P and Q be components of the force or weight W, and let the force or weight R, represented by A S, be equal to and directly opposite to W; then since the forces P and Q balance the force R, the force W will also balance R; whence, by the last corollary,

$$W : P :: \sin \alpha : \sin CAD \text{ or } \sin CAS$$

$$W : Q :: \sin \alpha : \sin BAD \text{ or } \sin BAS$$

$$\therefore R : P : Q :: \sin \alpha : \sin CAS : \sin BAS.$$

17. Cor. 7.—If the three sides of any triangle be parallel to three forces, which, acting on a point, keep it at rest, these three forces will be proportional to the sides of the triangle. For the forces P, Q, and R keep the point or particle A at rest, and these forces are proportional to the sides of the triangle A B D.

* Because angle ACD = 180° − α = sup. of BAC, and cos ACD = − cos α, which is the angle actually used in this and the following formulæ.

18. Cor. 8.—If two forces P and Q act in the same or in opposite directions, their resultant will be respectively equal to $P + Q$ or to $P - Q$. This is self evident from daily experience. This may also be proved by making the line $AC = Q$ revolve to the right or to the left till it coincide with, or be opposite to, the direction of $AB = P$, the resultant or diagonal being respectively $P + Q$ or $P - Q$; thus further establishing the truth of the doctrine of the composition of forces.

19. Scholium.—The proposition, (Art. 10.) which has just been demonstrated, is generally known by the name of the *parallelogram of forces*, and is the foundation of the whole doctrine of equilibrium. Various demonstrations of this important proposition have been given by the most eminent mathematicians, such as D. Bernouilli, Dalembert, Laplace, and Poisson; but they are all of too abstruse a nature to be introduced in a work of this kind. The demonstration of the same proposition by Duchayla, though of an elementary character, and founded on self-evident principles, is at the same time abstruse and circuitous. The author has, therefore, here introduced the proof usually given by English mathematicians, which, from its extreme simplicity, may be considered as well adapted to those who are only commencing the study of the subjects treated of in this work.

20. Problem.—*If any number of forces* P, P′, P″, *&c., act in the same plane, in given directions on the point* A, *it is required to find the magnitude and direction of a single force which shall be equal to them all.*

This force may be easily found by geometrical construction from Art. 11. First, describe a parallelogram the sides of which represent two of the forces, and its diagonal will be the equivalent or resultant of these two forces. Draw a new parallelogram, with this diagonal and the line which represents the third force for its sides, and the new diagonal will be the resultant of the three first forces. Proceed in this manner till all the forces be included, and the last diagonal will be the equivalent or resultant of all the forces. But the following method is much better adapted to calculation and general practical purposes.

Let A be the point on which all the forces act. Draw any two lines Ax, Ay, through A, in the plane of the forces at right angles to each other. Let the force P be represented in magnitude and direction by AP; through P draw PB, PC perpendicular to Ax, Ay respectively, then ABPC is a parallelogram, and the force AP is equal to the two forces AB, AC, acting in the direction Ax, Ay. Similarly each of the other forces P′, P″, &c., may be resolved into two others in the directions Ax, Ay. Let the angles PAx, $P'Ax$, &c., be respectively denoted by α, α', &c., then $AB = P \cos \alpha$, $AC = P \sin \alpha$, which are the components of the force P in the direction Ax, Ay, respectively. In like manner the components of P′, P″, &c. in the direction Ax, are $P' \cos \alpha'$, $P'' \cos \alpha''$, &c.; and the components of the same forces in the direction Ay are $P' \sin \alpha'$, $P'' \sin \alpha''$, &c. Now, by putting X for the sum of all the forces in the direction Ax, and Y for the sum of all the forces in the direction Ay, there will result

$$X = P \cos \alpha + P' \cos \alpha' + P'' \cos \alpha'' + \&c. \qquad (1)$$

$$\text{and } Y = P \sin \alpha + P' \sin \alpha' + P'' \sin \alpha'' + \&c. \qquad (2)$$

Put R = resultant of all these forces and ϕ the angle it makes with Ax, then

$$R = \sqrt{X^2 + Y^2}; \quad (3) \qquad \tan \phi = \frac{Y}{X}. \quad (4)$$

NOTE.—It must be remembered that if any of the component forces AB, AB′, &c., and AC, AC′, &c., be estimated in an opposite direction from A, they must be considered negative.

21. COR.—When there is an equilibrium, the resultant or equivalent of all the forces = 0, and $\therefore R = \sqrt{X^2 + Y^2} = 0$; hence X and Y are each = 0, or their values in equation (1) and (2) vanish, on account of the forces counteracting each other.

Ex. 1.—It is required to determine geometrically and by computation, the resultant and direction of the four pressures or forces, P, P′, P″, P‴, all applied to the point A and acting in the same plane; the several forces being $P = 24$ tons, $P' = 18$, $P'' = 32$, and $P''' = 30$, and the angles which their respective directions make with a given line B A C being 77°, 37°, 9°, and 312°.

Geometrically. The method of solving this question geometrically is already pointed out in Art. 20, *i. e.*, by finding the resultant of two of the forces, which may be considered as a new force, then by finding the resultant of this new force and the line which expresses the third force, and so on till all the four forces shall be reduced to one force, which will be the resultant of the four given forces or pressures, and will be found $= 73{\cdot}4$, the number of tons required, its inclination to B A C being 13°.

Calculation. The pressures P, P′, P″ being all in the first quadrant of the circle their sines and cosines must be *positive*, but the pressure P‴ being in the fourth quadrant its sine must be taken *negative;* then the values of X and Y, Form. (1) and (2), being substituted in Form. (3), there will result

$$R = \sqrt{X^2 + Y^2} = \sqrt{\{(24 \times {\cdot}9744 + 18 \times {\cdot}6018 + 32 \times {\cdot}1564 - 30 \times {\cdot}7431)^2 + (24 \times {\cdot}225 + 18 \times 7986 + 32 \times {\cdot}9877 + 30 \times 6691)^2\}} = 73{\cdot}427 \text{ tons},$$

and from Form. (4)

$$\tan \phi = \frac{X}{Y} = {\cdot}23105 = \tan 13^\circ,$$

which are the values of the resultant and its angle of inclination to the given line B A C.

Ex. 2.—Four forces in the same plane are 3, 4, 5, and 6 cwt., acting upon a given point, and are inclined to a given line at angles of 20°, 40°, 80°, and 150°, respectively;

required the magnitude and direction of another force which shall just counteract or balance these four forces.

Ans. The force is $11\frac{5}{8}$ cwt., and inclined $80^\circ\ 17'$ to the given line.

Ex. 3.—In pulling a weight along the ground by a cord, inclined to the horizon at an angle of 45°, a power of 80 lbs. was exerted; required the force with which the body was dragged horizontally.

Here the resultant, which represents the force of 80 lbs., is resolved into two other equal forces, the one parallel with, and the other perpendicular to, the horizon, which last force is wholly inefficient in acting on the body, and may, therefore, be considered as lost; whence the required force is readily found to be $56\frac{1}{2}$ lbs. nearly.

Ex. 4.—Two equal forces act at an angle 120°; prove that their resultant is equal to one of the equal forces.

Ex. 5.—A weight of 20 lbs., suspended by a cord from a fixed point, is drawn by the hand in the plane of suspension through an angle of 30°; required the pressure at the point of suspension, and the force exerted by the hand.

Ans. $23\frac{1}{10}$ lbs. and $11\frac{11}{20}$ lbs. nearly.

Ex. 6.—If any number of forces acting on a point, be represented by the sides of a polygon taken in order, these forces will keep the point at rest; required the proof when the polygon is in one plane.

Ex. 7.—A boat is fastened to a fixed point, and is acted on at the same time by the wind and the current. Now the wind is S. E., the direction of the current S., and the direction of the boat from the point P, S. 20° W., also the pressure on P is 300 lbs.; it is required to find the forces of the wind and the current.

Ans. Force of the wind 145 lbs.; of the current 384 lbs.

Ex. 8.—Two unequal forces P and Q act at an angle of 120°; prove that their resultant is $= \sqrt{P^2 - PQ + Q^2}$.

22. Prop.—*If the directions of three forces meet in one point, and if their magnitudes be represented by the three contiguous edges of a parallelopiped, their resultant will be represented, both in magnitude and direction, by the diagonal drawn from their point of meeting to the opposite angle of the parallelopiped.*

Let the magnitudes and directions of the three forces be represented by A B, A C, A D, and let the parallelopiped be A G. Then, since A B H C is a parallelogram, the force A H is the resultant of the two forces A B, A C; but A D G H is a parallelogram, and its diagonal A G is the resultant of the two forces A D, A H; that is, of the three forces A B, A C, A D.

23. Cor. 1.—If A S be prolonged till it be equal to A G, then A S represents the magnitude and direction of a force, that will hold the three forces A B, A C, A D in equilibrium, because it is equal and opposite to the resultant A G of these three forces.

24. Cor. 2.—If four forces in different planes act upon a point or body and keep it in equilibrium, these four forces are proportional to the three edges and diagonal of a parallelopiped, formed on lines respectively parallel to the directions of the forces.

25. Cor. 3.—Hence a single force may be resolved into three others in different planes; and each of these may be resolved again into others, either in the same or different planes, and so on to any extent.

26. Schol.—The properties in the preceding propositions and their corollaries hold good for all similar forces acting on one point or body, whether they act by drawing or pressing, or whether they be instantaneous or continual, as in the cases of percussion and gravity, and are of the utmost importance in the application of forces to mechanics and natural philosophy. The properties of several forces in different planes may be developed analytically by means of three co-ordinate planes as in Art. 20, where several forces in the same plane are developed by means of rectangular coordinates. This subject shall be resumed further on, in order that the student may proceed to those parts of statics, which are of real utility, and not requiring at the same time a knowledge of the geometry of three dimensions; his studies, the author trusts, will thus be rendered more easy and interesting.

THE PRINCIPLE OF THE EQUALITY OF MOMENTS.

27. DEFINITION 1.—The product of a force and the perpendicular distance of a given point from its direction, is called the *moment of the force* with respect to that point.

28. DEF. 2.—If through the point an axis be drawn perpendicular to the plane, passing through the point and the direction of the force, this product is called the *moment of the force* as it respects the axis.

29. PROP.—*The sum of the moments of any number of forces that tend to turn a body in one direction, is equal to the sum of the moments of any number of forces that tend to turn the body in the opposite direction, all the forces, in both cases, being supposed to be in equilibrium.*

Let the three forces A B, A D, A S, in the same plane, keep the body A in equilibrium; draw the parallelogram ABCD, the diagonal A C of which will be the resultant of A B, A D and equal and opposite, to SA. Let the point P be taken in the plane of the three forces, and join A P,

D P, C P, and from P let fall the perpendiculars Pa, Pb, Pc, Pd, on AB, AD, AS, or on their prolongations; then the quadrilateral PADC is = triangle APD + triangle PDC; and

the area of $\triangle PAC = PAD + PDC - ACD$ (1)

$$\triangle PAC = \frac{AC \times Pb}{2},$$

$$\triangle PAD = \frac{AD \times Pd}{2}$$

$$\triangle PDC = \frac{DC \times Pc}{2}$$

$$\triangle ADC = \frac{DC \times ac}{2}$$

Hence, by substituting these four values in (1.), there results

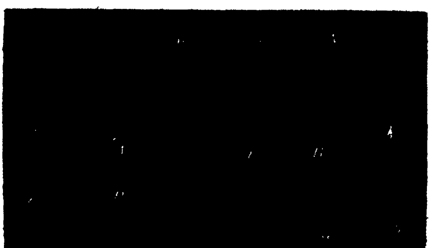
$$AC \times Pb = AS \times Pb = AD \times Pd + DC \times Pc - DC \times ac$$
$$\text{————————} = AD \times Pd + DC\,(Pc - ac)$$
$$\text{————————} = AD \times Pd + AB \times Pa.$$

Hence we see that when a body or point A is kept in equilibrium by three forces A B, A D, A S, the sum of the moments $AB \times Pa + AD \times Pd$, which tend to turn the body in one direction, is equal to the moment $AS \times Pb$, which tends to turn the body in the opposite direction, and since all those forces may be resolved into innumerable other forces, the proposition is true for any number of forces.

30. Cor. 1.—Hence the moment of the resultant is equal to the sum of the moments of its components.

31. Prop.—*To find the resultant of two parallel forces acting perpendicularly at the ends of a rigid straight rod, and the moments of these two forces.*

Let A B represent the rod, (supposed to be without weight,) A P, B Q the magnitudes of the two forces P, Q; and let two opposite and equal forces S, T, expressed by A S, B T be applied to the extremities A and B of the rod, and in the prolongation of its direction. Then, since these two assumed equal forces S, T evidently balance each other, the resultant of the two forces P, Q will be the same as that of the four forces P, Q, S, T. Complete the parallelograms $aPAS'$, $bQBT$; then the resultant of the two forces AP, AS is Aa and the resultant of the two forces BQ, BT is Bb; but, since the forces S, T counteract each other, the two forces P, Q are evidently equivalent to the two forces aA, bB, which act obliquely at the ends of the



rod A B. Prolong aA, bB till they meet at K, and through K draw MN parallel to AB, and KF parallel to AP or BQ. The force Aa will produce the same effect as an equal force at K acting in the direction KA: and this force may be resolved into two others, one in the direction KM, equal and parallel to AS, and the other in the direction KF, equal and parallel to AP. In the same way the force Bb may be removed to K, and resolved into two forces, one in the direction KN, equal and parallel to BT, and the other in the direction KF, equal and parallel to BQ. Thus the four forces P, Q, S, T, may be considered as acting at K, of which the two forces S, T, being equal and opposite, will counterbalance each other, and therefore produce no effect, while the other two P, Q, acting in the direction KF, will produce a resultant equal to their sum P + Q or equal to their representatives AP, BQ. Also by similar triangles.

$$AP : AS\ (= aP) :: KF : AF$$
$$BT\ (= Qb) :: BQ :: BF : KF$$
$$\therefore AP : BQ :: BF : AF.$$

But, AP = P and BQ = Q, therefore,

$$P : Q :: BF : AF \text{ and}$$
$$\therefore P \times AF = Q \times BF.$$

Hence, if the line or rod AB be divided in F inversely as the forces P and Q, their moments estimated from F will be equal, and if an axis pass through the point F, the forces P and Q will sustain each other in equilibrium on the the rod.

32. Cor.—If an axis pass through B, and P + Q = R = resultant of the forces P and Q, then, by compounding the last proportion,

$$P : R :: AF : AB.$$

33. Prop.—*The sum of the moments of two parallel forces is equal to the moment of their resultant.*

Let P and Q be two parallel forces, acting in the same direction on the line AB; and R their resultant, acting at F. From any point D draw Db perpendicular to their directions.

Put D$a = p$, D$b = q$, and D$f = r$, then by Art. 31.

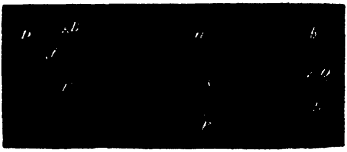
$$P : Q :: fb : af :: FB : AF;$$

$$\therefore P \times af = Q \times fb, \text{ or } P(r - p) = Q(q - r),$$

$$\text{whence } Pp + Qq = (P + Q)r = Rr.$$

And if the point D′ be taken any where between a and b, it will be found that $Qp - Pp = Rr$. In this case D′a or p is measured in the opposite direction from D′, and is therefore to be considered negative.

34. Cor. 1.—If one of the forces, as Q, act in an opposite direction, the force P being now removed to A, and the resultant or fulcrum to F nearest to D,



then $P \times af = Q \times fb$, or $P(p - r) = Q(q - r)$

$$\therefore Pp - Qq = (P - Q)r = Rr.$$

Hence in all cases the sum of the moments of two parallel forces is equal to the moment of their resultant, recollecting that the signs of those forces that act in opposite directions must be considered negative, as well as the signs of those forces that are estimated in an opposite direction from D.

35. Cor. 2.—Hence the resultant of any number of parallel forces may be easily found. Let R represent a force equal and opposite to the resultant of the parallel forces P, Q, S, &c.; if all these forces be moved parallel to their directions till they coincide with their resultant R, they will be in equilibrium with the force R, therefore,

$$R = P - Q + S + \&c.$$

$$\text{and } R \times D'D = P \times D'B - Q \times D'C + S \times D'E + \&c.,$$

$$\text{whence } D'D = \frac{P \times D'B - Q \times D'C + S \times D'E + \&c.}{R}$$

$$= \frac{P \times D'B - Q \times D'C + S \times D'E + \&c.}{P - Q + S + \&c.}$$

NOTE.—The equilibrium of forces acting in different planes shall be considered further on; we shall now, as an application of what is already done, proceed to the discussion of the following subjects.

THE MECHANICAL POWERS.

36. By the mechanical powers we are enabled to sustain a great weight, or overcome a great resistance, by a small force, or change the direction of any force.

The mechanical powers are usually considered six in number;—the *Lever*, the *Wheel and Axle*, the *Pulley*, the *Inclined Plane*, the *Wedge*, and the *Screw*.

The first three, when in a state of equilibrium, may be reduced to the lever; and the three last, may be referred to the inclined plane; so that, strictly speaking, we cannot reckon more than two simple mechanical powers.

37. When two forces act on each other by means of machinery, one of them is usually called the *power* and the other the *weight*. The resistance to be overcome is the weight; and the force, of whatever kind, which is employed to overcome that resistance, is called the power.

1. THE LEVER.

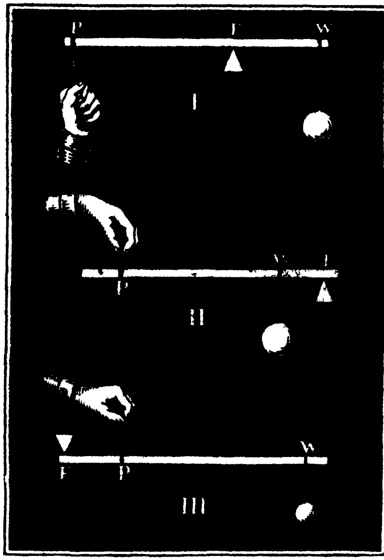
38. The *lever* is an inflexible rod movable in one plane about a point called the *fulcrum* or centre of motion. The parts of the lever, into which the fulcrum divides it, are called the *arms* of the lever. When the arms are in the same straight line, it is called a *straight* lever, otherwise a *bended*, or more commonly, a *bent* lever.

39. There are commonly reckoned three kinds of straight levers, depending on the position of the *points* of application of the *power* and the *weight* with respect to the *fulcrum*.

40. A lever of the first kind is represented in fig. I., in which the fulcrum F is situated between the power P and the weight W.

In a lever of the second kind, fig. II., the power P and the weight W act on the same side of the fulcrum F, the weight being between the fulcrum and the power.

In a lever of the third kind, fig. III., the power P and the weight W act on the same side of the fulcrum F, as in the latter case, but the power, in this case, is between the fulcrum and the weight.



41. Levers of the first kind are steel-yards, crowbars, pincers, &c. Levers of the second kind are nut-crackers, oars of a boat, where the water is considered the fulcrum, &c. Levers of the third kind are such as tongs, sheep-shears, &c.; the bones of animals are also considered as levers of the third kind, in which the joint is the fulcrum, the muscle near the joint the power, and the force exerted by the limb, at a greater distance from the joint, is the weight.

42. Prop.—*To find the conditions of equilibrium, when a power and weight act in the same plane on a lever.*

(1.) Let A F B be a lever, F the fulcrum, P and W the power and weight, acting respectively on the arms A F, B F of the lever by their gravity, the directions A P, P W, will be therefore parallel to one another. Now it is evident, from Art. 31, that if the resultant of these two forces passes through the fulcrum F, there will be an equilibrium, since the fulcrum is a fixed point; but if the resultant pass through any other point in A B, as F′, the force at F′ will be unsupported, and will cause the lever to move round F in the direction of this force. Hence it is evident, from Art. 31, that the lever A B must be divided in F, so that

$$P : W :: FB : FA.$$

$$\text{Whence } P \times FA = W \times FB.$$

(2.) But if the direction of the forces P and W be inclined to each other, let these directions meet each other in the point K; and let this point be rigidly connected with the lever A B, then the forces P and W may be considered to be applied at the point K, instead of the ends A, B of the lever, and therefore the resultant of these two forces will pass through K. But when there is an equilibrium it must also pass through the fulcrum F, as in the last case, hence K F will be the direction of the resultant. Let K f represent the pressure on the fulcrum F, and let the parallelogram K $a f b$ be completed, then K a, K b will express the two forces P and W. Draw F S, F T perpendicular to A K, B K, then

$$\begin{aligned} P : W &:: Ka : Kb = af. \\ &:: \sin Kfa : \sin aKf, \\ &:: \sin fKb : \sin aKf, \\ &:: FT : FS. \end{aligned}$$

43. Cor. 1.—Put A F $= a$, B F $= b$, the angle P A F $= \alpha$, and W B F $= \beta$, then since FS $= a \sin \alpha$, and F T $= b \sin \beta$, we shall have, by multiplying extremes and means,

$$P\, a \sin \alpha = W\, b \sin \beta.$$

44. Cor. 2.—This proposition is equally true for straight or bent levers of any figure, also from Art. 35, it is true for levers of the second or third kind; while the pressure on the fulcrum in levers of the second kind is evidently $= W - P$, and in those of the first and third kinds the pressure on the fulcrum is $= W + P$, the lever, in these cases, being considered to be without weight.

45. PROP.—*If any number of forces or weights* P, Q, &c.; *p*, *q*, &c., *acting upon the arms of a straight lever to turn it in opposite directions, round the fulcrum* F, *be such that*

t $P \times FM + Q \times FN + \&c. - p \times Fm + q \times Fn + \&c.$

here will be an equilibrium on the lever.

For the resultant of all these forces passes through F, and if we estimate the moments of these forces from F, we shall have, by Art. 35.

$$P \times FM + Q \times FN + \&c. - p \times Fm - q \times Fn - \&c. = R \times 0 = 0, \text{ whence}$$

$$P \times FM + Q \times FN + \&c. = p \times Fm + q \times Fn + \&c.$$

46. COR.—If any of these forces or weights act obliquely, such forces must be multiplied by the sine of the angle which their directions make with the lever, or if the lever be bent, such forces must be multiplied by the perpendiculars from the fulcrum on their respective directions.

47. *To find the fulcrum, when the power, weight, and length of lever are given.*

Returning to Art. 40, fig. I., we have, by Art. 42,

$$P : W :: FW : FP; \text{ whence by comp.,}$$

$$P + W : P :: FW + FP = PW : FW$$

$$P + W : W :: PW : PF$$

$$\therefore \quad FW = \frac{P \times PW}{P + W} \text{ and } PF = \frac{W \times A.W}{P + W},$$

whence the distance of the fulcrum from either end of the lever may be found.

48. When the power is required to be very great, and it is not convenient to construct a very long lever, a *compound* lever, or a *composition* of levers is used. In the composition of levers in

the annexed figure, the several levers act perpendicularly upon one another, as A B, B C, C D, the fulcrums of which are respectively F, F′ and F″; then

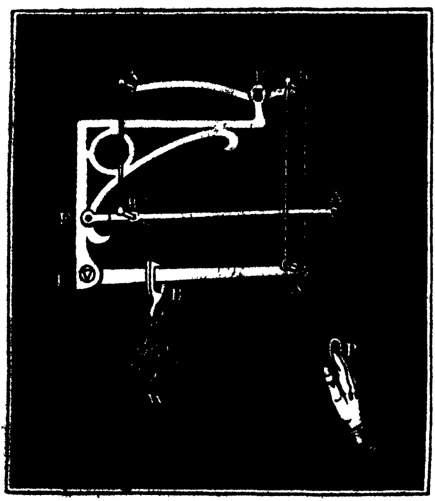
power P acting at A : weight at B :: B F : F A,
weight at B : weight at C :: F′C : F′B,
and weight at C : weight at W :: F″D : F″C.

Hence, by compounding these three proportions,

$$P : W :: FB \times F'C \times F''D : FA \times F'B \times F''C.$$

And generally, when a system of this kind is in equilibrium, the ratio of the power to the weight or load will be as the product of the alternate arms of each lever, beginning with the power, to the product of the alternate arms, beginning from the weight or load, of whatever kind the levers may be, recollecting that if any of the levers be bent, or the forces act obliquely, the arms must be considered as the perpendiculars let fall from the fulcrums on which such forces act.

49. A system or composition of levers may be conveniently arranged, as in the fig. annexed. Here we have three levers, two of the second, *i. e.*, A F, A″ F″, and one of the first kind, A′B′; and we will now consider the manner in which the power P is transmitted to the weight W. The power P acting upon the lever A F, produces a downward force at B $= \frac{P \times AF}{FB}$. The arm A′F′ of the second lever is, therefore, pulled down by the force $\frac{P \times AF}{FB}$, and this force, multiplied by



A′ F′ and then divided by F′ B′, will give $\frac{P \times AF \times A'F'}{FB \times F'B'}$ = force with which B′, and therefore A″, is drawn upwards. And lastly, we find in the same manner,

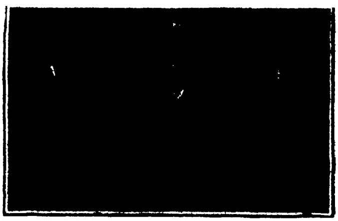
$$W = \frac{P \times AF \times A'F' \times A''F''}{FB \times F'B' \times F''B''}$$

Thus, for example, if AF, A′F′, A″ F″ be respectively 16, 20, and 18 inches, and B F, B′ F′, B″ F″ be respectively 2, 2, and 3 inches; then

$$W = \frac{P \times 16 \times 20 \times 18}{2 \times 2 \times 3} = 480 \times P,$$

or the weight is 480 times the power.

50. *The Balance.*—One of the most useful applications of the lever is to the balance, which consists chiefly of a lever of the first kind with equal arms, from the ends of which scales are suspended. This lever A B is called the *beam*, C is the *fulcrum* or centre of motion, g is the centre of gravity (which term will be hereafter particularly defined) of the beam and scales, this point is placed a little below the fulcrum, otherwise the beam would rest in any position; if, on the contrary, the point g were above the beam, the least disturbance would cause the beam to upset. The points of suspension A, B should be so situated that a straight line A B, joining them, may be perpendicular to the line joining the centre of gravity g with the point of support m.



In a perfect balance all the parts must be symmetrical with respect to the fulcrum c; that is, the parts on either side of this point must be exactly equal. Moreover, the scales must be in equilibrium when empty, and there must be as little friction as possible at the fulcrum c.

51. *The False Balance.*—This balance has its arms of unequal length, and is in equilibrium when charged with unequal weights. But the true weight of a body may be found

by a false balance in the following manner. First, weigh the body in one scale, and afterwards weigh it in the other; then the mean proportional between these weights will be the true weight. For let $x =$ true weight of the body, and W the number of ounces or pounds it weighs in the scale A, and w the ounces or pounds it weighs in the scale B; then, by Art. 39, we shall have

$$Ac \times x = Bc \times W,$$

$$\text{and } Bc \times x = Ac \times w.$$

By multiplying these equations, there results,

$$Ac \,.\, Bc \,.\, x^2 = Ac \,.\, Bc \,.\, W \,.\, w,$$

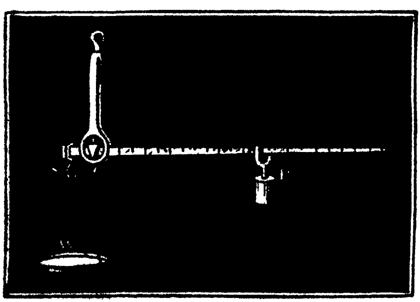
$$\therefore\ x^2 = W w,$$

$$\text{or}\quad x = \sqrt{W w}.$$

That is, the true weight is a mean proportional between the two false weights W and w.

52. ***The Common Steelyard.***—This is another useful application of the lever for ascertaining the weights of bodies. It is a lever with arms of unequal length, by means of which a single weight P is sufficient to determine, from its position, the weight of any other body W.

The beam of the steelyard is shewn in the annexed figure. C is its fulcrum. The body W, the weight of which is to be found, is suspended at the end S of the shorter arm, and the constant weight P is moved along the graduated arm till there shall be an equilibrium. Let us first assume that the scale and heavy ball at S keep the lever in equilibrium or horizontal, when the load W and



the weight P are removed, as is the case in some steelyards. Now, let W and P be applied to the steelyard so that they may balance each other, then $P \times CP = W \times CS$, or

$W = \frac{P \times CP}{CS}$, $\therefore$ when $CP = CS$, W will be $= P$, and when $CP = 2CS$, W will be $= 2P$, and so on. Therefore, if the longer arm of the lever be marked or graduated so that C1, C2, C3, &c., shall be equal to SC, 2SC, 3SC, &c., respectively; then, when P is at the 1st, 2nd, 3rd, &c., marks, the corresponding weights of W will be P, 2P, 3P, &c. Thus if $P = 1$ pound then W will be successively equal to 1, 2, 3, &c., pounds, when P is at the 1st, 2nd, 3rd, &c., marks or divisions on the longer arm of the steelyard. In the figure, P is shewn at the twelfth division on the longer arm, therefore, in this case, $W = 12P$; and, if $P = 1$ pound, $W = 12$ pounds.

53. If the lever do not balance itself, its weight must be taken into calculation by considering it to act at the centre of gravity; and if the lever be in the form of a prism, or an uniform bar of any kind, its centre of gravity will be at its middle point.

Let l be the whole length of such a lever or steelyard, and w its weight; then the distance of the centre of gravity of the lever from its fulcrum C will be $= \frac{1}{2}(l - 2SC)$ at which distance the weight w acts,

$$\therefore W \times SC = P \times PC + \tfrac{1}{2}(l - 2SC)w$$

$$\text{whence } P = \frac{W \times SC - \frac{1}{2}w(l - 2SC)}{PC}.$$

Ex. 1.—Let the whole length of a lever be $l = 8$ feet, its lesser arm $SC = 3$ feet, and its whole weight $w = 4$ lbs., and let a weight $W = 100$ lbs. be suspended in the scale at S; what weight P must be placed at the end of the longer arm to hold the lever in equilibrium?

By the formula given above

$$P = \frac{W \times SC - \frac{1}{2}w(l - 2SC)}{PC} = \frac{100 \times 3 - 2(8 - 6)}{5} = 59\tfrac{1}{5}\text{lbs.}$$

Ex. 2.—On a lever three feet in length a weight of 500 lbs. is suspended at one end, at $2\frac{1}{2}$ inches from its fulcrum; what weight at the other end will keep the lever in equilibrium, the lever being assumed to be without weight?

Ans. 40 lbs.

Ex. 3.—Required the force that will draw a carriage-wheel over an obstruction, assuming the whole weight of the carriage to be collected at the axis of the wheel.

Let O be the centre of the wheel D C W, C the obstruction, P the drawing power acting in the direction O P, W the weight of the load acting in the direction O W perpendicular to the horizon. Draw C m, C n perpendicular to OP, OW. Then the wheel, in turning over the obstruction, must turn round the point C; therefore O C may be considered as a lever, the fulcrum of which is C, and C m, C n are the perpendiculars from the fulcrum in the directions of the power and weight respectively.

Hence P : W : : C n : C m : : sin C O n : sin C O m.

Put the radius of the wheel O W $= r$, and the height of the obstruction $= n\text{W} = h$; then $\text{C}n = \sqrt{2rh - h^2}$, and $\sin \text{CO}n = \frac{\text{C}n}{\text{CO}} = \frac{\sqrt{2rh - h^2}}{r} =$ (since h is usually very small compared with r,) $\frac{\sqrt{2rh}}{r}$ nearly $= \sqrt{\frac{2h}{r}}$.

$$\therefore \text{P} : \text{W} :: \sqrt{\frac{2h}{r}} : \sin \text{CO}m.$$

If, therefore, W and $\sqrt{\frac{2h}{r}}$ be given, P will be least when sin C O m is greatest, that is, when C O m is a right angle, in which case its sine is = rad. = 1, whence

$$\text{P} = \text{W}\sqrt{\frac{2h}{r}}.$$

Ex. 4.—If weights of 2, 4, and 6 cwts., be suspended at the distances of 3, 6, and 9 feet from the fulcrum of one arm of a straight lever, and weights of 4, 6, and 8 cwts., be suspended at 2, 5, and 6 feet from the fulcrum on the opposite arm; where must a weight of ½ cwt. be placed to keep the lever in equilibrium?

Ans. 4 feet from the fulcrum on the first arm.

Ex. 5.—The arms of a bended lever P F W are of equal length, and make an angle at the fulcrum F of 135°; required the position in which the lever will rest, when two weights of 3 and 5 cwts. are placed at P and W?

Ans. PF makes an angle of 8° 37′ with the horizon.

Ex. 6.—Required the weight of the body W, when the power P, on the lever in the last example, is 3 cwts., and the arm PF is horizontal. *Ans.* 4¼ cwts. nearly.

Ex. 7.—A beam AB sustaining a weight W at the point F, is supported by two posts at A and B; it is required to determine what portion of the weight is sustained by each of the props or posts, the weight of the beam being neglected.

Supposing the beam to turn on B as a fulcrum we shall have

$$\text{Pressure on A} \times AB = W \times FB,$$

$$\therefore \text{ Pressure on A} = \frac{FB \times W}{AB}.$$

Similarly, by supposing the beam to turn on A as a fulcrum, there will result,

$$\text{Pressure on B} = \frac{AF \times W}{AB}.$$

Ex. 8.—Two men carry a weight of 2 cwt. hung on a pole, the ends of which rest on their shoulders; what part of the load is borne by each man, the weight hanging 6 inches from the middle of the pole, the whole length of which is 4 feet? *Ans.* 140 lbs. and 84 lbs.

Ex. 9.—Let the length of the beam AB in Example 7 be 30 feet, FB = 10 feet, and consequently AF = 20 feet, and the weight W = 18 cwts; required the pressures on the supports A and B.

$$\text{Here pressure on A} = \frac{FB \times W}{AB} = \frac{10 \times 18}{30} = 6 \text{ cwts.},$$

$$\text{and pressure on B} = \frac{AF \times W}{AB} = \frac{20 \times 18}{30} = 12 \text{ cwts.}$$

NOTE 1.—If two or more weights be suspended at different points of a beam, supported by posts or props, the pressures due to each weight must be found for each of the posts separately, and the sum of the pressures on each post will give the total pressure on each.

NOTE 2.—If the weight of the beam be taken into calculation, that weight must be considered as acting on its centre of gravity, which centre, if the beam be of uniform thickness, will be at the middle point of the beam.

Ex. 10.—A beam, the length of which is 18 feet, is supported at both ends; a weight of 18 cwts. is suspended at 3 feet from one end, and a weight of 12 cwts. at 8 feet from the other end; required the pressure at each point of support.

Ans. The pressures, by Note 1, are found to be 15 and $9\frac{2}{3}$ cwts.

Ex. 11.—If the weight of the beam in the last example be 12 cwts., required the pressure on each point of support.

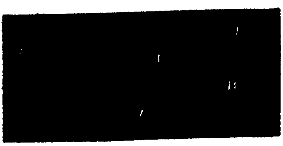
Ans. 21 and $15\frac{2}{3}$ cwts.

Ex. 12.—A uniform beam 40 feet in length, the weight of which is 4 cwts., is supported by two props A and B, 30 feet apart; now a weight of 24 cwts. is suspended on the beam at the distance of 10 feet from B, the beam projecting 8 feet over the prop at A, and 2 feet over that at B; required the pressure on each of the props.

Ans. $10\frac{2}{5}$ cwts. on A, and $17\frac{3}{5}$ cwts. on B.

TO GRADUATE THE LEVER OF A SAFETY VALVE.

The safety valve is for the purpose of preventing the bursting of boilers by the elastic force of the steam. A F is a graduated lever turning on F as a fulcrum; V is the valve, which is raised when the elastic force of the steam becomes too great for the pressure of the weight W, which presses down the valve by means of the lever A F.



Let A F = L, V F = l, W = weight at A, w weight of the lever A F, r = radius of the valve, and P = greatest pressure per square inch of steam in the boiler. Then πr^2 = area of the valve or its orifice, πr^2 P = pressure on the valve, and by the property of the lever.

$$\mathrm{L} \times \mathrm{W} + \tfrac{1}{2}\mathrm{L} \times w = l \times \pi r^2 \mathrm{P}, \text{ whence}$$

$$\mathrm{W} = \frac{\pi r^2 l \mathrm{P} - \frac{1}{2}\mathrm{L}\, w}{\mathrm{L}} \qquad (1)$$

The weight W at the end of the lever may be determined from Formula (1), after which the length L′, corresponding to any other given pressure P′, may be found from the following formula, which is derived by substituting L′ for L and P′ for P, in (1) and transposing which gives

$$L' = \frac{\pi r^2 l P' - \frac{1}{2} L w}{W}. \qquad (2)$$

Ex. 1.—Required the weight W when A F = 24, V F = 3 inches, weight of the lever 4 lbs., radius r of the valve = $1\frac{1}{2}$ inches, and the pressure P of the steam in the boiler 40 lbs. per square inch.

Here the area of the valve $\pi r^2 = 3{\cdot}1416 \times (1\frac{1}{2})^2 = 7{\cdot}07$ square inches nearly, which, by omitting the small decimal, may be taken as 7 square inches; whence

$$W = \frac{\pi r^2 l P - \frac{1}{2} L w}{L} = \frac{7 \times 3 \times 40 - 12 \times 4}{24} = 33 \text{ lbs.}$$

That is, 33 lbs. put at the end of the lever will give the required pressure. We have next to find the distance L′ from F, at which this weight must be put to give any other required pressure P′.

Ex. 2.—Let the pressure P′ be 20 lbs. per square inch, all the other dimensions and weights being as in the last example, required the distance A F = L′.

By Form. (2).

$$L' = \frac{\pi r^2 l P - \frac{1}{2} L w}{W} = \frac{7 \times 3 \times 20 - 12 \times 4}{33} = 11\tfrac{3}{11} \text{inches},$$

for the new distance A F at which the weight W must be suspended to give a pressure of 20 lbs. per square inch. Similarly, the distances on the lever may be found for any other pressures to complete the graduation of the lever.

Ex. 8.—A beam of timber, 24 feet in length, is found to balance itself on a prop 10 feet from the greater end; but on placing the middle of the beam on the prop, it requires a man's weight of 200 lbs. standing on the less end, and also a weight of 20 lbs., at a distance of 4 feet from this end, to balance the beam; what is the weight of the beam?

Ans. 11 cwts. 48 lbs.

THE WHEEL AND AXLE.

54. The wheel and axle consists of wheel with a cylindrical axis, passing through its centre perpendicular to the plane of the wheel. The power is applied to the circumference of the wheel, and the weight to the circumference of the axle.

55. Prop.—*The wheel and axle are in equilibrium when the power is to the weight as the radius of axle is to the radius of the wheel.*

Let CA, CB be the radii of the wheel and axle, at the extremities of which the power and weight act; then A B C may be considered as a lever, the fulcrum of which is C; and since the power P and the weight W, being suspended by cords, act perpendicularly to A C, we shall have

P : W : : C B : C A, that is

P : W : : rad. of axle : rad. of wheel.

Note.—The power may act by means of bars or handspikes inserted into the axle, and the wheel may be removed, as in the case of the *windlass* and *capstan*.

56. Cor. 1.—If the power p act in the direction $a\,p$, which cuts A C at right angles in D, then there will be an equilibrium when p : W : : C B : C D.

57. Cor. 2.—When P and W sustain each other by means of a wheel and axle, the thickness of the rope by which they are sustained must be taken into account; that is, we must add half the thickness of the rope to each of the distances at which P and W act. Therefore, if R = radius of the wheel, r = radius of the axle, and $2\,t$ = thickness of the rope, then we shall have

$$P : W :: r + t : R + t$$

$$\therefore W = \frac{P\,(R + t)}{r + t} \qquad (1)$$

$$P = \frac{W\,(r + t)}{R + t} \qquad (2)$$

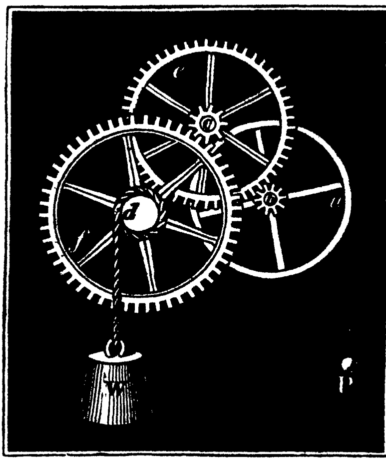
59. Cor. 3.—If the wheel be acted upon without a rope, the above proportion becomes

$$P : W :: r + t : R.$$

When P and W may be found as in the preceding case.

Note.—By increasing the size of the wheel in proportion to that of the axle, a very small force may be made to balance a very great weight, but as the weight is increased, the size of the wheel must also be increased to an inconvenient extent. Hence the use of a system or combination of wheels and axles. Now as the wheel and axle is only a modification of the lever, so also a system of wheels and axles is only a modification of the compound lever, already described in Art. 48. A system of wheels and axles are sometimes turned by simple contact with each other, and sometimes by cords, chains, or straps passing over them; in all such cases the friction of the surfaces prevents their sliding on each other; but the most usual method of transmitting power to complex machinery is by means of *teeth* or *cogs*, which are raised on the surfaces of the wheels and axles.

60. Prop.—*In a system of toothed wheels and axles, it is required to find the relation between the power and the weight, when they are in equilibrium.*



The power P is applied to the circumference of the first wheel a, which transmits its effect to the circumference of the first axle or pinion b; this acts on the circumference of the second wheel e; and so on through the pinion c to the wheel f, till the force is transmitted to the last axle d, which supports the weight W. This system or combination of wheels and axles is evidently the very same in principle as the combination of levers in Art. 48; therefore P is to W as the product of the radii of all the axles is to the product of the radii of all the wheels; and if the letters, referring to the wheels and axles in the annexed figure, denote the radii of those wheels and axles, we shall have

$$P : W :: bcd : aef$$

$$\text{whence } P = \frac{W\,bcd}{aef} \qquad (1)$$

$$\text{and } W = \frac{P\,aef}{bcd} \qquad (2)$$

61. COR.—Since the number of teeth in wheels are as their radii, P : W : : product of number of teeth in all the pinions : product of number of teeth in all the wheels; whence the number of teeth in the respective wheels and pinions may be substituted for their radii in the two preceeding formulæ.

NOTE.—By a combination of wheels and axles, such as that just referred to in Art. 60, a power to any extent whatever may be acquired.

Ex. 1.—A weight of one ton or 2240 lbs. is sustained by a rope of 2 inches in diameter, going round an axle 4 inches in diameter; what weight must be suspended at the circumference of the wheel, by a rope of the same thickness, to obtain an equilibrium, the radius of the wheel being 6 feet?

$$\text{By Form. (2), Art. 57,} \quad P = \frac{W\,(r+t)}{R+t}.$$

Here $W = 2240$, $r = 2$, $R = 72$, and $2\,t = 2$ or $t = 1$,

$$\text{whence } P = \frac{2240\,(2+1)}{72+1} = \frac{6720}{73} = 92\tfrac{4}{73} \text{ lbs.}$$

If the thickness of the rope had not been considered, then

$$P = \frac{W\,r}{R} = \frac{4480}{72} = 69\tfrac{2}{9} \text{ lbs.}$$

Ex. 2.—In a combination of wheels and axles there are given the radii of the wheels, 20, 26, and 48 inches, and the radii of the pinions and axle 4, 5, and 8 inches. Now, if a power of 1 cwt. be applied to the circumference of the first wheel, what weight will it be able to sustain at the circumference of the axle or last pinion?

By Form. (2), Art. 60.

$$W = \frac{P\,aef}{bcd} = \frac{112 \times 20 \times 26 \times 48}{4 \times 5 \times 8} = 112 \times 26 \times 6 \text{ lbs} = 156 \text{ cwt.}$$

Ex. 3.—The number of teeth in each of three successive wheels is 144, and the number of teeth in each of the axles or pinions is 6; what weight will this machine support with a power of 2 cwt.? *Ans.* 1384 tons 8 cwt.

Ex. 4.—A power of 10 lbs. balances a weight of 300 lbs. on a wheel, the diameter of which is 10 feet; what is the diameter of the axle, the thickness of the rope on the wheel being one inch, and that of the rope on the axle two inches?

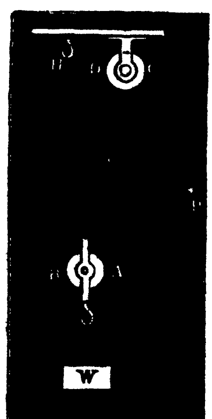
THE PULLEY.

62. A *pulley* is a small wheel moveable about an axis passing through its centre, in the circumference of the wheel is a groove to admit a rope or flexible chain. The pulley is called fixed or moveable, according as its axis is fixed or moveable.

63. Prop.—*In the single fixed pulley there is an equilibrium when the power and weight are equal.*

For through the centre C of the pulley draw AB, which represents a lever of the first kind, of which the fulcrum is C, and since the arms A C, C B are equal, the power and weight suspended at A and B must be equal when an equilibrium is obtained.

64. Prop.—*When the power sustains the weight by means of one moveable pulley, the power is just half the weight, if the portions of the sustaining cord be parallel.*



First, it is evident that the rope P C D A B H must have the same tension everywhere throughout its length, or the system would not be in equilibrium, and this tension must be equal to the power P, and since the tensions of the two parts of the rope A D, B H are each equal to P, the weight W, suspended from the axle of the pulley A B, must be necessarily equal to 2 P. —The same may be proved in the following manner, suppose A, B to be joined by a line passing through the axle of the moveable pulley, then the

Fig. 1.

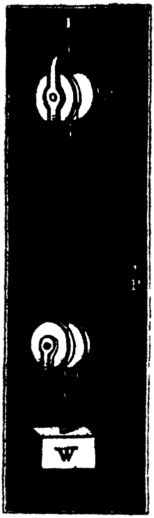
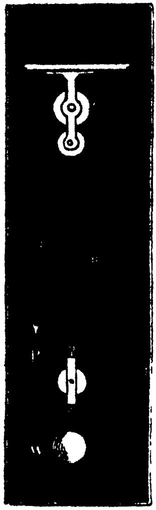


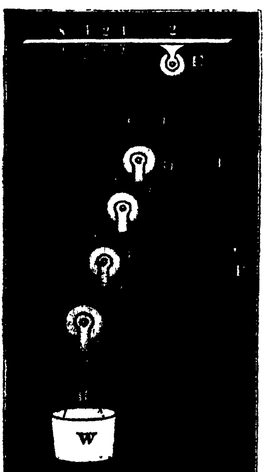
Fig. 2.



line A B may be considered as a lever of the second kind, the power P acting at A, the weight at the axle of the pulley, and the fulcrum at B; therefore P : W :: $\frac{1}{2}$AB (= rad.) : AB :: AB : 2AB, consequently 2 P = W, or P = $\frac{1}{2}$ W.

65. Cor. 1.—The same principle may be applied to a system or combination of pulleys, all drawn by one cord, passing over an equal number of fixed and moveable pulleys. In fig. 1, P : W :: 1 : number of parts of the cord passing over the moveable block. Therefore, in fig. 1, where the number of parts of the cord going over the moveable block is 4, we shall have P : W :: 1 : 4, whence 4 P = W, or P = $\frac{1}{4}$ W; and generally, if the number of these parts of the cord be n, we shall have P : W :: 1 : n, whence

$$P = \frac{W}{n} \text{ or } W = nP.$$



66. Cor. 2.—In fig. 2, the weight, being sustained by three cords, is equal to three times the power; and generally, if the number of the parts of the cord (passing over moveable pulleys) be n, we shall have P : W :: 1 :: $n + 1$, whence W = $(n + 1)$ P.

67. Prop.—*In a combination where each pulley hangs by a separate cord, and the cords are parallel* P : W : 1 : 2^n; *being the number of moveable pulleys.*

In this combination, a cord goes over the fixed pulley E, under the moveable pulley D, and is fixed to the hook at 1. Another cord is fixed to D, goes under the moveable pulley C, and is fixed to the hook at 2; and so on.

From Art. 63, the weight at $D = 2P$,

the weight at $C = 2 \times$ weight at $D = 2^2 P$.

the weight at $= B\ 2 \times$ weight at $C = 2^3 P$

and if the number of moveable pulleys be n, then

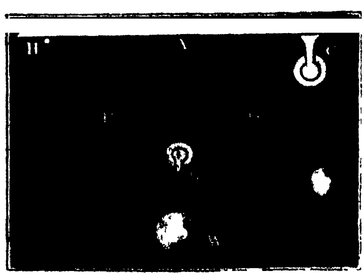
$$W = 2^n P.$$

68. Cor.—The tensions of each of the strings in this system is shewn by the numbers above the hooks; these tensions being P, 2 P, 4 P, &c.

69. Note.—Although the power increases rapidly in this system, being doubled by the addition of every moveable pulley; but this advantage over the common system is more than counterbalanced by the very limited range, since in the common blocks, the motion may be continued till the fixed and moveable block come into contact, but in this system the motion can only be continued till D and E come into contact, at which time the other pulleys will be far apart, because C rises only half as fast as D, B only one-fourth, and A only one-eighth as fast. Hence the longest possible range is but a small portion of the whole height occupied by the system, which accordingly entails a great waste of space, and is hardly of any practical use.

70. Prop.—*There will be an equilibrium on the single moveable pulley, when the power is to the weight as radius to twice the cosine of the angle which either string makes with the direction in which the string acts.*

A cord fixed at H passes under the moveable pulley B, and over the fixed pulley C, the power P being applied at the extremity of the cord. The weight W is suspended to the centre of the moveable pulley B, which in this case is assumed to be of very small radius. Draw the vertical line AB of such a length as to represent the weight W, and complete the parallelogram A D B E; then B D, B E will represent the tensions on the cord, which are evidently each equal to the power P, ∴ all the sides of the parallelogram are



equal. Now, conceive E D to be joined, by a line not shewn in the figure, then we shall have P : W : B D : AB : : rad. : 2 cos A B D, because the angle E D B is the complement of A B D. Whence

$$W = 2 P \cos ABD.$$

71. Cor.—If the weight of the moveable pulley be considered, and that weight $= w$;

then $P : W + w :: 1 : 2 \cos ABD$. Whence

$$W = 2 P \cos ABD - w.$$

Ex. 1—In a system of pulleys, such as shewn in fig. 1, Art. 65, the number of moveable pulleys being six, required the weight, the power being $\frac{1}{2}$ cwt. and the weight of the moveable block and pulleys being 36 lbs.

Ex. 2.—If the angle made by the horizon and a cord passing under a moveable pulley be 30°; what proportion does the power bear to the weight? *Ans.* $P = W$.

Ex. 3.—If the angle in the last example be 45°; what proportion does the power bear to the weight?

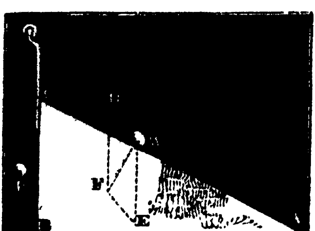
Ans. $P : W :: 1 : \surd 2$.

Ex. 4.—If w be the weight of each pulley in Art. 67, then prove that $W = 2^n P - (2^n - 1)w$.

THE INCLINED PLANE.

72. *The inclined plane* is considered in mechanical science as a smooth, perfectly hard and inflexible surface; the iron rails on an ascending or descending gradient of a railway may be regarded as a plane of this kind, or at least a near approach to it.

73. Prop.—*When a body is in equilibrium on an inclined plane,* P : W : : *the sine of the inclination of the plane to the horizon : the cosine of the angle which the sustaining cord makes with the plane.*



Let A B be parallel to the horizon, A C a plane inclined to it; W a body sustained on the plane by a power P acting in the direction WD. Draw W E perpendicular to A B, take W D, W E to represent the two forces P and W, and complete the parallelogram W E F D. Since P and W are in equilibrium, their resultant must be perpendicular to the plane A C, and they will be supported by the reaction of the plane. Hence the pressure R of the body on the plane will be represented by the diagonal WF of the parallelogram, and the three forces P, W, and R will, therefore be proportional W D, W E, W F, and

$$P : W :: W D : W E :: \sin W F D : \sin D W F.$$

But $\sin WFD = \sin FWE = \sin BAC$; and, since the weight W may be regarded as a point on the plane A C, and because WF is perpendicular to AC, $\sin DWF = \cos CWD$, hence

$$P : W :: \sin B A C : \cos C W D.$$

74. Cor. 1.—When $\cos C W D = \text{rad.} = 1$, then

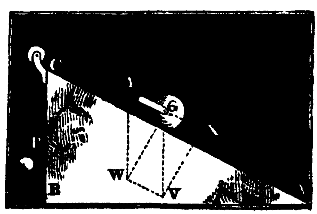
$$P : W :: \sin B A C : 1$$
$$:: \quad C B \quad : A C,$$

that is, P : W : : the height of the plane : its length; also
W : to pressure R against the plane : : A C : A B,
that is, W : R : : the length of the plane : its base.

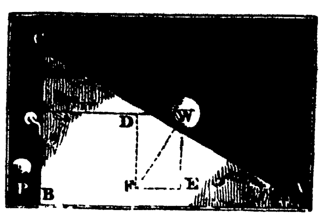
$$\text{Whence } P = \frac{W \times BC}{AC} \quad (1)$$

$$W = \frac{P \times AC}{BC} \quad (2)$$

$$R = \frac{W \times AB}{AC} \quad (3)$$



In this case the direction of the power is parallel to the inclined plane A C, as in the annexed figure, and the weight is the greatest that can be supported by the given power P.



75. Cor. 2.—If the power act parallel to the base A B, then the angle CWP = complement of D W F, and

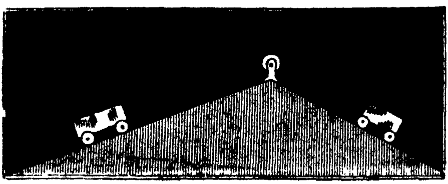
$$P : W :: \sin BAC : \cos BAC.$$
$$:: \quad CB \quad : \quad AB.$$

76. COR. 3.—If the power act perpendicular to the horizon the $CWD = 90° - BAC$, therefore $\cos CWD = \sin BAC$, and consequently

$$P : W :: 1 : 1, \text{ or } P = W.$$

In this case the weight is entirely supported by the power, and there is no pressure on the plane, which is also self-evident from the nature of forces.

77. COR. 4.—Hence it is easily seen that, if two weights balance each other on two inclined planes of the same height, as in the annexed figure, the weights must be directly proportional to the lengths of the planes on which they rest.

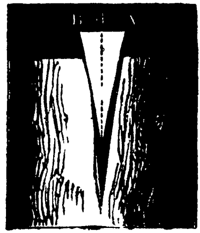


NOTE.—Examples would have been given here, as has already been done after the exposition of the principles of the other mechanical powers; but as the *wedge* and the *screw* are only modifications of the inclined plane, the screw being chiefly combined with one or more of the other mechanical powers, it is thought best to treat immediately of the principles of these two powers and afterwards to give copious examples on the inclined plane which, as a power on railways, roads, &c., performs most important effects, second only to those of the steam engine. Also, previous to giving these examples, it will be proper to explain the nature and define the power of various working agents by which forces are imparted.

THE WEDGE.

78. *The wedge* is commonly used for separating bodies that are strongly bound or pressed together, as for cleaving timber, in which case it is urged by percussion. The force impressed by percussion, or a blow on the back of the wedge, has an effect incomparably greater than any mere pressure or dead weight, such as is used in the other mechanical powers. But as the force of percussion cannot be measured, we shall only here compare its effects on wedges of different inclinations.

79. *In the wedge* A B C, *the power acting perpendicular to the back* AB *is to the forces acting perpendicular to either side* A C *or* B C *as the breadth of the back* A B *is to the length of the side* A C *or* B C, *the wedge will be in equilibrium.*



For when three forces are in equilibrium, they are as the corresponding sides of a triangle, drawn perpendicular to the directions in which these forces act. But A B is perpendicular to dC the direction of the force against the back, and A C, B C are perpendicular to the forces acting against them; therefore, if P = force on the back, and W = pressure on A C or B C,

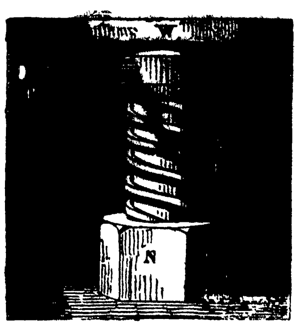
$$\text{then} \quad P : W :: AB : AC :: 2 \sin ACB : 1.$$

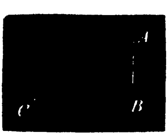
$$\therefore P = 2 W \sin ACB.$$

THE SCREW.

80. *The screw* is a spiral groove winding round a cylinder so as to cut all the lines drawn on its surface parallel to its axis at right angles. The screw is, therefore, nothing more than an inclined plane, wrapped round the surface of the cylinder, the base of the plane being equal to the circumference of the cylinder's base, and coinciding with it, and the height of the plane equal to the distance A B between two of the threads.

81. Prop.—*In a vertical screw, when there is an equilibrium,* $P : W :: d : 2\pi r$; *in which* d *is the distance between two contiguous threads measured in a direction parallel to the axis of the screw, and* $2\pi r$ *is the circumference of the circle which* P *describes.*



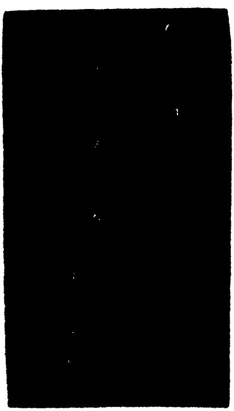


For, since the screw is nothing more than an inclined plane A B C, unwrapped from the cylinder, (see the small fig.) the base B A of the plane being equal to the circumference of the cylinder's base, and the height A B of the plane equal to the distance between two of the threads of the screw, and since the power in this case acts parallel to the base, it follows from Art. 75, Cor. 2, that P : W :: AB : BC :: the distance between two of the threads : the circumference described by the power, which in this case is the circumference of the cylinder, to which the power P is supposed to be applied, the weight W resting on the top of the screw, as shewn in the larger figure.

$$P : W :: AB : \text{the circumference described by } P,$$
$$:: d : 2\pi r,$$

82 Cor.—Instead of considering the screw to raise a weight W by acting vertically, we may suppose it to be applied to produce a pressure W in any other direction, and the proportion between P and W will be the same as before.

83. Prop.—*In the endless screw there will be an equilibrium, when* $P : W :: d \times \rho \times \rho' \times \rho'' \times$ &c. $: 2\pi r \times r' \times r'' \times r''' \times$ &c., *where d is the distance between the threads and* $2\pi r$ *the circumference described by the power, as before, and* r', r'', r''', *&c. the radii of all the wheels, and* ρ, ρ', ρ'', *&c. the radii of all the axles.*



The endless screw is so combined with the wheel and axle, or with a system of wheels and axles, that the threads of the screw may work in the teeth of the first wheel, the teeth of the pinion of this wheel working the teeth of the next wheel, and so on, the weight being supported by the axle of the last wheel. Let P′ = power produced by the screw at the circumference of the wheel E, the power P acting on the winch AC at A, then, by Art. 81,

$$P : P' :: d : 2\pi r,$$

and in the wheel and axle, Art. 60,

$$P' : W :: \rho, \rho', \rho'', \&c. : r, r', r'', \&c.$$

$$\therefore P : W :: d\rho, \rho', \rho'', \&c. : 2\pi r, r', r'', r''', \&c.$$

84. Cor.—In the figure to Art. 83, there are three wheels E, F, and G, and their pinions or axles, whence in the system referred to

$$P : W :: d\,\rho\rho'\rho'' : 2\pi\, r r' r'' r'''.$$

Note.—The number of teeth in the wheels and pinions may be substituted for their radii, as in Art. 61.

Ex. 1.—The distance PW (fig. Art. 81.) at which the power acts is 6 feet, and the distance between two of the threads of the screw is 2 inches; what weight will a man be able to raise, when he acts at P with a force of 150 lbs.?

Here the power acts 72 inches from the centre, hence $2\pi r = 2 \times 72 \times 3{\cdot}1416 = 452{\cdot}39$ inches = circumference described by the power; whence, by Art. 81,

$$P : W :: 2 : 452{\cdot}39 \text{ in which } P = 150 \text{ lbs.}$$

$$\therefore W = \frac{452{\cdot}39 \times 150}{2} = 33929\tfrac{1}{4} \text{ lbs.}$$

Ex. 2.—If the endless screw be turned by a winch AC of 18 inches, the threads of the screw being distant ½ an inch each, the screw turns a toothed wheel E, the pinion of which turns another wheel F, the pinion of this another wheel G, and on the pinion or axle of which is sustained the weight W; now the radii of the wheels to be 18 inches, those of the pinions and axle 2 inches, and the length of the winch AC = 22 inches; what weight will a man be able to sustain who acts at the handle of the winch with a force of 150 lbs.? From Art. 84.

$$P = 150 : W :: \tfrac{1}{2} \times 2^3 :: 2\,3 \times {\cdot}1616 \times 22 \times 18^3, \text{ whence}$$

$$W = \frac{150 \times 2 \times 3{\cdot}1416 \times 22 \times 18^3}{\frac{1}{2} \times 2^3} = 30231630 \text{ lbs.} =$$

13496 tons. nearly.

VIRTUAL VELOCITIES.

85. Prop.—*If a power and weight be in equilibrium in any machine, and the whole be put in motion; the power : weight :: the weight's velocity : the power's velocity.*

One proof of this important proposition may be simply derived from an enumeration of the cases of the different mechanical powers.

(1.) *The lever.* Let A F B be a lever, kept at rest by the power P and the weight W; and let the lever move through a very small angle to the position aFb; then A and B will evidently describe circular arcs Aa, Bb, which will be as the velocities of the points A and B, and ultimately these arcs may be taken for straight lines perpendicular to A B, and because the triangles AFa, BFb have the equal angels at F, we shall have

$$Aa : Bb :: AF : FB,$$

$$\therefore \text{P's vel.} : \text{W's vel.} :: W : P.$$

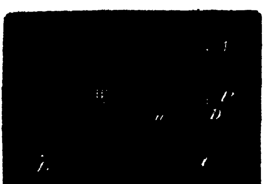
(2.) *In the wheel and axle*, if the power be made to descend through a space equal to the circumference of the wheel, the weight will be made to ascend through a space equal to the circumference of the axle in the same time, and since these circumferences are as their radii, we shall have

$$\text{P's vel.} : \text{W's vel.} :: \text{rad. axle} : \text{rad. wheel} :: W : P.$$

(3.) *In the single moveable pulley with parallel strings*, if the weight ascend 1 inch, each of the strings is shortened 1 inch, whence the power descends 2 inches, therefore

$$\text{P's vel.} : \text{W's vel.} :: 2 : 1 :: W : P.$$

(4.) *In the system of pulleys* (Art. 65), if the weight ascend 1 inch, each of the strings at the lower block will be shortened 1 inch, and the power will descend n inches; therefore



$$\text{P's vel.} : \text{W's vel.} :: n : 1 :: W : P.$$

(5.) *In the inclined plane*, let the weight W be raised through a small space Ww and let W A be drawn in the direction of the power, causing P to descend; then Ww, wn are as the velocities of power and weight, therefore

$$\text{P's vel.} : \text{W's vel.} :: Ww : wn :: AB : AC :: W : P.$$

(6.) *In the screw*, if the power P describe the circumference of a circle, the radius of which is PW, (Art. 81) the weight W will be raised through a distance equal to two adjacent threads of the screw; therefore

$$P\text{'s vel.} : W\text{'s vel.} :: 2\pi \times PW : AB :: W : P.$$

86. COR. 1.—Hence not only in the mechanical power, taken singly, but in any combination of them, we shall have generally

$$P\text{'s vel.} : W\text{'s vel.} :: W : P.$$

87. COR.—The weight of a body multiplied by its velocity is called its *momentum;* and it is hence evident that, in general, when there is an equilibrium, the *momenta of the power and weight are equal.* On this principle the power of a machine may be estimated, and frequently this will be found the simplest method.

ON FRICTION.

88. In the investigations of the problems in equilibrium the surfaces of bodies have been assumed to be perfectly smooth; but, in practice, all bodies are found to be more or less rough, and therefore the results that have been deduced will be more or less modified by the effects of this roughness, which produces a retarding force called friction. It has been found by experiment that this retarding force or resistance, on a given surface, is a certain proportional part of the weight of the body moved, and that it is not affected by the rate of motion, nor by the extent of the rubbing surface. Thus, if the weight W rest on the horizontal plane AB, and it be drawn horizontally by a weight F attached to a cord passing over a pulley P, then the weight F, which is just sufficient to draw W along the plane, will measure the friction of W on the plane. If W be 1 ton, then, in the case of a well made, smooth Macadamized road, the resistance of friction is found to be about $\frac{1}{30}$ of the whole load, or F is about 75 lbs. to the ton; so that a horse drawing 1 ton along such a road, must pull with a force of 75 lbs.; which is called the traction of the horse. In the case of a railway, where the

friction is probably the smallest in all ways, being about $\frac{1}{280}$ of the weight, therefore, if W be 1 ton, then F will be $\frac{W}{280} = 8$ lbs., and if $\frac{1}{n} = \frac{1}{280}$, then, generally, F will be $= \frac{W}{n}$. The fractions $\frac{1}{30}$ and $\frac{1}{280}$ are called the co-efficients of friction.

89. *If the inclination of the plane, on which a body is moved, is small*, as on the ascending and descending gradients of railways, and the ascents and descents of common roads, the pressure on the plane will evidently be very nearly equal to the weight of the body; hence the resistance produced by friction may be calculated with sufficient accuracy after the manner explained in the last article.

90. Now, let P = power requisite to draw a weight W, including its friction, along a plane with a rise of h feet in 100 feet, Q = power requisite to draw W along the plane exclusive of friction, and let the friction F be $= \frac{W}{n} =$ an n part of the weight; required the relation between P and W.

By Art. 74. $\quad Q : W :: h : 100,$

$$\text{whence } Q = \frac{h\,W}{100};$$

$$\text{but } P = Q + F = \frac{h\,W}{100} + \frac{W}{h} = \frac{h\,n + 100}{100\,n} W, \qquad (1)$$

$$\therefore W = \frac{100\,n\,P}{h\,n + 100}. \qquad (2)$$

Ex. 1.—If a train of 30 tons be moved along a level railway; what power will be required to overcome the resistance of friction at the rate of 8 lbs. per ton, or $\frac{1}{280}$ of the weight?

Here the required power P is equal to the resistance of friction, that is, by Art. 88,

$$P = F = \frac{W}{n} = \frac{30 + 2240}{280} = 260 \text{ lbs.},$$

$$\text{or,} \quad P = 8\,W = 8 \times 30 = 240 \text{ lbs.}$$

Ex. 2.—The gradient of a railway rises 2 feet in 100; what power will be required to draw a train of 50 tons up the gradient, the coefficient of friction being $\frac{1}{280}$ or $n = 280$?

By Form. (1), Art. 89,

$$P = \frac{h\,n + 100}{100\,n}\,W = \frac{5 \times 280 + 100}{100 \times 280} \times 50 \times 2240 = 2640 \text{ lbs.}$$

Ex. 3.—The ascent of a turnpike road is 5 feet in 100; what power will be requisite to draw a load of 6 tons thereon, the coefficient of friction being $\frac{1}{24}$ or $n = 24$?

$$\text{Here } P = \frac{h\,n + 100}{100\,n}\,W = \frac{5 \times 24 \times 100}{100 \times 24} 6 \times 2240 =$$

$$1344 \text{ lbs.} = 112 \text{ cwt.}$$

THE USEFUL EFFECT OR MODULUS OF A MACHINE.

90. The useful effect or modulus of a machine is, the fraction which expresses the value of the work compared with the power applied, which is expressed by unity. Thus, if a machine only perform $\frac{2}{3}$ of the work applied to it, in this case $\frac{1}{3}$ of the work or power applied is lost by friction, and $\frac{2}{3}$ is called the modulus of the machine. The work that is thus lost depends on the nature and extent of the rubbing surfaces. The work thus lost in the screw, the inclined chain-pump, &c., is very great. The following is a table of the moduli of machines for raising water, with examples of their application.

Inclined chain pump............	$\frac{2}{5}$
Upright chain pump............	$\frac{1}{3}$
Bucket wheel	$\frac{2}{3}$
Pumps for draining mines ...	$\frac{2}{3}$

Ex. 1.—A power of 150 lbs. is applied to the winch which turns the axle of an inclined chain pump; what weight of water will this power raise, the length of the winch being 20 inches and the radius of the axle 4 inches?

By the property of the lever, or wheel and axle, in conjunction with the table,

$$4\,W = \tfrac{2}{5} \times 20 \times P, \text{ whence}$$

$$W = \frac{\frac{2}{5} \times 20 \times 150}{4} = 300 \text{ lbs.}$$

Ex. 2.—The piston of a steam engine draws the rod of a pump for draining a mine with a force of 6 tons; what weight of water will be raised by the piston?

Here the engine is supposed to act with a lever with equal arms.

$$\therefore \quad W = \tfrac{2}{3} P = \tfrac{2}{3} \times 6 = 4 \text{ tons.}$$

THE PRACTICAL APPLICATION OF STATICS TO THE WORK OF LIVING AGENTS AND TO MACHINERY.

In applying the principles already laid down, to estimate and compare the different kinds of work performed under different circumstances, it becomes necessary to have a distinct measure or unit of work by which the various results can be estimated and compared.

91. *The English unit of work is the power necessary to raise one pound through a space of one foot.* Thus, if one pound be raised one foot either by a living agent or by a machine, then one unit of work has been performed; if 1 lb. be raised 5 feet, then 5 units of work has been performed; if 4 lbs. be raised 6 feet, then $4 \times 6 = 24$ units of work has been performed, and so on. *Hence the units of work performed are measured by product of the weight of the body in pounds, and the space or height in feet through which it is raised;* also, pressures or resistances of every kind, in whatever direction they are exerted, may be expressed in pounds, and therefore measured by the unit of work here described.

Ex. 1.—How many units are required to raise a corf of coals of 5 cwt. from a pit, the depth of which is 60 fathoms?

Weight of coals in pounds $= 112 \times 5 = 560$ lbs.
Depth of pit in feet........ $= 60 \times 6 = 360$ feet.
$\therefore$ the units of work required $= 560 \times 360 = 201{,}600$.

Ex. 2.—The ram of a pile engine weighs 9 cwt., and it has a fall of 21 feet, required the units of work exerted in raising the ram?

$$\text{Units of work} = 9 \times 112 \times 21 = 21{,}168.$$

Ex. 3.—How many units of work will be required to pump 6000 cubic feet of water from a mine, the depth of which is 80 fathoms?

A cubic foot of water weighs $62\frac{1}{2}$ lbs., hence,

$$\text{Weight of water} = 62\tfrac{1}{2} \times 6000 = 375{,}000$$
$$\therefore \text{ units of work} = 375{,}000 \times 80 \times 6 = 180{,}000{,}000.$$

Ex. 4.—A horse, moving at the rate of 3 miles an hour, draws a bucket of water weighing 100 lbs., out of a well, by means of a rope passing over a pulley; required the units of work done per minute.

$$\text{Space passed over per minute} = \frac{5280 \times 3}{60} = 264$$
$$\therefore \text{ units of work per minute} = 264 \times 100 = 26{,}400.$$

Ex. 5.—How much labouring force will be required to raise 1000 gallons of water from a well, the depths of which is 50 fathoms?

Ex. 6.—How many units of work will be performed by a man descending a mine 50 fathoms deep, and drawing up a weight of 140 lbs. over a fixed pulley, the man's weight being supposed slightly to exceed the given weight?

Ans. 42,000 lbs.

SOURCES OF LABOURING FORCE AND THE WORK OF LIVING AGENTS.

92. The chief sources of labouring force are *animals*, including *man*, *water*, *wind*, and *steam:* the labouring force or work of animals varies according to the manner in which they exert their strength, and it is estimated by *the number of units of work* which they can raise, or move by drawing, or by pressure in any direction, in one minute. The following table shews the amount of effective work, that can be performed by several of the most common living agents.

Work done per minute.

Duration of labour, eight hours per day.

Horse power	33,000 units.*
A man turning a winch	2,600 ,,
—— drawing horizontally	3,200 ,,
—— raising materials with a pulley	1,600 ,,
—— throwing earth to the height of 5 feet	560 ,,

* This is the number of units of work assigned by Watt to a horse, but by recent experiments, it has been found to be considerably too much, $\frac{2}{3}$ of which, or 22,000 is considered to be the work of a horse of average strength; however, the number given in the table is still retained by engineers as the number of units of a horse's power.

The work of a mule and an ass are respectively estimated at $\frac{2}{3}$ and $\frac{1}{4}$ of that of a horse.

Work of water.—When water acts on the float boards of a wheel, the quantity of work which it is capable of performing is equal to the product of the weight of the water and the height through which it falls.

Work of steam.—Steam acts by its elastic force against the surface of the vessel in which it is contained, and the measure of its pressure is the number of pounds it will raise upon one square inch. Thus, if in a steam engine, the surface of the safety valve be one square inch, and have a weight of 40 lbs. placed upon it, and the steam be just able to raise the valve, and to escape, the pressure of the steam is said to be 40 lbs. per square inch.

Examples of manual power.

Ex. 1.—How many tons of earth will a man raise in 8 hours working with a winch, (wheel and axle,) from a mine 20 fathoms deep?

Here the time of work is $8 \times 60 = 480$ minutes, hence
Units of work per day .... $= 2600 \times 480$ (see table.)
Units of work in raising 1 ton to the height of 20 fathoms, or 120 feet .... $\} = 2240 \times 20 \times 6$

$$\therefore \text{ Number of tons raised } \ldots\ldots = \frac{2600 \times 480}{2240 \times 20 \times 6} = 4{\cdot}64.$$

Ex. 2.—How many cubic feet of earth of 100 lbs. per foot, will a man throw to the height of 5 feet in a day of 8 hours?

$$\text{No. c. ft.} = \frac{560 \times 60 \times 8}{100 \times 5} = 537\tfrac{3}{5}.$$

Ex. 3.—How many tons of earth will a man raise with a single pulley in a day of 8 hours, from a mine 80 feet in depth?

$$\text{No. of tons} = \frac{1600 \times 60 \times 8}{2240 \times 80} = 4\tfrac{2}{7}.$$

Examples of horse power.

Ex. 4.—How many horse powers will it require to raise 5 cwt. of coals per minute from a mine 100 fathoms deep?

Weight of coals raised per minute $= 5 \times 112 = 560$ lbs.
Depth of mine in feet .... $= 6 \times 100 = 600$ feet.
$\therefore$ units of work per minute ...... $= 560 \times 600 = 336{,}000$.

Now a horse does 33,000 units of work per minute, (see table, Art. 92.)

$$\therefore \text{ Horse powers, or HP} = \frac{336000}{33000} = 10\tfrac{2}{11}.$$

Ex. 5.—How many horse-powers will be required to lift 10000 cubic feet of water per hour, from a mine 80 fathoms deep?

$$\text{Weight of water} = 62\tfrac{1}{2} \times 10000 \text{ lbs.}$$

$$\text{Depth of mine...} = 6 \times 80 \text{ feet.}$$

$$\therefore \text{ units of work per min.} = \frac{62\frac{1}{2} \times 10000 \times 6 \times 80}{60},$$

$$\text{and HP} = \frac{62\frac{1}{2} \times 10000 \times 6 \times 80}{60 \times 33000} = 151\tfrac{17}{33}.$$

Ex. 6.—How many cubic feet of water will an engine of 10 horse-powers raise per hour, from a mine 80 fathoms deep?

Ex. 7.—Required the number of cubic feet of water which an engine of 60 horse-powers will raise per hour, from a mine 80 fathoms deep, supposing $\frac{1}{6}$ of the work to be lost by friction.

Ex. 8.—A forge hammer weighing 5 cwt. makes 60 lifts of 2 feet each in one minute; what is the horse-power of the engine that moves the hammer?

WORK IN MOVING A CARRIAGE OR RAILWAY TRAIN ON A HORIZONTAL PLANE.

NOTE.—When a locomotive engine commences its motion, its power exceeds the resistance, and therefore the speed of the engine continues to increase until the resistance becomes equal to the power of the engine, then the speed of the train will be uniform, which is commonly called a steady speed, or the greatest or maximum speed, the work destroyed by the resistance being now exactly equal to the power exerted by the locomotive engine. The same may be said of all other machines; and it is on this principle that the following investigations are made.

93. By Art. 88, the friction on a horizontal plane is $\frac{w}{n}$, or the nth part of the weight w of the carriage or train;

$\frac{1}{n}$ being the coefficient of friction, therefore the whole resistance to motion on the plane is also $= \frac{w}{n}$. Let P = power or units of work required to move the train, s = space in feet moved over in the time t in minutes, and HP = number of horse-powers in P; then P = 33000 HP = units of work or pounds moved one foot in one minute, and $\frac{s}{t}$ = feet moved in one minute by the weight $\frac{w}{n}$, whence $\frac{w}{n} \times \frac{s}{t}$ = units of work required in moving the carriage or train;

$$\therefore \; P = 33000 \text{ HP} = \frac{w}{n} \times \frac{s}{t}$$

$$\text{whence} \quad \text{HP} = \frac{W s}{33000 \, n \, t} \qquad \text{(A)}$$

In railway calculations of this kind w and s are usually given in tons and miles, which are to be reduced to pounds and feet by multiplying them respectively by 2240 and 5280, also n is most commonly = 280; if, therefore, we substitute 2240 W for w, 5280 S for s, and 280 for n, in Form. (A), we shall have, after reduction,

$$\text{HP} = \frac{128 \, W \,.\, S}{100 \, t} = \frac{1{\cdot}28 \, W \,.\, S}{t} \qquad (1)$$

$$\text{whence} \quad W = \frac{\text{HP} \,.\, t}{1{\cdot}28 \, S} \qquad (2)$$

$$S = \frac{\text{HP} \,.\, t}{1{\cdot}28 \, W} \qquad (3)$$

$$\text{and} \quad t = \frac{1{\cdot}28 \, W \,.\, S}{\text{HP}} \qquad (4)$$

Ex. 1.—Required the horse-power (HP) of a locomotive engine, which moves with a steady speed of 50 miles per hour, on a level railway, the weight of the train being 45 tons, and the friction $\frac{1}{280}$ of the weight of the train, the resistance of the air not being considered.

By Form. (1),

$$\text{HP} = \frac{1{\cdot}28\,\text{W}\,.\,\text{S}}{t} = \frac{1{\cdot}28 \times 45 \times 50}{60} = 48 \text{ horse powers.}$$

Ex. 2.—An engine of 40 HP moves with a steady speed of 35 miles per hour on a level railway; required the weight of the train, the friction being as usual.

By Form. (2),

$$\text{W} = \frac{\text{HP}\,.\,t}{1{\cdot}28\,\text{S}} = \frac{40 \times 60}{1{\cdot}28 \times 35} = 53\tfrac{4}{7} \text{ tons.}$$

Ex. 3.—In what time will an engine of 50 HP, moving a train of 60 tons, complete a distance of 40 miles?

By Form. (4),

$$t = \frac{1{\cdot}28\,\text{W}\,.\,\text{S}}{\text{HP}} = \frac{1{\cdot}28 \times 60 \times 40}{50} = 61\tfrac{11}{25} \text{ min.} = 1 \text{ h. } 1\tfrac{11}{25} \text{ min.}$$

Ex. 4.—How many miles per hour will a train of 40 tons be drawn by an engine of 35 HP, the friction being as usual?

By Form. (3),

$$\text{S} = \frac{\text{HP}\,.\,t}{1.28\,\text{W}} = \frac{35 \times 60}{1{\cdot}28 \times 40} = 41\tfrac{1}{64} \text{ miles.}$$

Ex. 5.—To how many pounds per ton does the friction amount in Example 1., the engine being 48 HP?

By transposing Form. (A)

$$\frac{1}{n} = \frac{33000\,t\,.\,\text{HP}\,.}{w\,2} = \frac{33000 \times 60 \times 48}{2240 \times 40 \times 5280 \times 50} = \tfrac{1}{280} \text{ of the}$$

weight of the train, or 8 lbs. per ton, the values of w and s being in pounds and feet respectively.

Ex. 6.—If 4 horses draw a load of 6 tons, 2 miles per hour, on a road of which the coefficient of friction is $\frac{1}{20}$; how many units of work will each horse perform?

By transposing Form. (A), and putting U instead of 33000, we shall have

$$\text{U} = \frac{w\,s}{n\,t\,.\,\text{HP}} = \frac{6 \times 2240 \times 2 \times 5280}{20 \times 60 \times 4} = 29568 \text{ units of work.}$$

Ex. 7.—What must be the effective HP of a locomotive engine, which moves with a uniform speed of 50 miles per hour, on a level railway, the weight of the train being 30 tons, and the friction as usual? *Ans.* 32 HP.

Ex. 8.—In what time will a locomotive engine of 50 HP, which moves a train of 135 tons, complete a journey of 80 miles on a level rail? *Ans.* 3 h. $4\frac{8}{37}$ min.

Ex. 9.—At what rate per hour will a train of 100 tons be drawn by an engine of 50 HP on a level rail?

Ex. 10.—The maximum speed of a locomotive engine of 50 HP is 40 miles per hour on a level rail; required the weight of the train.

WORK IN OVERCOMING THE JOINT RESISTANCES OF FRICTION AND GRAVITY ON AN INCLINED RAILWAY OR COMMON ROAD.

94. Let P = power and w = weight in pounds of a carriage or train, and h = rise of the inclined plane in every 100 feet of its length; then, by Art. 89, Form. (1), $P = \frac{100 + h\,n}{100\,n}\,w$; and let HP, s and t respectively represent the horse-powers, space in feet, and time required in moving the weight $\frac{100 + h\,n}{100\,n}\,w$, as in the last article; then P = 33000 HP = units of work in pounds, and $\frac{s}{t}$ = feet moved in one minute by the weight $\frac{100 + h\,n}{100\,n}\,w$; whence $\frac{100 + h\,n}{100\,n}\,w \times \frac{s}{t}$ = units of work required in moving the weight, which must be equal to the units of work in the power,

$$\therefore \quad P = 33000\ \text{HP} = \frac{100 + h\,n}{100\,n}\,w \times \frac{s}{t}$$

$$\text{whence} \quad \text{HP} = \frac{(100 + h\,n)\,s\,w}{33000 + 100\,n\,t}. \qquad \text{(A)}$$

Now, let W = weight moved in tons and S = space in distance moved in miles, as usually given in railway calculations; then $w = 2240\,W$, and $s = 5280\,S$; these values being substituted in Form. (A), and n being taken = 280 as in the last article, there will result, after reduction,

$$\text{HP} = \frac{256\,(5 + 14\,h)\,W\,.\,S}{1000\,t} \qquad (1)$$

$$\text{whence } W = \frac{1000\,t\,.\,\text{HP}}{256\,(5 + 14\,h)\,S} \qquad (2)$$

$$S = \frac{1000\,t\,.\,\text{HP}}{256\,(5 + 14\,h)\,W} \qquad (3)$$

$$t = \frac{256\,(5 + 14\,h)\,W\,.\,S}{1000\,\text{HP}} \qquad (4)$$

and $$h = \frac{1000\,t\,.\,\text{HP} - 1280\,W\,.\,S}{3584\,W\,.\,S}. \qquad (5)$$

NOTE.—In all these formulæ h must be taken negatively, when the weight or train descends the plane, in which case gravity assists the moving power, it also appears that when h is negative and equal $\frac{5}{14}$ of a foot, then no power is required to move the train, for the value of HP vanishes, since in this case $5 + 14\,h$ becomes $= 0$.

Ex. 1.—A train of 40 tons ascends a railway gradient, rising 2 feet in 100, with a uniform speed of 15 miles per hour; required the HP of the locomotive engine, the friction being as usual.

By Form. (1),

$$\text{HP} = \frac{256(5 + 14h)W\,.\,S}{1000\,t} = \frac{256(5 + 28) \times 40 \times 15}{1000 \times 60} = 84\tfrac{12}{25}.$$

Ex. 2.—Required the HP, as in the last Example, when the weight of the train is 60 tons, the rise 1 in 200 or $\frac{1}{2}$ in 100, and the rate of motion 30 miles per hour.

Ans. 92 HP.

Ex. 3.—An engine of 75 HP ascends a gradient, rising $\frac{3}{4}$ in 100, with a uniform speed of 20 miles per hour; required the weight of the train.

By Form. (2),

$$W = \frac{1000\,t\,.\,\text{HP}}{256(5 + 14h)S} = \frac{1000 \times 60 \times 75}{256\,(\frac{21}{2} + 5) \times 20} = 56{\cdot}7 \text{ tons}.$$

Ex. 4.—A train of 120 tons descends a gradient, rising $\frac{1}{4}$ in 100, with a uniform speed of 50 miles per hour; what is the HP exerted by the engine?

Here h must be negative, because the train descends the gradient; hence

By Form. (1),

$$\text{HP} = \frac{256\,(5 - 14\,h)\,S\,.\,W}{1000\,t} = \frac{256\,(5 - \frac{7}{2}) \times 50 \times 120}{1000 \times 60} = 38\tfrac{2}{5}.$$

Ex. 5.—A train of 50 tons ascends a railway gradient, having a rise of 1 in 600; what is the speed of the engine when its HP = 40?

By Form. (3) the speed is found = $25\frac{1}{2}$ miles per hour.

Ex. 6.—At what rate per hour will a train of 50 tons be drawn by an engine of 60 HP up a gradient rising 1 in 800?

Ex. 7.—An engine of 40 HP draws a train of 50 tons, with a uniform speed of $25\frac{1}{2}$ miles per hour up a gradient; required the rise per cent. of the gradient.

By Form. (5), the rise of the gradient is found to be $\frac{1}{6}$ per cent.

FORCES ACTING IN ANY DIRECTION IN THE SAME PLANE ON A RIGID BODY.

(A). PROP.—*To find the resultant of any number of forces acting in the same plane on a rigid body.*

Let P, P, &c. be any number of forces, acting in the directions P Q, P′ Q′, &c. on the body in the plane x A y; let A x, A y be drawn in this plane at right angles to each other; and let x, y; x', y'; &c. be the co-ordinates of the points M, M′, &c.; also, let α, α', &c. be the angles at which the forces P, P′, &c. are inclined to the axis A x, at Q, Q′, &c.

Now let each of the forces P, P′, &c. be resolved into two others X, Y; X′, Y′; &c. parallel to the axes A x, A y; then there will result

$$X = P\cos\alpha, \quad X' = P'\cos\alpha', \quad \&c.;$$

$$\text{and} \quad Y = P\sin\alpha, \quad Y' = P'\sin\alpha', \quad \&c.$$

Let X_1 be the resultant of the first set of forces, and m its distance from the axis Ax; and let Y_1 be the resultant of the second set, and n its distance from the axis Ay; then by Art. 20, there will result

$$X_1 = X + X' + \&c.$$

$$Y_1 = Y + Y' + \&c.$$

$$X_1 m = X y + X' y' + \&c.$$
$$Y_1 n = Y x + Y' y' + \&c.$$

From these four equations X_1, Y_1, m, n, can be determined. Take $AC = m$, $AB = n$, and draw C S, T B respectively parallel to Ax, Ay; then the forces X_1, Y_1, acting in the directions C S, T B, may be assumed to act at their point of intersection D; now let R be their resultant, and ϕ the angle which its direction makes with the axis Ax; then, we shall have

$$R = \sqrt{(X_1^2 + Y_1^2)},$$
$$\text{and} \quad \tan\phi = \frac{Y_1}{X_1};$$

whence the position and magnitude of the resultant of all the forces P, P′, P″, &c. becomes known,

(B). Cor.—When there is an equilibrium among the forces P, P′, P″, &c., one or more of them must act in contrary directions to the others, in which cases the sines and cosines of their directions must be negative according to the quadrant in which they fall, as estimated from the axis Ax. Hence it is evident that, in this case,

$$R = \sqrt{(X_1^2 + Y_1^2)} = 0, \quad \text{and consequently}$$
$$X_1 = 0, \quad \text{and} \quad Y_1 = 0, \quad \text{or}$$
$$X + X' + \&c. = 0, \quad \text{and} \quad Y + Y' + \&c. = 0.$$

Ex. 1.—ABCD is a square, the side of which is 20 inches; and four forces of 8, 10, 12, and 16 cwts. act in the plane of the square, at the points A, B, C, D, and make respectively with A B the angles 30°, 45°, 60°, and 150°; what is the magnitude and position of a force, which acting on AB, shall keep the square in equilibrium?

Ans. Force 28·45 cwts., acting at an angle of 80° 26′, and at a perpendicular distance of 16⅞ inches, form (A).

Ex. 2.—Required the same, as in the last Example, when, instead of a square, the figure is a rhombus, the acute angles of which are each 60°.

Ex. 3.—If any number of forces, in the same plane, act on a point and keep it at rest, they may be represented by the sides of a polygon taken in order; required the proof.

ON FORCES WHICH ARE NOT IN THE SAME PLANE.

(C.) Prop.—*To find the resultant of any given number of forces, acting in any given directions upon a point.*

Let AP be any force acting at the point A; from A draw the three rectangular co-ordinates Ax, Ay, Az, and complete the parallelopiped A P; join P B, A m; then mB is a parallelogram, hence the force A P, acting at the point A, may be resolved into two forces represented by AB, Am, acting at A; also, because CD is a parallelogram, the force Am may be resolved into the two forces AC, AD, acting at A. Thus the force A P is resolved into the three forces A B, A C, AD, acting in the directions of the three co-ordinates.—Let A P be represented by Q, and let α, β, γ be the angles which AP makes with Ax, Ay, Az respectively; then, because A B is perpendicular to the parallelogram P B, it is also perpendicular to the line PB, and ∴ AB = AP cos α = Q cos α; similarly to A C = Q cos β, and A D = Q cos γ. —Now, let Q′, Q″, &c., be any other forces acting at A, and making with A x angles α', α'', &c.; with A y angles β', β'', &c.; and with A z angles γ', γ'', &c.; then, if all these forces Q′, Q″, &c., be resolved each into three others acting at right angles to each other at A, and if we make

$$Q\cos\alpha + Q'\cos\alpha' + Q''\cos\alpha'' + \&c. = X,$$

$$Q\cos\beta + Q'\cos\beta' + Q''\cos\beta'' + \&c. = Y,$$

$$\text{and} \quad Q\cos\gamma + Q'\cos\gamma' + Q''\cos\gamma'' + \&c. = Z.$$

X, Y, Z will be respectively equal to the sum of the resolved forces, acting in the directions of A x, A y, A z. Now let R be the resultant of X, Y, Z, and π, ρ, σ the angles it makes with A x, A y, A z; and let it now be assumed that A B, A C, A D shall represent X, Y, Z; then we shall have

$$AP^2 = AB^2 + BP^2 = AB^2 + Am^2 =$$
$$AB^2 + AC^2 + AD^2; \quad \text{whence}$$
$$R = \sqrt{(X^2 + Y^2 + Z^2)} \qquad (1)$$

also $AB = AP \cos \pi = R \cos \pi$, $AC = R \cos \rho$, and
$$AD = R \cos \sigma; \quad \text{whence}$$
$$\cos \pi = \frac{X}{R}, \quad \cos \rho = \frac{Y}{R}, \quad \cos \sigma = \frac{Z}{R}. \qquad (2)$$

(D). COR. 1.—Since $R^2 = X^2 + Y^2 + Z^2 = R^2 \cos^2 \pi + R^2 \cos^2 \rho + R^2 \cos^2 \sigma$, there results
$$\cos^2 \pi + \cos^2 \rho + \cos^2 \sigma = 1. \qquad (3)$$
Therefore only two of the angles π, ρ, σ are required to determine the position of the resultant A P.

The equations (1) and (2) are available to find the magnitude and position of the resultant R of three forces, acting on a point at right angles to each other; which equations are usually given for this purpose in works on mechanics.

(E.) COR. 2.—When there is an equilibrium, we shall have
$$R = \sqrt{(X^2 + Y^2 + Z^2)} = 0,$$
consequently $X = 0$, $Y = 0$, $Z = 0$, as in former cases.

(F.) PROP.—*To find the resultant of any given number of parallel forces, acting on a rigid body, and not in the same plane.*

This may be most readily done by finding the resultant of any two of the forces (Art. 20), then by considering this resultant as one force, and next finding the resultant of this force and a third force, and so on till the resultant of all the forces be found.

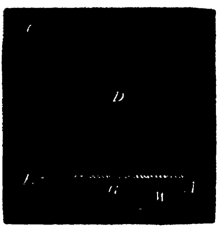
(G.) PROP.—*To determine the conditions of equilibrium of any number of forces, acting in any directions on a rigid body.*

Let P, P′, P″, &c., be the forces, acting upon a rigid body at the points Q, Q′, Q″, &c.; and let A x, A y, A z be three axes at right angles to each other; x, y, z the co-ordinates of the point Q; x', y', z' those of the point Q′, and so on. Also let α, β, γ; α', β', γ', &c., be the angles which the directions of the forces P, P′, P″, &c., make with lines parallel to A x, A y, A z respectively. Now, let the force P be resolved into three others X, Y, Z, parallel to the three axes; similarly let the force P′ be resolved into the three forces X′, Y′, Z′; and so on. Thus the forces P, P′, &c., may be resolved into three sets of forces, acting at the points Q, Q′, &c., and parallel to A *x*, A *y*, A *z*, respectively; whence the conditions of equilibrium may be readily found, as in Articles A and C and consequently the magnitude and position of their resultant.

PROBLEMS ON ALL THE PRECEDING ARTICLES.

(H.) PROB. 1.—*A bar of iron A B, 12 feet long, resting on a ledge at B, is supported in a horizontal position at the other end A by a chain A C, fastened to a hook at* C, *which is* 16 *feet directly above* B; *a weight* W *of* 12 *cwts. is suspended from the beam at* E, 8 *feet from* B; *required the tension of the cord, the weight of the iron bar being* 4 *cwts.* = *w*.

Perpendicular to A C draw B D, which is readily found to be 9·6 feet, which is the perpendicular distance, from the fulcrum B of the force or tension acting in the direction A C; and the weight of the iron bar acts at its centre of gravity G, the middle point of A B. Now, put T = tension of the chain and W = weight of the iron bar; then, either by the property of the bent lever Art. 46, or by the equality of moments, we shall have



$$\mathrm{BD} \times \mathrm{T} = \mathrm{BE} \times \mathrm{W} + \mathrm{BG} \times w = \mathrm{BE} \times \mathrm{W} + \tfrac{1}{2}\mathrm{AB} \times w,$$

$$\text{whence } \mathrm{T} = \frac{\mathrm{BE} \times \mathrm{W} + \frac{1}{2}\mathrm{AB} \times w}{\mathrm{BD}} = \frac{8 \times 12 + 6 \times 4}{9{\cdot}6} = 12\tfrac{1}{2}\text{cwts.}$$

(I.) PROB. 2.—Required the weight W, in the last Problem, which will be necessary to break the chain A C, when it can just sustain 2 tons or 40 cwts.

By transposing the formula in the preceding problem, there will result

$$\mathrm{W} = \frac{\mathrm{BD} \times \mathrm{T} - \frac{1}{2}\mathrm{AB} \times w}{\mathrm{BE}} = \frac{9{\cdot}6 \times 40 - 6 \times 4}{8} = 45 \text{ cwts.}$$

(J.) PROB. 3.—*A cone* B C, *of given dimensions and weight, rest on one edge of its base at* B, *on the horizontal* *plane* A E, *and is sustained by the cord* A C; *now* A C *and* A B *are given to determine the tension of the cord* A C.

From B draw B D perpendicular to A C and from G, the centre of gravity of the cone, draw G E perpendicular to A E; then B D, BE are respectively the perpendicular distances from the fulcrum B, at which the force of tension on A C and the weight of the cone act, whence by the property of the bent lever,

$$\text{B D} \times \text{tension on A C} = \text{B E} \times \text{wt. of cone},$$

$$\therefore \text{ tension on A C} = \frac{\text{B E} \times \text{wt. of cone}}{\text{B D}};$$

in which B D and B E are readily found by geometry, the distance of G from the base of the cone being $\frac{1}{4}$ of its height. Art. 114.

(K.) Prob. 4.—A *and* B *are two fixed points, and* W *is a weight suspended at a loop* C *in the cord* A B C, *it is required to find the tensions on the parts* A C, B C *of the cord.*

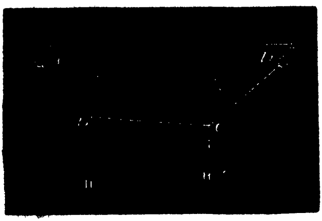
The point C of the cord is kept at rest by three forces or tensions; *i. e.*, the weight W acting by the cord C W, and the tensions or forces of the cords C A, C B acting in the directions of the cords. From any point *c* in the line C W prolonged, draw *c b*, *c a* parallel to C A, C B respectively. Now that the weight W may be supported in equilibrium, the resultant of the tensions of the cords C A, C B must be in the vertical direction *c* C, and must be equal to the tension exerted by W, and the tensions of the cords C A, C B will be respectively C *a*, C *b* (Art. 11). Hence, if P and Q represent the tensions of the cords C A, C B, we shall have

P : Q : W :: Ca : Cb : Cc :: sin BCc : sin ACc : ACB.

whence P and Q become known.

(L.) Cor.—If the figure be inverted, and A C, C B represent two props supporting a weight at C, the pressures on the two props will evidently be the same as the tensions of the cords which have been just determined.

(M.) Prob. 5.—*A cord* A B C D, *of which the ends* A, D *are fixed, and at two loops* B *and* C *are suspended two weights* W *and* W'; *required the conditions of equilibrium.*



Let the cord make with the vertical lines at the points B, C, the angles α, α', and β, β' respectively; and let B the tension of B C, which will evidently be the same both at B and C. The point B is kept at rest by three forces or tensions, *i. e.*, the weight W, the tension A in the direction AB, and the tension B in the direction B C. Whence by Art. K.

$$W : P :: \sin ABC : \sin ABW :: \sin(\alpha + \alpha') : \sin \alpha$$

$$\text{Hence } W = \frac{B \sin(\alpha + \alpha')}{\sin \alpha} = B \sin \alpha' (\cot \alpha + \cot \alpha').$$

In like manner we shall have for the point C,

$$W' = B \sin \beta (\cot \beta + \cot \beta');$$

and since β is the supplement of α', $\sin \beta = \sin \alpha'$, therefore

$$W : W' :: \cot \alpha + \cot \alpha' : \cot \beta + \cot \beta'.$$

In a similar manner we should find that, if there were more angles, the weights would be proportional to the sum of the cotangents of the angles which the supporting cords make with a vertical line.

* Note.—A cord, when kept at rest in this way, is called a *funicular polygon.*

Problems for Exercise.

Prob. 6.—The height of a cone is double the diameter of its base; what is the inclination of its axis with the horizon when it is on the point of falling? *Ans.* 45°.

PROB. 7.—The ends of a cord 10 feet long are fastened to two fixed points A and B, 6 feet apart, and in the same horizontal line; where must two weights in the proportion of 3 to 5 be hung on the cord, so that, when at rest, they shall be in the same horizontal line, at a distance of 3 feet perpendicularly below A B?

Ans. 3 feet $1\frac{3}{4}$ inches, and 3 feet $4\frac{3}{4}$ inches from the ends of the cords.

PROB. 8.—The position of the centre of gravity of a right cone being given to find the centre of gravity of a frustum thereof.

PROB. 9.—Required a solution to Prob. 3, when a given weight is suspended to the vertex of the cone.

PROB. 10.—Two uniform beams of equal length are connected by a joint at one extremity, and placed across a given cylinder; required their position when in equilibrium.

PROB. 11.—How many tons of coals will a man, working with a wheel and axle, draw in a day of 8 hours from a mine, the depth of which is 165 feet?

PROB. 12.—How many cubic feet of water will a man raise in a day of 8 hours by a pump to the height of 35 feet, supposing that he can perform 2500 units of work exclusive of the friction of the pump?

PROB. 13.—A train of 80 tons ascends a gradient rising 1 in 100; required the maximum speed, the engine being of 60 horse-powers?

PROB. 14.—What power does the engine exert and in what direction, when the train in the last example descends the gradient with a constant speed of 45 miles per hour?

PROB. 15.—The area of the piston of a high-pressure engine is 1200 square inches, the length of the stroke $8\frac{1}{2}$ feet, and the pressure of the steam upon the piston 32 lbs. per square inch and the number of strokes per minute 18; required the number of cubic feet which the engine will raise from a mine 60 fathoms deep, the friction being estimated at 1 lb. per square inch *plus* the pressure of the atmosphere.

THE CENTRE OF GRAVITY.

95. Def.—*The centre of gravity* of a body or system of bodies is that point on which the body or system will balance itself in any position whatever; or it is that point, which being supported, the body or system will be also supported; hence the whole weight of a body may be considered as collected at its centre of gravity.

96. The centre of gravity of a body is not always situated within the substance of the body. Thus the centre of gravity of a bow is somewhere in the concavity of the bow and not in its substance, and the centre of gravity of a ring is in the centre of its circumscribing circle.

97. Prop.—*If a body be suspended at any point, it will not remain at rest, till the centre of gravity be in a vertical line passing through the point of suspension.*

Let S be the point of suspension of the body S B T, G its centre of gravity; then the effect of the weight of the body to put it in motion is the same as if its matter were collected at G. Join S G, and prolong it to K, through S, G draw S T, G *g* perpendicular to the horizon; take G *g* to represent the weight of the body, draw *g* K perdendicular to S K, and complete the parallelogram G K *g* H. Then the force or weight G *g* is equivalent to the two forces G K, G H, of which G K is sustained by the reaction of the fixed point S, and G H tends effectively to move the centre of gravity in a direction perpendicular to S G, therefore the point G cannot remain at rest till G H vanishes, that is, when S G coincides with S T.

98. Cor. 1.—When G is in the vertical line S T, *below the point of suspension* S, the weight of the body will be effective in *drawing* the point S. But if G be in S T *above* S, the body will produce a *pressure* on S. In both cases the body will be at rest; but there is this important difference in the two cases, for if the body be moved from the position

of rest in the former case, it will have a tendency to return to it; but in the latter case, if the position of the body be the least changed, it will tend to move further from its position of rest. In the first case it is called *stable* equilibrium, and in the second *unstable* equilibrium.

99. Cor. 2.—If the body be suspended by a cord P S, when there is an equilibrium the line P S will be vertical, and will pass, when prolonged, through the centre of gravity of the body, which centre will evidently descend to the lowest point.

100. Cor. 3.—Hence the following *experimental* method of finding the centre of gravity. Let the body be suspended by a string, and be at rest, then the centre of gravity will be somewhere in the vertical line passing through the point of suspension. Again let the body be suspended from some other point, then a vertical line drawn through this point will also pass through the centre of gravity, which is, therefore, in the intersection of these two lines.

101. Cor. 4.—If a body be at rest upon a plane, whether horizontal or inclined, and if a line, drawn perpendicular to its centre of gravity, fall within its points of support, or within its base, the body will be at rest; but if the perpendicular fall without the points of support or base, the body will fall. For, since all the matter of the body may be considered as collected in the centre of gravity, in the former the body is supported, and in the latter not supported.

102. Cor. 5.—In like manner, it is evident, that if a body be placed on an inclined plane, and be prevented from sliding by friction, the body will rest or roll down the plane, accordingly as the vertical line passing through the centre of gravity falls within or without the base.

TO FIND THE CENTRES OF GRAVITY OF CERTAIN BODIES GEOMETRICALLY.

103. Axiom.—The centre of gravity of a material straight line of uniform thickness and density is in the middle of the line.

104. Prop.—*To find the centre of gravity of a triangle* A B C.

Bisect A B, A C in M, N; join C M, B N, cutting each other in G; then G is the centre of gravity of the triangle.

For the triangle may be considered to be composed of lines, of uniform thickness and density, drawn parallel to A B, such as *a b*; then by similar triangles,

AM : *am* :: CM : C*m* :: BM : *bm*,

and A M = B M, therefore *a m* = *b m*; hence the line *a b* will balance itself on C M. Similarly every other line parallel to A B will be in equilibrium on C M; therefore the whole triangle A B C will balance itself on C M, and consequently the centre of gravity of the triangle is in C M. In like manner it may be proved that the centre of gravity of the triangle is in the line B N; therefore G the point of intersection of C M, B N is the centre of gravity of the triangle. Join M N; then, since A M = M B and A N = N C, M N is parallel to B C, therefore,

$$BC : MN :: AB : AM :: 2 : 1;$$

also the triangles B G C, M G N are similar, and

$$CG : GM :: BC : MN :: 2 : 1$$

$$\therefore CG = 2MN, \text{ and consequently } CM = 3GM,$$

which determines the position of the centre of gravity of the triangle.

105. Cor.—In the same manner the centre of gravity of a parallelogram may be found by bisecting all its four sides, and joining the points of bisection of the opposite sides; the intersection of these joining lines will be the centre of gravity required.

106. Prob.—*To find the centre of gravity of two bodies* A *and* B, *connected by an inflexible line* A B *without weight.*

Divide A B in G so that A : B :: G B : G A; then G is the centre of gravity. For, if G be the fulcrum of a lever A B supporting the bodies A, B in equilibrium, the above proportion will hold, Art. 42, therefore G is the centre of gravity.

107. Cor. 1.—Hence A + B : A :: A B : G B.

108. Cor. 2.—Hence the centre of gravity of any number of bodies connected by inflexible right lines without weight

may be found. Let A, B, C represent there bodies of which the centre of gravity is required: join any two of them, as A and B by the line A B; which divide in g, by Art. 107, Cor. 1, so that A + B : B :: A B : A g; then g will be the centre of gravity of A and B. Now suppose the sum of the bodies A and B to be collected at g; join g and C by the line g C, and divide it at G, so that A + B + C : C :: g C : g G; then G will be the centre of gravity of the three bodies A, B and C.

109. Cor. 3.—In like manner the centre of gravity of any number of bodies may be found, by finding as in the preceding Corollaries, the consecutive centres of gravity of 2, 3, 4, &c., bodies.

110. Cor. 4.—If all the bodies be in a right line, their common centre of gravity may be found by finding the fulcrum on which they will all be in equilibrium, as in Art. 47.

111. Cor. 5.—Hence the centre of gravity of any plane rectilineal figure may be found by dividing it into triangles, and first finding the centres of gravity of each of the triangles; and supposing each of them to be collected at its centre of gravity, the centre of gravity of the whole will be found by Art. 108 and 109.

112. Prob.—*To find the centre of gravity of any irregular plane figure* A F f a.

Divide the base A F into any number n of equal parts, in A, B, C, &c., and draw the ordinates B b, C c, &c., at right angles to A F; then, if the ordinates be sufficiently near to one another, the parts $a\,b$, $b\,c$, $c\,d$, &c. may be regarded as straight lines without material error. Now, conceive the diagonals a B, b C, c D, &c. to be drawn, then the figure will be divided into $2\,n$ triangles; and putting A a = a, B b = b, &c., and A B = B C = &c. = h, the areas of these triangles will be

$$\tfrac{1}{2}\,a\,h,\ \tfrac{1}{2}\,b\,h,\ \tfrac{1}{2}\,b\,h,\ \tfrac{1}{2}\,c\,h,\ \tfrac{1}{2}\,c\,h,\ \tfrac{1}{2}\,d\,h,\ \tfrac{1}{2}\,d\,h,\ \&c.,$$

and the distances of the centres of gravity of these triangles from A a are respectively,

$\frac{1}{3}h$, $n-\frac{1}{3}h$, $n+\frac{1}{3}h$, $2n-\frac{1}{3}h$, $2n+\frac{1}{3}h$, $3n-\frac{1}{3}h$, $3n+\frac{1}{3}h$, &c., and multiplying each of these areas by the distance of its centre of gravity from A a, and adding them together, there results

$$h^2\left[(b+2c+3d----+\tfrac{1}{2}nf)+\tfrac{1}{6}(a-f)\right].$$

Now put the distance of the centre of gravity of the whole from A $a = x$; then the sum of the products just found will be equal to the area of the whole figure $\times\ x = x \times \frac{1}{2}h\,(a + 2b + 2c + 2d + ----f)$. Hence

$$x = 2h\,\frac{b + 2c + 3d-----\frac{1}{2}nf + \frac{1}{6}(a-f)}{a + 2b + 2c-----f}.$$

CENTRES OF GRAVITY OF DIFFERENT BODIES.

(See *Integral Calculus.* Weale's Elementary Series.)

113. The centre of gravity of a cylinder, prism, or any other body, the parallel sections of which are equal, is in the middle of the axis of that body.

114. In a cone or any other pyramid, the distance of the centre of gravity from the base is $\frac{1}{4}$ of the axis.

115. In a conic frustum, or in the frustum of any regular pyramid, the distance on the axis to the centre of gravity from the less end is

$$\tfrac{1}{4}\,\mathrm{L}\,.\,\frac{3\,\mathrm{R}^2 + 2\,\mathrm{R}\,r + r^2}{\mathrm{R}^2 + \mathrm{R}\,r + r^2},$$

where L denotes the length or axis, and R and r the radii of the greater and less ends in the conic frustum, or the sides of the two ends in any regular pyramid.

116. In the paraboloid the distance of the centre of gravity from the vertex is $\frac{2}{3}$ of the axis.

117. In the frustum of the paraboloid, the distance on the axis from the centre of the less end is $\frac{1}{3}\,\mathrm{L}\,.\,\dfrac{2\,\mathrm{R}^2 + r^2}{\mathrm{R}^2 + r^2}$, where L is the axis and R and r the radii of the greater and less ends.

118. In a hemisphere the distance of the centre of gravity is $\frac{3}{8}$ of the radius from the centre.

Examples for practice.

Ex. 1.—The weights of two bodies are 5 and 2 cwts., and their distance apart 21 feet; at what distance from the larger body is their common centre of gravity?

By Art. 107. $5 + 2 : 2 :: 21 : 6$ feet.

Ex. 2.—If three equal bodies, considered as points, be placed at the three angles of a triangle, then the common centre of gravity of these bodies is the same as that of the triangle; required the proof.

Since the bodies A, B and C are all equal, the centre of gravity of two of them, as A, B will be at g, the middle point of the side A B; then two bodies must now be considered as collected at g, and let C g be joined; then by Art. 107. $A + B = 2C : C :: CG : Gg :: 2 : 1$; hence $CG = 2Gg = \frac{2}{3} Cg$; $\therefore$ by Art. 104, G is the centre of gravity of the triangle A B C.

Ex. 3.—Four bodies, considered as points, the weights of which are 6, 8, 10 and 12 lbs., are placed at the successive angles of a square whose side is 3 feet; required the distance of their common centre of gravity from the largest body.

Ex. 4.—Two spheres of given diameter touch one another internally; required the centre of gravity of the solid included between the surfaces of the two spheres.

Ex. 5.—A iron rod of uniform thickness, 8 feet in length and weighing 80 lbs., has a weight of 60 lbs. suspended at one end; what point in the rod will be the centre of gravity?

As the rod is of uniform thickness its centre of gravity is at its middle point, that is 4 feet from the point where the weight is suspended; whence

$80 + 60 : 60 :: 4 : 1\frac{5}{7}$ feet from the less end.

Ex. 6.—One of the sides of a given right angled triangle rest on a horizontal plane, and the other side is vertical; reguired the greatest isosceles triangle which can be described on the hypothenuse as a base, so that the whole figure shall not fall.

Ex. 7.—The height of a cylinder is double the diameter of its base; required the angle of inclination of its base with the horizon, when it is just ready to fall. *Ans.* 30°.

Ex. 8.—Seven equal weights are placed at seven of the angles of a cube; required the distance of their common centre of gravity from the remaining angle.

Ex. 9.—A hemisphere and a cone abut from a common base; required the centre of gravity of the solid included by their surfaces.

PART II.

DYNAMICS.

DEFINITIONS.

119. *Dynamics* treats of the action of forces producing motion, and of the laws of motion.

120. *Motion* is the act of a body's changing its place; and is divided into two kinds, *absolute* and *relative.*

A body is said to be in *absolute motion,* when it is transferred from one point of fixed space to another; and to be in *relative motion,* when it changes its situation with respect to surrounding bodies.

121. *Uniform motion* is when a body passes over equal spaces in equal times.

122. *Accelerated motion* is when a body continually increases its motion over successive portions of space in equal times, and it is called *retarded motion,* when the spaces described continually decrease.

123. *Velocity* is the degree of swiftness or slowness of a body's motion, and it is *measured* by the space *uniformly described* in a unit of time, as in one second.

124. The *momentum* of a body is the product of its velocity and quantity of matter, which last is in the compound ratio of its density and magnitude.

125. *Accelerating force* is measured by the velocity uniformly generated in a given time, without regard to the quantity of matter moved.

126. *Moving force* is measured by the momentum uni-

formly generated in a given time, and it is equal to the product of the accelerating force and the quantity of matter.

127. Let s be the space described in the time t, with the uniform velocity v; then by Def. 123,

$$s = tv, \quad \text{whence} \quad t = \frac{s}{v}, \quad \text{and} \quad v = \frac{s}{t}.$$

Thus, if a body move uniformly at the rate of 5 feet per second, and is 2 minutes or 120 seconds in motion; then

$$s = tv = 5 \times 120 = 600 \text{ feet},$$

the space or distance passed over by the body.

128. Let M be the momentum of a body, W its quantity of matter or weight, and v its velocity; then by Def. 124,

$$M = Wv, \quad \text{whence} \quad W = \frac{M}{v}, \quad \text{and} \quad v = \frac{M}{W}.$$

Thus, if a body weighing 20 lbs. moves with a velocity of 6 feet per second, then

$$M = Wv = 20 \times 6 = 120 = \text{the momentum}.$$

NEWTON'S LAWS OF MOTION.

129. *First.* A body in motion, and not acted upon by any external force, will move with a uniform velocity in a straight line.

130. *Second.* When a force acts upon a body in motion, the change of motion in quantity and distance, is the same as if the force acted upon the body at rest.

131. *Third.* When pressure produces motion in a body, the momentum generated in a given short time is proportional to the pressure.

132. These three laws of motion are the simplest principles to which dynamics can be reduced, and on them the whole theory rests. These laws, however, do not admit of accurate proof by experiment, on account of the many causes of error, which are impossible to exclude; but are firmly established, from the following considerations. By assuming these laws to be true, and applying them to the investigation of the motions of the heavenly bodies, innumerable accurate results have been deduced, by operations more or less complex; and these results have, in every case, been found to

agree with actual observation: it follows, therefore, that these laws must be true.

133. Prop.—*When the force accelerates uniformly, the velocity generated in a given time is equal to the product of the force and time.*

Let f be the accelerating force; then $f =$ velocity generated in one second of time; and, since the force is uniform, f will also be the velocity to be added in the end of the next second; hence $2f$ will be the whole velocity generated in two seconds. Similarly $3f$ will be the velocity at the end of three seconds; and generally tf will be the velocity at the end of t seconds; therefore, putting $v =$ velocity, there results

$$v = ft, \qquad \text{hence} \quad t = \frac{v}{f}.$$

Thus, if $f =$ force of gravity at the earth's surface $= 32\frac{1}{6}$ feet, and the time of motion be three seconds, then

$$v = ft = 32\tfrac{1}{6} \times 3 = 96\tfrac{1}{2} \text{ feet.}$$

134. Prop.—*If a body be urged by a constant and uniform force, the space which it describes, from the beginning of the motion, is equal to half the product of the force and the square of the time.*

Let the time be divided into an indefinite number of equal parts; then, in each of these equal parts of time, the space described will be equal to the velocity gained; that is, by Art. 133, = force multiplied by the time from the commencement of motion; and the sum of all these spaces, or the whole space passed over, will be equal to the force multiplied by the sum of all these equal parts of time from the beginning of motion. Put $t =$ whole time, $s =$ the whole space described, and $f =$ force; then

$$s = (1 + 2 + 3 + 4 + \&c.\dots t)f.$$

But the sum of the arithmetical series $1 + 2 + 3 + 4 + \&c.\dots t$ is $= \frac{t+1}{2}t$; therefore, $s = \frac{t+1}{2}tf$; but when t is indefinitely great compared with the indefinitely small parts or units into which it was assumed to be divided, we shall have ultimately

$$\frac{t+1}{2}\,tf = \tfrac{1}{2}\,t^2 f,$$

therefore $\quad s = \tfrac{1}{2}\,t^2 f;$

whence $\quad t = \sqrt{\dfrac{2\,s}{f}},$

and $\quad f = \dfrac{2\,s}{t^2}.$

135. COR.—Since $v = ft$, $t = \dfrac{v}{f}$, whence,

by substitution $\quad s = \tfrac{1}{2}\,t^2 f = \dfrac{v^2}{2f},$

therefore $\quad v = \sqrt{2\,sf}.$

136. PROP.—*If a body, urged by a constant and uniform force, move through any given space, it will move through twice that space in the same time by the velocity acquired.*

For, by Art. 133, $v = ft$ and, by the last Art. $s^2 = \tfrac{1}{2}\,t^2 f = \tfrac{1}{2}\,t \times tf$; therefore, by substitution $s = \tfrac{1}{2}\,t\,v$; but the space described in the time t with the last velocity v is $t\,v$; therefore, the space described by the last velocity is twice the space described in the same time by the accelerating force.

137. PROP.—When a body is projected with a given velocity V, and acted upon in the same direction by a constant force f : to find the space s described in the time t.

By Art. 135, the space described in the time t by the velocity $V = V\,t$; and by Art. 134, the space described in the same time by the constant force f, will be $= \tfrac{1}{2}\,t^2 f$: but by the second law of motion, when any force is exerted on a body in motion, the effect is the same as if it acted upon a body at rest; therefore, the whole space described will be equal to the sum of the spaces described by each motion separately; consequently

$$s = Vt + \tfrac{1}{2}\,t^2 f = (V + \tfrac{1}{2}\,tf)\,t.$$

138. COR. 1.—If the body be projected in a direction opposite to that in which the force acts, we shall have, for a like reason,

$$s = (V\,t - \tfrac{1}{2}\,t^2 f) = (V - \tfrac{1}{2}\,tf)\,t.$$

139. Cor. 2.—In the same manner it may be proved that

$$v = V + tf,$$

when the body is projected in the direction of the force, and

$$v = V - tf,$$

when projected in the opposite direction.

140. Gravity at the earth's surface or terrestrial gravity, is that force by which bodies are urged towards the centre of the earth, and it is measured by the velocity it generates in a second of time. Experiments shew that a falling body descends $16\frac{1}{12}$ feet in the first second, neglecting the resistance of the air, and that it has then acquired a velocity of $2 \times 16\frac{1}{12} = 32\frac{1}{6}$ feet, which is the true measure of the force of gravity; this quantity is usually represented by the letter g, which being substituted for f in the Formulæ of Articles 134 and 135, there will result

$$s = \tfrac{1}{2} g t^2 = \frac{v^2}{2g} = \tfrac{1}{2} l v. \qquad (1)$$

$$v = \sqrt{2gs} = 2gt = \frac{2s}{t}, \qquad (2)$$

$$\text{and} \quad t = \sqrt{\frac{2s}{g}} = \frac{v}{g} = \frac{2s}{v}. \qquad (3)$$

Ex. 1.—Find the space through which a heavy body falls in 10 seconds, and the velocity acquired, g being $= 32\frac{1}{6}$,

Here $s = \frac{1}{2} g t^2 = \frac{1}{2} \times 32\frac{1}{6} \times 10^2 = 1608\frac{1}{3}$ feet, the space,

and $v = 2gt = 2 \times 32\frac{1}{6} \times 10 = 643\frac{1}{3}$ ft. the vel. per second.

Ex. 2.—How far must a body fall to acquire a velocity of 120 feet per second?

$$\text{Here} \quad s = \frac{v^2}{2g} = \frac{120^2}{64\frac{1}{3}} = 223\frac{4}{5} \text{ feet.}$$

Ex. 3.—In what time will a body be in falling through a space of 100 feet?

$$\text{Here} \quad t = \sqrt{\frac{2s}{g}} = \sqrt{\frac{200}{32\frac{1}{6}}} = 2\tfrac{1}{2} \text{ seconds nearly.}$$

Ex. 4.—How far must a body fall to acquire a velocity of 1000 feet per second?

Ex. 5.—An arrow, shot perpendicularly upwards, returned again in 10 seconds; required the velocity with which it was shot, and the height to which it rose, taking $g = 32\frac{1}{5}$.

Ans. Velocity $= 161$ feet; height $= 402\frac{1}{2}$ feet.

Ex. 6.—A body is projected vertically with a velocity of 1000 feet per second; required its situation at the end of 10 seconds.

Ex. 7.—With what velocity must a body be projected downwards, from a height of 150 feet, that it may pass over that space in 2 seconds?

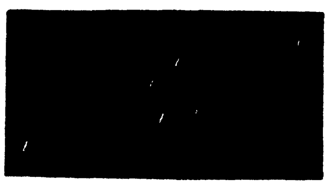
This is done by Art. 137, and the vel. is found to be $42\frac{4}{5}$ ft.

Ex. 8.—If a body be projected perpendicularly downwards with a velocity of 50 feet per second; where will the body be at the end of 10 seconds? *Ans.* $1429\frac{1}{3}$ feet.

Ex. 9.—A stone is dropped into a well, and after 3 seconds it is heard to strike the water; required the depth of the surface of the water, the velocity of sound being about 1127 feet per second. *Ans.* $135\frac{1}{2}$ feet, nearly.

Ex. 10.—A body has fallen through m feet when another body begins to fall from a point n feet below it; required the distance the latter body will fall, before it is passed by the former. *Ans.* $\frac{n^2}{4m}$.

141. *To determine the conditions of motion on an inclined plane through the effect of gravity.*



Let A B be an inclined plane, B C its horizontal base, A C its height, and P a body descending on the plane; from P, the centre of gravity of the body, draw Pp perpendicular to B C to represent the pressure of P occasioned by gravity; draw also P e parallel and Pf perpendicular to A B, and complete the parallelogram ef, then the force Pp is equivalent to the two Pe, Pf, of which Pf is sustained by the reaction of the plane; the force Pe is wholly efficient in accelerating the motion of the body P. Let this force be represented by f and Pp by g force of gravity, then by similar triangles.

$$f : g :: Pe : Pp : AC : AB,$$

$$\therefore \quad f = \frac{AC \times g}{AB}.$$

Now, put $AB = l$, $AC = h$, and the angle $ABC = \alpha$, then the force, which produces motion on the inclined plane. becomes

$$f = g\frac{h}{t} = g \sin \alpha.$$

Hence the accelerating force on an inclined plane is constant, and the equations of motion will be obtained by substituting this value of f for g in equations (1), (2), and (3) of Art. 140.

$$s = \frac{g h t^2}{2 l} = \frac{l v^2}{2 g h} = \tfrac{1}{2} t v \qquad (1).$$

$$v = \frac{2 s}{t} = \frac{g h t}{l} = \sqrt{\frac{2 g h s}{l}} \qquad (2),$$

$$\text{and} \quad t = \frac{2 s}{v} = \frac{l v}{g h} = \sqrt{\frac{2 l s}{g h}} \qquad (3).$$

$$\text{Also} \quad \sin \alpha = \frac{h}{l} = \frac{v}{g t} = \frac{2 s}{g t^2} = \frac{v^2}{2 g s} \qquad (4),$$

$$s = \tfrac{1}{2} g t^2 \sin \alpha = \frac{v^2}{2 g \sin \alpha} \qquad (5),$$

$$v = g t \sin \alpha = \sqrt{2 g s \sin \alpha} \qquad (6),$$

$$\text{and} \quad t = \frac{v}{g \sin \alpha} = \sqrt{\frac{2 s}{g \sin \alpha}} \qquad (7).$$

Cor. 1.—If s be taken $= l$ in Form. (2), $v = \sqrt{\frac{2ghs}{l}}$, it becomes $v = \sqrt{2gh}$; hence, the velocity acquired is the same as would be acquired in falling through the height of the plane.

Cor. 2.—If a body be projected down or up an inclined plane with a given velocity v then the distance s, which the body will be from the point of projection in a given time t, will be respectively

$$s = tv + \frac{g h t^2}{2 l} = \frac{t}{2 l}(2 l v + g h t) \qquad (8),$$

and $$s = tv - \frac{2 h t^2}{2 l} = \frac{t}{2 l}(2 l v - g h t) \qquad (9).$$

Ex. 1.—The length of an inclined plane is 200 feet, and its height 25 feet; through what space will a body descend on it in 6 seconds?

By Form. (1),

$$s = \frac{\frac{1}{2} g h t^2}{l} = \frac{16\frac{1}{12} \times 25 \times 6^2}{200} = 72 \text{ feet } 4\tfrac{1}{2} \text{ inches}$$

Ex. 2.—The length of an inclined plane is 100 feet, and its angle of inclination 60°, required the time of falling down it, and the velocity acquired?

By Form. (7),

$$t = \sqrt{\frac{2 s}{g \sin \alpha}} = \sqrt{\frac{200}{32\frac{1}{6} \times \cdot 866}} = 2\cdot 68 \text{ seconds}$$

and by (6) $v = g t \sin \alpha = 32\frac{1}{6} \times 2\cdot 68 \times \cdot 866 = 74\cdot 6$ feet.

Ex. 3.—If an inclined plane rise $2\frac{1}{2}$ feet in 100, in what time will a body, descending down this plane, acquire a velocity of 5 feet per second?

By Form. (3),

$$t = \frac{l v}{g h} = \frac{100 \times 5}{32\frac{1}{6} \times 2\frac{1}{2}} = 6\cdot 22 \text{ seconds.}$$

Ex. 4.—If a body be projected up an inclined plane, which rises 1 in 6, with a velocity of 50 feet per second; required its place and velocity after 6 seconds.

Ans. $203\frac{1}{3}$ feet from the bottom, and velocity $17\frac{4}{5}$ feet.

Ex. 5.—If a body be projected with a velocity of 40 feet per second down an inclined plane, which rises 1 in 3; what space will it have moved through at the end of 6 seconds?

Ans. $144\frac{1}{3}$ yards.

142. Prop.—*If a circle be in a vertical plane, the times of descent down all its cords, drawn from both extremities of*

its vertical diameter, are equal; and the velocities acquired in falling down the chords are proportional to their lengths.

Let A C B be a circle, A B its vertical diameter, A C any chord drawn through A, and C D perpendicular to A B; then by the last article.

$$\text{time down } AC = \sqrt{\frac{2\,AC^2}{g\,.\,AD}}$$

$$= \sqrt{\frac{2\,AB}{g}} \text{ since by geo. } \frac{AC^2}{AD} = AB.$$

This result being independent of the position of C, the times of descent down all chords are equal, and also equal to time of falling freely down the diameter A B.

Also by the last article the velocity acquired down A C is equal to

$$\sqrt{2g \times AD} = \sqrt{\frac{2g\,AC^2}{AB}} = AC\sqrt{\frac{2g}{AB}}.$$

In which, since g and A B are constant, the velocity acquired down any chord A C is as the length of A C.

In the same manner this proposition may be proved with respect to any other plane C B.

Lemma.—Previous to discussing the following propositions, it will be proper to give Atwood's experiment, for examining the motions of bodies, when acted upon by constant forces. The machine he used for this purpose was a single fixed pulley, with its axis placed on wheels to diminish the friction. Two equal weights P, P are placed in two similar and equal boxes, connected with a string passing over the pulley; then these weights will exactly balance each other. Now, let another weight p be added to each of them separately, and it will then be found that the velocity generated in a given time is always proportional to $\frac{p}{2P+p}$; that is, if the whole mass or weight moved be the same, that is, $2P+p$, the velocity as p, the weight that puts the whole system in motion; and if p be constant, the velocity is inversely as $2P+p$ which is the whole mass or weight moved. Since, then, the velocity is proportional to

$\frac{p}{2P+p}$, it follows that p is proportional to $(2P+p) \times$ velocity, and therefore is proportional to the momentum generated in a given time, or varies as the moving force. This establishes the truth of Newton's third law of motion in a most satisfactory manner.

143. PROP.—*When two unequal bodies are connected by a cord hanging over a fixed pulley, to determine the nature of their motion, the cord and pulley being considered without weight.*

Let P and Q be the two bodies; then it is evident that the moving force is in this case proportional to the excess of P over Q, that is, to $P-Q$; but the accelerating force is as the moving force divided by the quantity of matter moved, by the third law of motion, and is therefore as $\frac{P-Q}{P+Q}$. When Q $=0$, the body falls freely, and the accelerating force is the force of gravity g; hence the accelerating force in this case is $f=\frac{P-Q}{P\times Q}g$. This value of the accelerating force being substituted for g in the formulæ Art. 140, will shew the relation between the space, velocity, and time of the two bodies.

144. PROP.—*To find the accelerating force when one body draws another along an inclined plane.*

Let a body P descend down the inclined plane, and draw the body Q up another inclined plane, (see fig. Art. 77); and let α and β be the respective angles of elevation of the planes, on which P and Q are in motion; then the force of P in the direction of its plane is equal to $P \sin \alpha$, and the force of Q in the direction of its plane is equal to $Q \sin \beta$. Now, if these forces be equal, the bodies P and Q will be in equilibrium; but if $P \sin \alpha$ be greater than $Q \sin \beta$, P will descend and draw Q up the inclined plane. Since the difference of these forces, *i. e.*, $P \sin \alpha - Q \sin \beta$ produces motion, and the whole mass moved is $P+Q$, it may be shewn, as in the last article, that the accelerating force is

$$f=\frac{P \sin \alpha - Q \sin \beta}{P+Q} g.$$

145. COR. 1.—When P hangs vertically, then $\alpha = 90°$, and therefore,

$$f = \frac{P + Q \sin \beta}{P + Q} g.$$

146. Cor. 2.—When P hangs vertically, and Q is on a horizontal plane, then $\alpha = 90°$ and $\beta = 0$, therefore,

$$f = \frac{P}{P+Q} g.$$

Ex. 1.—P and Q hang over a fixed pulley, P = 97 lbs. and Q = 96 lbs.; required the space descended by P in 10 seconds.

Here $f = \frac{P-Q}{P+Q} g = \frac{97-96}{97+96} g = \frac{g}{193}$, which, being substituted for g in $s = \frac{1}{2} g t^2$, gives $\frac{\frac{1}{2} g t^2}{193} = \frac{193 \times 100}{12 \times 193} =$ $8\frac{1}{3}$ feet = space descended by P.

Ex. 2.—A weight P of 1 lb. drags a weight Q of 99 lbs. along a smooth horizontal table; required the distance descended in 10 seconds.

Here $f = \frac{P}{P+Q} g = \frac{1}{100} g$, this substituted for g in $s = \frac{1}{2} g t^2$, gives $s = \frac{\frac{1}{2} g t^2}{100} = \frac{16\frac{1}{2} \times 100}{100} = 16\frac{1}{12}$ feet = space descended by P.

Ex. 3.—There are two equal weights, one of which descends vertically and draws the other up a plane, inclined 30° to the horizon; required the accelerating force.

Ans. $f = \frac{1}{4} g = 4\frac{1}{48}$ feet.

ON MOTION UPON A CURVE AND THE VIBRATIONS OF PENDULUMS.

147. Prop.—*If a body fall from rest down a perfectly smooth curved surface, the velocity acquired is equal to that which would be acquired in falling through the same perpendicular height.*

Let A B C D be a system of planes; draws D T parallel and A T perpendicular to the horizon, also draw B b, C c, parallel to D T. The velocity acquired in falling from A to B is equal to that which would be acquired in falling from A to b (Art. 141, Cor. 1), and supposing no velocity is lost in passing from one plane to another, the body will begin to descend down B C with the velocity acquired in falling through A B consequently the velocity acquired at C will be the same as in falling perpendicularly through A *c*. Similarly it may be shown that the velocity acquired at D will be equal to that which would be acquired in falling through the perpendicular distance A T, supposing no velocity to be lost in passing from one plane to another. Now, let the number of planes be increased indefinitely, then the angles at B, C, &c., will be diminished indefinitely, therefore the velocity lost is diminished indefinitely, and the system of planes approximates to a curve A B C D as its limit, in which, therefore, no velocity will be lost. Hence the whole velocity acquired in falling down the curve A B C D is equal to that which would be acquired in falling down the same perpendicular altitude A T.

148. COR. 1.—If a body be projected up a curve, the perpendicular altitude to which it will ascend is equal to that through which it must fall to acquire the velocity of projection ; since the body in ascending will be retarded by the same degrees that it was accelerated in descending.

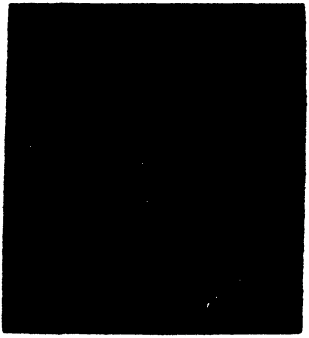
149. COR. 2.—This proposition is true when the body is retained in the curve by a string or cord, which is at every point perpendicular to the curve, since the string will, in this case, support that part of the weight of the body, which was before supported by the curve.

THE PENDULUM.

150. DEF. 1.—A *pendulum* consists of a heavy body, suspended by a thread or slender wire, and made to vibrate in a vertical plane. When the body is considered as a point, and the thread or wire without weight, it is called a *simple pendulum.*

151. DEF. 2.—The time from the commencement of motion till all the velocity is lost by the ascent of the body, is called the *time of oscillation;* and the angles through which the body moves is called the amplitude.

152. Prop.—*To find the time of oscillation of a pendulum in a small circular arc.*



Let a body descend from A, and be kept in the circular arc AOB by the thread CA, which is supposed to be without weight; let the body descend to M, and let the arc M N be indefinitely small; then we may consider this arc with the velocity at M uniformly continued. Put this velocity $= v$, then the time through $\text{M N} = \frac{\text{M N}}{v}$; but the velocity at M is equal to the velocity acquired in falling from D to P (Art. 147), therefore

$$v^2 = 2g \times \text{D P} = 2g(\text{O D} - \text{O P})$$

$$= 2g \cdot \frac{(\text{chord A O})^2 - (\text{chord M O})^2}{2\text{C O}}$$

$$= \frac{g}{l}\left((\overline{\text{arc A O}})^2 - \overline{\text{arc M O}})^2\right) \text{ very nearly;}$$

since the arc AO is supposed never to exceed 2 or 3 degrees, l being the length of the pendulum $=$ A C. Now, take $\text{O}a =$ arc O A, and describe the semicircle adb, with the radius $\text{O}a$; draw mh, nd perpendicular, and hc parallel to $\text{O}a$, and join $\text{O}h$; then

$$v^2 = \frac{g}{l}(\text{O}a^2 - \text{O}m^2) = \frac{g}{l}mh^2; \text{ and } v = mh\sqrt{\frac{g}{l}};$$

$$\therefore \text{ time through M N} = \frac{\text{M N}}{v} = \frac{mn}{mh}\sqrt{\frac{l}{g}}.$$

Now M N or mn being indefinitely small, hd may be considered as a straight line, and the triangles $\text{O}hm$, chd will be ultimately similar, therefore

$$\text{O}h : mh :: hd : he \text{ or } mn;$$

$$\text{whence } \frac{mn}{mh} = \frac{hd}{\text{O}h} = \frac{hd}{\text{O}a};$$

$$\therefore \text{ time through } MN = \frac{hd}{Oa} \sqrt{\frac{l}{g}} = \frac{hd}{OA} \sqrt{\frac{l}{g}}.$$

As this is true for every indefinitely small part of AO or Oa, and as the sum of all the portions hd is evidently equal to the semi-circumference adb, the time of moving through $AOB = \frac{adb}{Oa} \sqrt{\frac{l}{g}}$; and since

$$adb : Oa = OA :: \pi : 1 \quad \text{there results}$$

$$\pi = \frac{adb}{OA}$$

$$\therefore \text{ time of describing } AOB = \pi \sqrt{\frac{l}{g}};$$

which, putting t for the time of oscillation, becomes

$$t = \pi \sqrt{\frac{l}{g}}.$$

153. Cor. 1.—Hence, if L and l be the lengths of two pendulums, and T and t the times of their vibrations; then, since π and g are constant, we shall have

$$T : t :: \sqrt{L} : \sqrt{l},$$

$$\text{or} \quad T^2 : t^2 :: L : l.$$

From which proportions, if the length of a pendulum and the time of its vibration be given, the length of any other pendulum to vibrate in a given time may be found, and *vice versa.*

154. Cor. 2.—In the latitude of London the length of the seconds pendulum is found by experiment to be $39\frac{1}{8}$ inches, nearly; substituting this for l in the Form., $t = \pi \sqrt{\frac{l}{g}}$, taking $t = 1$ second, and transposing, there results

$$g = \pi^2 l = 32\tfrac{1}{6} \text{ feet, nearly.}$$

155. Cor. 3.—If n be the number of seconds in a day, and s the number of seconds in the same time; then

$$n = \frac{s}{t} = \frac{s}{\pi} \sqrt{\frac{g}{l}};$$

hence, if g be given n, the number of vibrations varies inversely as the square of the length l of the pendulum.

156. Cor. 4.—If l be increased by a small quantity λ, and n be diminished by a corresponding quantity ν; then

$$\frac{n-\nu}{n} = \sqrt{\frac{l}{l+\lambda}} = 1 - \frac{1}{2}\frac{\lambda}{l} \quad \text{nearly,}$$

$$\text{whence} \quad \frac{\nu}{n} = \frac{1}{2}\frac{\lambda}{l}, \quad \text{or} \quad \nu = \frac{n\lambda}{2l}.$$

157. Cor. 5.—If l be given, and g be increased by a small quantity γ, also let ν be the corresponding increment of n; then

$$\frac{n+\nu}{n} = \sqrt{\frac{g+\gamma}{g}} = 1 + \frac{1}{2}\frac{\gamma}{g} \quad \text{nearly,}$$

$$\text{hence} \quad \nu = \frac{n\gamma}{2g}.$$

158. Cor. 6.—The force of gravity above the earth's surface varies inversely as the square of the distance, in the same latitude; therefore, if r = radius of the earth, h = height of any place above the surface, γ the gravity at that height, and ν the number of seconds, which a pendulum vibrating seconds at the earth's surface, loses in a day; then

$$\frac{n-\nu}{n} = \sqrt{\gamma} = \frac{r}{r+h} = 1 - \frac{h}{r} \quad \text{nearly;}$$

$$\therefore \quad \nu = \frac{nh}{r}.$$

159. Note.—The force of gravity has been found to vary in different latitudes; the increment of its force above its force at the equator being nearly as the square of the sine of latitude.

Ex. 1.—The length of a pendulum is 60 inches; in what time will it vibrate?

By Cor. 1. $\quad T^2 : t^2 :: L : l$,

but l has been found (Cor. 2) to be $39\frac{1}{8}$ inches, when it vibrates seconds; hence

$$T^2 : 1^2 :: 60 : 39\tfrac{1}{8}$$

$$\therefore \quad T = \sqrt{\frac{8 \times 60}{313}} = \sqrt{\frac{480}{313}} = 1{\cdot}24 \text{ seconds, nearly.}$$

Ex. 2.—Required the length of a pendulum, that makes 80 vibrations in a minute. *Ans.* 22 inches.

Ex. 3.—If a clock loses 1 minute in 24 hours, how much must the pendulum be shortened to make it keep true time?
Ans. $\frac{1}{18}$ of an inch, nearly.

Ex. 4.—A seconds pendulum is carried to the top of a mountain 1 mile high; what number of seconds will it lose in a day, the radius of the earth being 4000 miles?

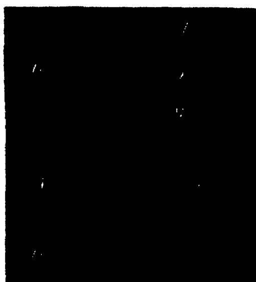
By Cor. 6, $\nu = \frac{nh}{r} = \frac{86400 \times 1}{4000} = 21\frac{3}{5}$ seconds.

Ex. 5.—A pendulum vibrating seconds at the equator, when carried to the pole, gains 5 minutes per day; find the proportion of the equatorial and polar gravity.
Ans. 144 : 145.

ON THE MOTION OF PROJECTILES.

160. Prop.—*A body projected obliquely to the horizon, will describe a parabola; supposing that the motion is not effected by the resistance of the air.*

Let a body be projected from A in the direction A T; through A draw D B perpendicular to the horizon, and let A T be the space the body would describe with the velocity of projection continued uniformly during the time t, and A B the space through which gravity would cause it to descend in the same time; complete the parallelogram A M; then, since the motion in the direction AT neither accelerates nor retards the approach of the body to the line B M, the body will be in the line B M at the end of the time t. By like reasoning the body will be in the line T M at the end of the same time, and therefore it will be at M, the point of their intersection at the end of the time t. Let V be the velocity of projection, then, because A T is the space which would be described in the time t with the velocity V continued uniformly,



$AT = Vt$; and, since AB is the space through which the body would fall by gravity in the time t, $AB = \frac{1}{2} g t^2$; hence

$$t^2 = \frac{2 AB}{g} = \frac{AT^2}{V^2} = \frac{BM^2}{V^2}; \quad (\text{since } AT = Vt.)$$

$$\therefore \quad BM^2 = \frac{2 V^2}{g} AB. \qquad (1)$$

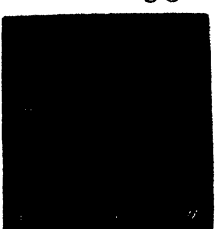
Hence the curve AM is a parabola, of which AB is the diameter, BM an ordinate, and $\frac{2 V^2}{g}$ the parameter. (See *Hann's Analytical Geometry, Weale's Series.*)

161. COR. 1.—If AD be taken $= \frac{1}{4}$ of the parameter at $A = \frac{1}{4} \times \frac{2 V^2}{g} = \frac{V^2}{2g}$, and DE be drawn perpendicular to BD, DE, will be the directrix of the parabola. For the distance from the directrix to any point in a parabola is equal to the distance of this point from the focus. (See *Hann's Analyt. Geo. Art.* 25, *Par.*).

162. COR. 2.—The horizontal velocity of the body is uniform, since it is evidently not affected by gravity.

163. COR. 3.—Because $AD = \frac{V^2}{2g}$, AD is the space through which a body would fall to acquire the velocity of projection V. AD is usually called the *impetus*, or height due to the velocity in the curve.

164. COR. 4.—AT is a tangent to the parabola at A, because it is parallel to BM; (see *Hann's Analy. Geo.*), therefore, if the angle TAC be made $= TAD$, and $AC = AD$, C will be the focus of the parabola; and, the focus and directrix being given, the parabola can be constructed.



165. PROP.—*To find the equation of the parabolic curve, described by the projectile, referred to horizontal and vertical co-ordinates.*

Let $h = AD =$ impetus $=$ height of the directrix, $AP = x$, $PM = y$, the angle $PAm = \alpha$, $V =$ velocity of projection, and $t =$ time of describing AM; then

$$AP = Am \cos\alpha; \quad \text{or} \quad x = Vt\cos\alpha;$$

since $Am = Vt$; also $Pm = x\tan\alpha$; and $Mm = \frac{1}{2}gt^2$;

$$\therefore \quad y = Pm - Mm = x\tan\alpha - \tfrac{1}{2}gt^2;$$

by substituting the value of t, from the equation $x = Vt\cos\alpha$, in the above equation, there results

$$y = x\tan\alpha - \frac{gx^2}{2V^2\cos^2\alpha};$$

and by substituting h for $\frac{V^2}{2g} = AD$ (Cor. 3), this equation becomes

$$y = x\tan\alpha - \frac{x^2}{4h\cos^2\alpha},$$

which is the equation of the curve, from which all the properties relating to projectiles may be readily derived.

166. NOTE.—If the curve meet the horizontal plane AH, passing through the point of projection A, the distance AH is called the *horizontal range*, and the time of describing the curve A M H is called the *time of flight*.

167. PROP.—*The velocity and direction of projection being given, to find the horizontal range, the time of flight, and the greatest height to which the body will rise above the horizontal plane.* (See last figure.)

Let the body be projected from A in the direction A T; also let A M H be the parabolic path of the projectile, and AH the horizontal range; then

$$TH = AT\sin\alpha = Vt\sin\alpha; \quad \text{but} \quad TH = \tfrac{1}{2}gt^2;$$

$$\text{hence} \quad Vt\sin\alpha = \tfrac{1}{2}gt^2$$

$$\therefore \quad t = \frac{2V\sin\alpha}{g}. \qquad (1)$$

Again, $AH = AT\cos\alpha = \frac{2V}{g}\sin\alpha \times V\cos\alpha$; and if AH be put $= R$, $\frac{V^2}{2g} = h$, and $2\sin\alpha\cos\alpha = \sin 2\alpha$; then there will result

$$R = 2h\sin 2\alpha. \qquad (2)$$

Lastly, if the point C bisect A H, the maximum height H is evidently $CB = \frac{1}{4} Ch$ (see *Hann's Analyt. Geo.*) $= \frac{1}{4} TH$; but $TH = \frac{1}{2} g t^2 = \{\text{from equa. } (1)\} \dfrac{2V^2}{g} \sin^2 \alpha = 4h \sin^2 \alpha$;

hence $$H = h \sin^2 \alpha. \qquad (3)$$

168. COR. 1.—When the impetus or velocity of projection is given, the range varies as $\sin 2\alpha$, and is consequently the greatest when $2\alpha = 90°$, or the angle of elevation $45°$; in which case $R = 2h$.

169. COR. 2.—When the velocity of projection is given, the elevation requisite to hit a given mark, on the horizontal plane, will be found from Equa. (2); and since $\sin 2\alpha = \sin (180° - 2\alpha)$, there will always be two values of α, or two elevations, which will satisfy this condition.

170. PROP.—*The velocity and direction of projection being given, to find the time of flight, and range on an oblique plane passing through the point of projection.*

Let the body be projected from A in the direction AT; also let AI be the inclined plane, the curve AI the path of the projectile, and AH a horizontal line. Put the angle of elevation $TAH = \alpha$, the angle $IAH = \beta$, the range $AI = R$, and the time of describing the curve $AI = t$; then by trigonometry

$$TI : AT :: \sin TAI : \sin AIT;$$

but $TI = \frac{1}{2} g t^2$, $AT = Vt$, $\sin TAI = \sin (\alpha - \beta)$;

and $\sin AIT = \sin AIH = \cos \beta$; hence

$$\tfrac{1}{2} g t^2 : Vt :: \sin (\alpha - \beta) : \cos \beta,$$

$$\therefore \quad t = \frac{2V \sin (\alpha - \beta)}{g \cos \beta} \qquad (4)$$

Again, $AI : AT :: \sin ATI : \sin AIT$; but

$$AT = Vt = \frac{2V^2 \sin (\alpha - \beta)}{g \cos \beta} = 4h \frac{\sin (\alpha - \beta)}{\cos \beta},$$

$\sin ATI = \cos TAH = \cos \alpha$, and $\sin AIT = \cos \beta$; hence

$$R : 4h\frac{\sin(\alpha-\beta)}{\cos\beta} :: \cos\alpha : \cos\beta,$$

$$\therefore \quad R = 4h\frac{\sin(\alpha-\beta)\cos\alpha}{\cos^2\beta}. \qquad (5)$$

171. Cor.—If the plane be a descending one, as A I′, the angle β must be considered *negative*.

172. Scho.—The theory of projectiles, just given, depends on three suppositions, which are all in some degree inaccurate: Firstly, that the force of gravity in every point of the curve described by the projectile is the same. Secondly, that it acts in parallel lines. And thirdly, that the motion takes place in a non-resisting medium. The two former of these suppositions, however, differ insensibly from the truth; but the resistance of the air affects the motions of all bodies, especially when their velocities are great, so very materially as to make the parabolic theory almost useless in practice. From experiments made with great care, it appears, that when the velocity is about 2000 feet per second, the resistance of the atmosphere is about 100 times as great as the weight of the ball; and that the maximum horizontal range is less than a mile, while according to the theory, it ought to be above 23 miles.—Another great irregularity in the firing of balls is the deflection of its path to the right or to the left of the vertical plane passing through the axis of the gun. Deviations of this kind usually take place when there is considerable windage, *i. e.*, when the ball is too small for the calibre of the gun. This deviation has been found to be, in some cases, as much as 300 or 400 yards in a range of a mile, or extending from $\frac{1}{6}$ to $\frac{1}{4}$ of the whole range.

Dr. Hutton has deduced from experiments various rules to remedy these deviations of the theory from actual practice; for which, see his *Tracts*, *Vol. III.*

173. The following rule, obtained from experiment, has been given, to find the velocity of any shot or shell, when the weight of the charge of powder and that of the shot are given.

Rule.—Divide thrice the weight of the powder by the weight of the shot, both in the same denomination; extract the square root of the quotient, multiply the root by 1600, and the product will be the velocity in feet.

That is, if p be the weight of the powder, w that of the ball, and v its velocity; then

$$v = 1600 \sqrt{\frac{3p}{w}}.$$

Ex. 1.—It is required to find with what velocities the following shells, weighing 90, 48 and 16 lbs., with the respective charges of 4, 2 and 1 lbs. of powder, will be discharged.

By the rule just given, the respective velocities are found to be 584, 565 and 693 feet per second.

Ex. 2.—Required the time in which a shell will range 3250 feet, at an elevation of 32°.

Ans. $11\frac{1}{4}$ seconds nearly.

Ex. 3.—How far will a ball range on a plane which ascends $8\frac{1}{4}°$, and on another which descends $8\frac{1}{4}°$; the velocity or impetus being 3000 feet, and the elevation $32\frac{1}{2}°$?

Ans. 4244 feet on the ascent and 6754 feet on the descent.

ON THE ROTATION OF BODIES.

174. Prop.—*In a rigid system of material particles m, m', m'', &c., in the same horizontal plane* PQ, *and moveable round a vertical axis* RS, *a moving force* F *acts at the point* P, *in the same plane, to turn the system; to determine the accelerating force at any point.*

Put $m + m' + m'' + \&c. = M$; and let f, f', &c. be moving forces which, acting separately on the particles m, m', &c., would produce the same velocities as they would acquire by the action of the force F, when they are connected together; and let v be the angular velocity imparted to the system by the force F in the indefinitely small time t, to radius $= 1$, and r, r', r'', &c., the distances of the particles m, m', &c., from the axis R S at C; then rv will be the velocity given to the particle m.

Now, since the moving force f is supposed to give the velocity $r\,v$ to m in the time t, we shall have $r\,v = \frac{f}{m} t$, by the laws of motion, or $f\,t = m\,r\,v$. Let the force $-\phi$, acting at P, balance f, acting at m, then the force ϕ will produce the same effect on m as f does; for if the forces ϕ and $-\phi$ act on the system, they will produce no change in the motion; and because f and $-\phi$ counteract each other, the only efficient force is ϕ; hence $\mathrm{P\,C} \times \phi = \mathrm{C\,M} \times f$, and (putting $\mathrm{P\,C} = a$) $\phi = f\frac{r}{a}$; $\therefore \phi\,t = \frac{r}{a} f\,t = \frac{m\,r^2\,v}{a}$. Similarly $\phi'\,t = \frac{m'\,r'^2\,v}{a}$, $\phi''\,t = \&c.$ Now, because the forces ϕ, ϕ', &c., produce the same motion in the system as the force F, we shall evidently have $\mathrm{F} = \phi + \phi' + \phi'' + \&c.$; therefore,

$$\mathrm{F}\,t = (\phi + \phi' + \&c.)\,t = \frac{m\,r^2\,v}{a} + \frac{m'\,r'^2}{a} + \&c.$$

$$\therefore v = \frac{\mathrm{F}\,a\,t}{m\,r + m'\,r'^2 + m''\,r''^2 + \&c.}.$$

But the velocity of any particle $m = r\,v =$ accelerating force at $m \times t$,

$$\therefore \text{ accelerating force at } m = \frac{\mathrm{F}\,a\,r}{m\,r^2 + m'\,r'^2 + \&c.} \quad (1)$$

$$\text{accelerating force at P} = \frac{\mathrm{F}\,a^2}{m\,r^2 + m'\,r'^2 + \&c.} \quad (2)$$

$$\text{and accel. force at 1 from R S} = \frac{\mathrm{F}\,a}{m\,r^2 + m'\,r'^2 + \&c.} \quad (3)$$

175. Cor. 1.—When the moving force F is a weight W, connected to the system by a cord passing over a fixed pulley $\mathrm{F} = \mathrm{W}\,g$; and since W must be one of the bodies m, m', &c.; hence

$$\text{accelerating force at F} = \frac{\mathrm{W}\,a^2\,g}{\mathrm{W}\,a^2 + m\,r^2 + m'\,r'^2 + \&c.}.$$

176. Cor. 2.—When the particles m, m', &c., are not in one plane perpendicular to R S, a plane may be taken passing through the centre of gravity of the system, perpendicular to R S, and the whole system may be considered to be projected

on this plane by lines parallel to R S; then, since each point is by this means kept at the same distance from R S, the effect produced by the motion will not be changed; therefore Formulæ (1), (2) and (3) will still hold.

NOTE 1.—The denominator of the fraction, which expresses the accelerating force on any given point of a system, is *the sum of each particle multiplied by the square of its distance from the axis;* this sum is called the *Moment of Inertia* with regard to this axis, and continually occurs in considering the rotation of bodies.

NOTE 2.—Nearly in the manner just given, D'Alembert has made all the most abstruse parts of dynamics to depend on the principle of equilibrium. This is commonly known by the name of *D'Alembert's Principle.*

177. PROP.—*To find the centre of gyration of any system of material particles.*

DEF.—*The centre of gyration* of a system of bodies, revolving round an axis, is that point in which, if all the matter of the system were collected, the same moving force would produce the same angular velocity in the system.

Let T be the centre of gyration; and put $CT = d$; then the accelerating force at the point $P = \dfrac{F a^2}{m r^2 + m' r'^2 + \&c.}$; and if all the matter of the system be concentrated at T, the accelerating force at P will be $= \dfrac{F a^2}{M d^2}$; and because the same angular velocity is produced in both cases, these accelerating forces must be equal; hence

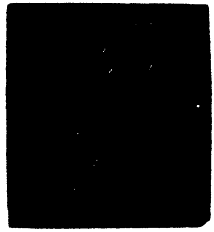
$$M d^2 = m r + m' r'^2 + \&c.$$

$$\therefore d = \sqrt{\frac{m r^2 + m' r'^2 + m'' r''^2 + \&c.}{M}} = \sqrt{\frac{m r^2 + m' r'^2 + \&c.}{m + m' + \&c.}}$$

178. PROP.—*To find the centre of oscillation of any system of material particles, moveable round a horizontal axis.*

DEF. *The centre of oscillation* is that point in a system at which, if the whole system be concentrated, it would vibrate in the same time as the whole system would do.

Let m, m', &c. be any number of particles connected together, and let them all be projected perpendicularly on a plane, which passes through G, their centre of gravity, and which is also perpendicular to the axis of suspension P C, also let O be the centre of oscillation, and P its pro-



jection in the axis P C; then since each particle is thus held at the same distance from the axis P C, the accelerating force will be the same distance from the axis as before. The moving forces may in this case be considered the same as the weights m, m', &c.; and the distances at which they act from C are Cp, Cp', &c., therefore, by Art. 175, the accelerating force on any point O, arising from each of these bodies, will be $\frac{m \cdot Cp \cdot CO \cdot g}{m r^2 + m' r'^2 + \&c.}$, $\frac{m' \cdot Cp' \cdot CO \cdot g}{m r^2 + m' r'^2 + \&c.}$, &c.; hence the accelerating force at O, resulting from all the particles acting together, will be

$$\frac{(m \cdot Cp + m' \cdot Cp' + \&c.) \cdot CO \cdot g}{m r^2 + m' r'^2 + \&c.} = \frac{M \cdot CG \cdot CO \cdot g \sin \alpha}{m r^2 + m' r'^2 + \&c.};$$

for $m \cdot Cp + m' \cdot Cp' + \&c. = (m + m' + \&c.)\ CI = M \cdot CG \sin \alpha$, α being the angle C G I. Similarly the accelerating force of a particle m, placed at O, is $\frac{m \cdot CP \cdot CO \cdot g}{m \cdot CO^2}$ $= g \sin \alpha$; and since O is the centre of oscillation, these forces must be equal; hence

$$M \cdot CG \cdot CO \cdot g \sin \alpha = (m r^2 + m' r'^2 + \&c.)\, g \sin \alpha,$$

$$\therefore CO = \frac{m r^2 + m' r'^2 + \&c.}{M \cdot CG}.$$

179. Cor. 1.—Because $m r^2 + m' r'^2 + \&c. = M \cdot CG \cdot CO = M \cdot CT^2$ by the preceding prop.; we have $CG \cdot CO = CT^2$; hence the centre of gyration is a mean proportional between the centre of gravity and the centre of oscillation.

180. Cor. 1.—Because the accelerating force of the whole system at the point O, is the same as that of a single particle placed at O, the time of oscillation of the system will be the same as the time of oscillation of a simple pendulum, the length of which is C O; therefore, if $CO = l$, the time of a very small oscillation will be $= \pi \sqrt{\frac{l}{g}}$.

TO FIND THE MOMENT OF INERTIA AND THE CENTRE OF OSCILLATION BY THE DIFFERENTIAL CALCULUS.

181. If it be assumed that the particles m, m', &c., that make up a body are all equal and their number indefinitely great, m will be ultimately proportional to $d\text{M}$, the differential of the mass of the body, therefore by the principle of the differential calculus, r being the distance of m or $d\text{M}$ from the axis,

$$\text{moment of inertia} = \int r^2 \, d\text{M};$$

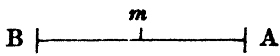
and, if k be the distance of the centre of gyration from the same axis,

$$k^2 \text{M} = \int r^2 \, d\text{M},$$

$$\text{and } k^2 = \frac{\int r^2 \, d\text{M}}{\text{M}}.$$

In finding the moment of inertia of lines, planes, and solids, they are supposed to be made up of an indefinite number of particles of matter uniformly diffused over them.

182. Prop.—*To find the moment of inertia of the right line* A B, *revolving round an axis perpendicular to it at* B.

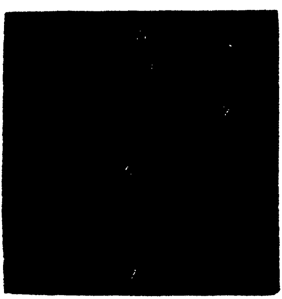


Put $\text{AB} = a$, $\text{B}m = r$, then the differential of the mass M, or $d\text{M}$, is proportional to $d\,r$, therefore by the last article

$$k^2 = \frac{\int r^2 dr}{r} = \frac{\frac{1}{3} r^3}{r} = \frac{1}{3} r^2$$

$$\text{and when } \quad r = a, \quad k^2 = \tfrac{1}{3} a^2, \quad \text{and}$$

$$\text{the moment of inertia} \quad k^2 \text{M} = \tfrac{1}{3} a^2 \text{M}.$$



183. Prop.—*To find the moment of inertia of a circle* A B, *revolving round its centre* G *in its own plane.*

Put the radius $\text{A G} = a$, the radius $m\,\text{G} = r$; then the circumference $m\,n = 2\pi r$, and the differential of the area of the circle $m\,n = 2\pi r\,dr = d\text{M}$; hence

$$k^2 = \frac{\int r^2 d\mathrm{M}}{\mathrm{M}} = \frac{\int 2\pi r^3 dr}{\int 2\pi r\,dr} = \tfrac{1}{2} r^2,$$

and when $r = a$, $k^2 = \frac{1}{2} a^2$, and $k^2 \mathrm{M} = \frac{1}{2} a^2 \mathrm{M}$.

184. PROP.—*To find the moment of inertia of a circle revolving round an axis lying in its own plane.*

Let R S be the axis of rotation, G R, a line perpendicular to RS, *pmq* any line parallel to RS. Put $\mathrm{RG} = b$, $\mathrm{AG} = a$, $\mathrm{R}m = r$, $\mathrm{G}m = x$, and $pm = mq = y$; then

$d\mathrm{M} = 2y\,dx\,\sqrt{a^2 - x^2}$, and $r^2 = (b - x)^2$, therefore

$$\int r^2 d\mathrm{M} = \int 2dx\,(b - x)^2 \sqrt{a^2 - x^2}, \quad \text{hence}$$

$$k^2 = \frac{\int 2dx\,(b-x)^2 \sqrt{a^2 - x^2}}{\int 2dx\,\sqrt{a^2 - x^2}} = \frac{(b^2 + \frac{1}{4}a^2)\int 2dx\,\sqrt{a^2 - x^2}}{\int 2dx\,\sqrt{a^2 - x^2}},$$

by taking the integral by parts between $x = -a$, and $x + a$, therefore

$$k^2 = a^2 + \tfrac{1}{4} b^2, \quad \text{and} \quad k^2 \mathrm{M} = (a^2 + \tfrac{1}{4} b^2)\,\mathrm{M}.$$

185. PROP.—*To find the moment of inertia of a sphere* A q B p *revolving round a diameter* A B.

Let $p\,q$ be a section of the sphere perpendicular to the axis of rotation A B; put $\mathrm{A\,G} = a$, $\mathrm{M\,G} = x$, and $m\,p = m\,q = y$; then, because the section $p\,q$ is a circle revolving round the axis A B, which passes through its centre, the moment of inertia of this circle $= \frac{1}{2} y^2\ \mathrm{M} = \frac{1}{2}\pi y^4$; therefore the moment of inertia of the circle $p\,q$, when its thickness is the indefinitely small space dx, is $\frac{1}{2}\pi y^4\,dx$; hence

$$k^2 = \frac{\int \frac{1}{2}\pi y^4\,dx}{\int \pi y^2\,dx} = \frac{\int \frac{1}{2}\pi (a^2 - x^2)^2 dx}{\int \pi (a^2 - x^2)\,dx} = \frac{\frac{8}{15}\pi a^5}{\frac{4}{3}\pi a^3} = \tfrac{2}{5} a^2,$$

the integral being taken between $x = -a$, and $x = a$,

$$\therefore \quad k^2 \mathrm{M} = \tfrac{2}{5} a^2 \mathrm{M}.$$

The following are the moments of inertia of several regular solids revolving round their axes.

186. In a cylinder $k^2 \mathrm{M} = \frac{1}{2} a^2 \mathrm{M}$.

187. In a paraboloid $k^2\ \mathrm{M} = \frac{1}{3} a^2 \mathrm{M}$.

188. In a cone $k^2 \mathrm{M} = \frac{3}{10} a^2 \mathrm{M}$, a being the radius of the base.

189. In an ellipsoid $k^2 \mathrm{M} = \frac{1}{5}(a^2 + b^2)\,\mathrm{M}$, a and b being

the semiaxes of the largest section perpendicular to the axis of rotation.

190. Prop.—*To find the centre of oscillation in lines, planes, and solids.*

By Art. 178. $CO = \frac{\textit{moment of inertia}}{CG.M} = \frac{k^2}{CG}$,

and k^2 is found by the preceding propositions.

The following are the distances of the centre of oscillation from the point of suspension in certain given figures.

191. In a straight line, vibrating at its extremity, $CO = \frac{2}{3}a$.

192. In a circle, vibrating about its axis in its own plane $CO = d + \frac{r^2}{4d}$, CG being $= d$, and the radius of the circle $= r$.

193. In a sphere, $CO = d + \frac{2r^2}{5d}$, in which d and r are the same as in the preceding article.

PART III.

HYDROSTATICS.

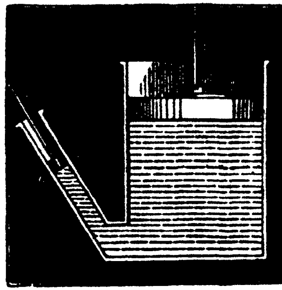
194. *Hydrostatics* is that branch of Statics which treats of the equilibrium of fluids.

195. *Fluids* yield without resistance to the smallest force impressed on them; they are divided into *elastic* and *non-elastic* fluids. An *elastic* fluid is one the dimensions of which are diminished by increasing the pressure upon it, and increased by diminishing the pressure, such as common air, gases, and vapours. A *non-elastic* fluid is one the dimensions of which are very little affected by any pressure, however great; such as water, mercury, spirits, &c.

196. Prop.—*Any pressure communicated to a fluid at rest is equally transmitted throughout the whole fluid.*

(This proposition, which is commonly made the basis of the doctrine of hydrostatics is proved by experiment.)

Let *c d e f* be a closed box filled with water; P a piston fitted into the upper face of the box, and allowed to move as freely as possible, and suppose another piston Q* with a transverse section equal to that of P, also fitted into the upper face of the box; then, if a weight be placed on P, an equal weight must be placed on Q, to preserve the equilibrium, thus shewing that the weight on P is transmitted through the fluid to the under surface of Q, and also with equal force, because it requires an equal weight on Q to balance this pressure. Again, if a piston *p* equal to P be fitted into a lateral opening in the side of the box, as at *a b*, it will be found that a pressure must be exerted at *a b* to retain the fluid in the box, before any pressure is applied at P; if then a weight be placed on P, an additional pressure equal to the weight on P, must be applied on *p* to maintain the equilibrium; thus proving that the pressure upon the surface at P is transmitted with equal force through the whole mass of the fluid.



NOTE.—One of the most extraordinary properties of fluids is that of transmitting pressures in every direction; this property can be conceived to arise only from the perfect freedom with which the particles of a fluid move amongst each other. This, in a mechanical point of view, is the characteristic distinction between fluids and solids: a solid imparts pressure only in the direction in which the force is exerted, while a fluid imparts pressure in all directions.

197. PROP.—The pressure at any point *q* in the interior of a fluid, the density of which is uniform, and which is acted on by no force but gravity, is equal to the weight of the vertical column *p q*.

Assume all the fluid in the vessel A B to be solid, except the vertical column *p q*; then it is evident that the particle *q* will be precisely in the same state as before; it is also evident that on the particle *q* the pressure

* The piston Q is not shewn in the figure.

is equal to that of the column above it $p\,q$; hence, when the whole is fluid, the particle is equally pressed in all directions, by a force equal to the weight of the vertical column above it.

198. Cor. 1.—If the point r be not directly under the surface of the fluid, draw $q\,r$ parallel to the surface of the fluid; then by the preceding proposition, the pressure at q is transmitted along the line $q\,r$, therefore the pressure at r must be equal to the pressure at q, otherwise the equilibrium would be destroyed; hence the pressure at r is equal to the weight of the vertical column $p\,q$.

199. Cor. 2.—Since it is well known that the surface of a fluid at rest is horizontal, it follows that a fluid in a system of vessels in free communication with each other, cannot be at rest except the surfaces of the fluid in all these different vessels be horizontal.

200. Cor. 3.—Hence it also follows that the surfaces of all perfect fluids are perpendicular to the direction of gravity.

201. Cor. 4.—In fluid surfaces of small extent, gravity may be considered to act in parallel lines; but in surfaces of great extent, such as the surfaces of large lakes, seas and oceans, the directions of gravity converge to a point at the earth's centre, and in these cases the surface of the fluid is a portion of a spherical surface having that point for a centre. Since the distance of this centre is known, the deviation of any portion of the earth's surface from the level may be readily calculated. See the Author's *Principles and Practice of Levelling in his Land and Engineering Surveying. Weale's Series.*

202. Prop.—*If the fluid in any vessel* A q B *be at rest, through the action of gravity alone, the pressure on an indefinitely small area* $q\,r$, *at any point in the bottom or sides, is perpendicular to the plane of that area, and equal to the weight of the vertical column* $p\,q$, *the base of which is* $q\,r$.

The pressure exerted on $q\,r$ perpendicularly, is equal to the weight of the fluid $p\,r$; let P = perpendicular pressure on qr, and W = weight of the fluid pr; take $qd = pq$ to represent the perpendicular pressure of any particle against $q\,r$; then this pressure may be resolved into two eq, fq; and eq is the part of the pressure which acts perpendicularly; and since qr is indefinitely small, we shall have

$$\text{P} : \text{W} :: \text{area}\ q\,r \times q\,e : \text{area}\ q\,s \times p\,q;$$

$$\text{but area}\ q\,r : \text{area}\ q\,s :: q\,d : q\,e,$$

$$\therefore \text{area}\ q\,r\,.\,q\,e = \text{area}\ q\,s\,.\,q\,p, \text{ and hence } \text{P} = \text{W}.$$

203. Prop.—*The pressure of a fluid on any surface is equal to the weight of a column of the fluid, the base of which is the surface pressed, and the height equal to the depth of its centre of gravity below the surface of the fluid.*

Let the whole surface S be divided into an indefinite number of parts s, s', &c., the distances of which from the surface of the fluid are respectively x, x', &c.; then the pressure of the fluid upon the indefinitely small portion s of the surface is equal to the weight of a column of the fluid, the base of which is s and the height x, by the last prop.; and if d be the density of the fluid, or specific weight of each unit in bulk, the pressure on s will be $= s\,x \times d$, and consequently the sum of all the pressures $= (s\,x + s'\,x' + \&c.)d$: but, by the nature of the centre of gravity, $s\,x + s'\,x' + \&c. = \text{S}\,h$, h being the distance of the centre of gravity of S from the surface of the fluids; hence the whole pressure upon the surface $\text{S} = \text{S}\,d \times h =$ a column of the fluid the base of which is S and height h.

204. Cor. 1.—Hence the pressure against one of the perpendicular sides of a cubical vessel, filled with fluid, is equal to half the pressure against the bottom, or equal to half the weight of the fluid; and the whole pressure against the bottom and sides of the vessel is equal to thrice the weight of the fluid.

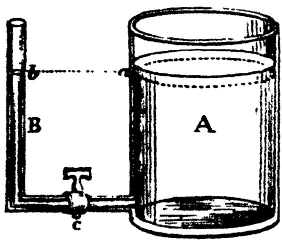
205. Cor. 2.—If h be the height of a cylinder and r the radius of its base; then the pressure against the base $= \pi\,r^2.\,h\,.\,d = r\,d\,r^2\,h$; and the pressure against the upright curved surface $2\,\pi\,r\,h\,.\,\frac{1}{2}\,h\,.\,d = \pi\,d\,r\,h^2$; therefore the two pressures are

$$\text{as}\ \pi\,d\,r^2\,h : \pi\,d\,r\,h^2,$$

$$\text{or as}\ r : h.$$

206. Cor. 3.—On this principle *Bramah's hydrostatic*

press may be explained; let pistons be fitted into the large and small cylinders A and B, which are connected together, as shewn in the figure, there being a valve at C to admit the water from B to A. A pump-piston in the cylinder B forces the water through the valve c into the cylinder A, and thus raises its piston. Now, let the diameter of the cylinder A = D inches, and that of the cylinder B = d inches; then the area of the piston is A = $\frac{1}{4}\pi D^2$, and the area of the pump-piston in B = $\frac{1}{4}\pi d^2$, therefore the areas are



$$\text{as } d^2 : D^2$$

$$\text{or as } 1 : \frac{D^2}{d^2}$$

Now, if D = 20 inches and $d = \frac{1}{2}$ an inch; then

$$1 : \frac{D^2}{d^2} :: 1 : \frac{20^2}{(\frac{1}{2})^2} = 1600.$$

Therefore, if a force be applied to the pump piston in B, it will produce an effect on that in A as 1 to 1600. Now, suppose the pump piston be pressed down by a lever with a force of 5 cwts.; then the large piston will ascend with a force of $1600 \times 5 = 8000$ cwts. = 400 tons.

207. Cor. 4.—On the same principle the nature of the *hydrostatic paradox* may be explained.

208. Prop.—*If fluids of different densities, such as water and mercury, be made to communicate, the heights to which they will rise in the limbs of the pipe* A B, *will be in the inverse ratio of their densities.*

Let the bend be first filled with mercury, and water be then poured into A; and let H = height of the fluid in A, and D = its density; also let h = height of the fluid in B, and d = its density; then, since the bore of the pipe is supposed to be the same throughout, there will result in the case of equilibrium.

$$H \cdot D = h \cdot d, \text{ hence}$$

$$H : h :: d : D.$$

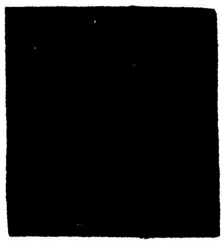
Since the density of water to that of mercury is nearly as $1 : 13\frac{1}{2}$, we shall have

$$H : h :: 13\tfrac{1}{2} : 1,$$

that is, if the height of the mercury be one inch, the height of the water will be $13\frac{1}{2}$ inches.

209. Prop.—*To find the centre of pressure upon a plane surface.*

Def. *The centre of pressure* is that point in the surface pressed by any fluid, to which, if the whole pressure were applied, the effect would be the same as when the pressure is diffused over the whole surface; and if a force equal to the whole pressure be applied in a contrary direction to this point, it will keep the surface at rest.



Let A B C be the level surface of the fluid pressing on the plane C G R, C R the intersection of these planes, and, P the centre of pressure. Suppose the whole area C G R to be divided into an indefinite number of small portions m, m', m'', &c., and draw $m\,q$, $q\,n$ perpendicular to C R, also $m\,n$ perpendicular to $q\,n$. Then because C R is perpendicular to $q\,m$, $q\,n$, it is also perpendicular to the plane $m\,q\,n$, and the planes A B G, $m\,q\,n$ are therefore perpendicular to each other, and $m\,n$ is vertical. Now, let $\phi =$ angle $m\,q\,n =$ inclination of the surface plane A B and the plane C G R; then the pressure on the indefinitely small surface m is proportional $m \cdot m\,n$.

But $m \cdot m\,n = m \cdot m\,q \sin \phi = m\,s \sin \phi$, s being put for $m\,q$.

Hence the effect of the pressure to turn the plane about the line C R will be as $m\,s \sin \phi \times s = m\,s^2 \sin \phi$; and the effect of all the pressures to turn the plane about C R will be proportional to

$$(m\, s^2 + m'\, s'^2 + m''\, s''^2 + \&c.) \sin \phi.$$

Put $\mathrm{M} = m + m' + \&c.$ = area C G R, and $\mathrm{HG} = d$; then the pressure on C G R will be as $\mathrm{M}\,.\,\mathrm{GI} = \mathrm{M}d \sin \phi$; and the effect of the pressure at P to turn the plane about C R will be as $\mathrm{M}\, d \sin \phi \times \mathrm{PR}$; hence

$$\mathrm{M}\, d \sin \phi \times \mathrm{PR} = (m\, s^2 + m'\, s'^2 + m''\, s''^2 + \&c.) \sin \phi,$$

$$\therefore \mathrm{PR} = \frac{m\, s^2 + m'\, s'^2 + m''\, s''^2 + \&c.}{\mathrm{M}\, d}.$$

Also, the effect of the pressure $m\, s \sin \phi$ to turn the plane about G H will be as $m\, s \sin \phi \times \mathrm{H}\, q$, and the effect of the pressure $\mathrm{M}\, d \sin \phi$ at P to turn the plane about C R will be as $\mathrm{M}\, d \sin \phi \times \mathrm{HR}$; hence

$$\mathrm{M}\, d \sin \phi \times \mathrm{HR} = m\, s \sin \phi \,.\, \mathrm{H}\, q + m' s' \sin \phi \,.\, \mathrm{H}\, q' + \&c.$$

$$\therefore \mathrm{HR} = \frac{m\, s\,.\,\mathrm{H}\, q + m' s'\,.\,\mathrm{H}\, q' + \&c.}{\mathrm{M}\, d}.$$

From the above value of P R, it appears that the centre of pressure P of the plane C R G is the same as the centre of oscillation of this plane, when moving round the axis C R; see Art. 178.

210. **Prop.**—*The centre of pressure against the rectangle* B F *is at* $\frac{2}{3}$ *of the depth* B D *from the surface* A B. (See last figure.)

Put $\mathrm{BD} = a$, $\mathrm{FD} = b$, and let B D be divided into n indefinitely small parts, each equal to λ, so that $a = n\, \lambda$; and conceive lines to be drawn through these divisions parallel to D F; then the area B F will be divided into n indefinitely small rectangles or lamina, each equal to $b\, \lambda$. Now supposing each of these lamina to be parallel to the surface A B of the fluid, we shall evidently have

$$ms^2 + m's'^2 + \&c. = b\lambda \times \lambda^2 + b\lambda \times (2x)^2 + \&c. \text{ to } b\lambda \times (n\lambda)$$

$$= b\lambda^3(1^2 + 2^2 + 3^2 + \&c. \text{ to } n^2)$$

$$= \frac{b\lambda^3(n+1)(2n+1)}{6} = \frac{bn^3\lambda^3}{3}\left(1+\frac{1}{n}\right)\left(1+\frac{1}{2n}\right)$$

$$= \frac{a^3 b}{3}\left(1+\frac{1}{n}\right)\left(1+\frac{1}{2n}\right),$$

putting a for its equal $n\, \lambda$.

Now, since n is indefinitely large, the fractions $\frac{1}{n}$, $\frac{1}{2n}$ are indefinitely small, consequently the above value of

$$m s^2 + m' s'^2 + \&c. = \tfrac{1}{3} a^3 b.$$

Hence, supposing the plane C G R to be rectangular, we shall have

$$P R = \frac{\frac{1}{3} a^3 b}{M d} = \frac{\frac{1}{3} a^3 b}{a b \times \frac{1}{2} a} = \tfrac{2}{3} a,$$

the point P being, in this case, evidently equidistant from the sides of the rectangle.

211. COR.—If B D = a, and B E = h; then the distance of the centre of pressure of the rectangle E F from the surface of the fluid will be equal to

$$\frac{\frac{1}{3} a^3 b - \frac{1}{3} h^3 b}{(a-h) b \times \frac{1}{2}(a+h)} = \tfrac{2}{3} \frac{a^2 + a h + h^2}{a + h}.$$

212. COR. 2.—If a = depth BD, b = breadth DF in feet, and S = specific gravity of the fluid (see fig. to Art. 204); then by Art. 203, the pressure P of the fluid against the vertical rectangular plane B F = $\frac{1}{2} a^2 b$ S, that is,

$$P = \tfrac{1}{2} a^2 b \, S,$$

and by Art. 210, the power P, being applied at $\frac{2}{3}$ of the depth of the fluid, will sustain the plane.

Ex. 1.—Required the pressure on a flood gate of a canal, the breadth of which is 12 feet and depth 6 feet.

$$P = \frac{6^2 \times 12 \times 62{\cdot}5}{2} = 13500 \text{ lbs.} = 6\tfrac{3}{112} \text{ tons.}$$

Ex. 2.—The depth of water, pressing against an embankment 100 feet long, is 9 feet, required the pressure thereon in tons.

$$P = \frac{9^2 \times 100 \times 62{\cdot}5}{2} = 253125 \text{ lbs.} = 113\tfrac{1}{448} \text{ tons.}$$

Ex. 3.—Required the pressure on the staves of a cylindrical vessel filled with water, the depth being 6 feet and the diameter of the base 5 feet.

Here the curved surface of the vessel must be considered as a plane; hence

$$P = \tfrac{1}{2} a^2 b \pi S = \frac{6^2 \times 5 \times 3{\cdot}1416 \times 6{\cdot}25}{2} = 17671\tfrac{1}{2} \text{ lbs.}$$

Ex. 4.—The depth of a cylinder filled with fluid is 3 feet; required its diameter when the pressures against the staves and bottom are equal.

$$\text{Put } x = \text{diameter of the cask,}$$

$$\text{then } P = \tfrac{1}{2}\, 3^2\, \pi\, S\, x = \tfrac{1}{4}\, 3\, \pi\, S\, x^2,$$

$$\text{whence } x = 6 \text{ feet.}$$

213. Prop.—*A perpendicular embankment or wall* ABCD *sustains the pressure of the water* BCEF, *required the conditions of equilibrium, when the wall is just on the point of overturning on* D *as a centre.*

Let KI be a vertical line passing through G the centre of gravity of the wall, P the centre of pressure of the water, the distance CP being $= \frac{1}{3}$ BC, by Art. 210. Draw PL perpendicular to AD in H; then since the section AC of the wall is here considered to be rectangular, the centre of gravity G is at the middle point of the wall, and therefore DI $= \frac{1}{2}$ DG $= \frac{1}{2}$ AB. Now, HDI may be considered as a bent lever, the fulcrum of which is D, the weight of the wall acting in the direction of the centre of gravity G on the arm DI, and the pressure of the water on the arm DH, or what amounts to the same thing, a force equal to that pressure drawing in the direction HL. Put P = pressure of the water and W weight of the wall; then

$$P \times DH = P \times \tfrac{1}{3} BC = W \times \tfrac{1}{2} DC,$$

$$\text{or } P = \frac{3\,DC\,.\,W}{2\,BC}.$$

When this equation holds, the wall or embankment will just be on the point of overturning; but in order that the wall may have complete stability, this equation ought to give a much larger value of P than its actual amount. The following formulæ are for embankments of one foot in length, because, if they have stability for that length, they will be stable for any other length.

Put a = BC depth of water and embankment, which are here supposed to be equal, b = DC = breadth of the embankment, S = specific gravity of water, and s = that of the wall; then by Art. 204, $P = \frac{1}{2} a^2 \times 1 \times S$, also $W = a \times b \times 1 \times s$, each value being for 1 foot in length, which being substituted in the above equation, there will result

$$\tfrac{1}{2} a^2 S = \frac{3b \times abs}{2a}.$$

$$\text{or,} \quad a^2 S = 3 b^2 s,$$

$$\text{or,} \quad a = b \sqrt{\frac{3s}{S}},$$

$$\text{and} \quad b = a \sqrt{\frac{S}{3s}},$$

which gives the breadth of an embankment or retaining wall that will just sustain the pressure of the water, the wall must therefore be made at least 1 foot thicker than shewn by this equation, to give it due stability.

Ex. 1.—Let the height of the wall B C = depth of the water = 12 feet, and the respective specific gravities of water and the wall be 62·5 lbs. and 120 lbs. per cubic foot; required the thickness of the wall, so that it may have complete stability to sustain the pressure of the water.

$$b = a \sqrt{\frac{S}{3s}} = 12\sqrt{\frac{62{\cdot}5}{3 \times 120}} = 12\sqrt{\frac{625}{3600}} = 12 \times \frac{25}{60} = 5 \text{ feet},$$

the thickness that will just sustain the pressure of the water, therefore 1 foot must be added to this thickness to give the wall complete stability,

hence $5 + 1 = 6$ the required width of the wall.

Ex. 2.—Let D C or A B = 3 feet, and the weight of a cubic foot of the wall = 150 lbs., required the height of the wall when it is on the point of being overturned, the water being at the top.

$$a = b \sqrt{\frac{3s}{S}} = 3 \sqrt{\frac{3 \times 150}{62{\cdot}5}} = \frac{18}{5} \sqrt{5} = 8{\cdot}05 \text{ feet}.$$

Ex. 3.—Required the thickness of a rectangular embankment or retaining wall, when its height is 12, and the weight of a cubic foot of the material is 125 lbs., so that it may just be on the point of being overturned, the water standing at the brim.

214. Prop.—*The section* B C D *of an embankment or retaining wall is triangular, the face* B C *being vertical; required the condition of equilibrium, when the wall is just on the point of being overturned on* D *as a centre.*

Draw D n bisecting B C in n, from the centre of pressure P draw P H perpendicular to B C cutting Dn in G, which is the centre of gravity of the triangular section of the wall, also draw G I, D H respectively perpendicular to DC, PH; then H D I may be considered as a bent lever, the pressure of the water acting at H, and the weight of the wall acting at I. Put B C $= a$, D C $= b$, and the specific gravities of the wall and water as in the last problem;

then P C = G I = H D $= \frac{1}{3} a$, and, by the nature of the centre of gravity, D I $= \frac{2}{3}$ D C $= \frac{2}{3} b$; the weight of 1 foot in length of the wall $= \frac{1}{2} a b s$, and the pressure at P of the same length of water $= \frac{1}{2} a^2$ S; hence by the property of the bent lever,

$$\tfrac{2}{3} b \times \tfrac{1}{2} a b s = \tfrac{1}{3} a \times \tfrac{1}{2} a^2 \text{S},$$

$$\text{whence } b = a \sqrt{\frac{\text{S}}{2s}}, \text{ and } a = b \sqrt{\frac{2s}{\text{S}}}.$$

215. Cor. 1.—If $x =$ B $r =$ any variable depth of the water, and $y = r s =$ the corresponding width of the embankment; then, these values being substituted for a, b respectively in the equation $b = a \sqrt{\frac{\text{S}}{2s}}$, give

$$y = x \sqrt{\frac{\text{S}}{2s}},$$

an equation of the first degree, which is therefore the equation of the straight line B D, and consequently the triangular embankment B C D is equally strong throughout.

216. Cor. 2.—By comparing the values of b in this and the preceding problem, it will be seen that an embankment or retaining wall with a triangular section is stronger than

one with a rectangular section, when the quantity of material in these two forms of the embankment is considered; for when the walls have the same quantity of material in both cases, the base of the wall in Art. 214, must be twice the width of the base of the wall in Art. 213; if, therefore, we put P = pressure the wall sustained in Art. 213, and P′ = pressure sustained by the wall in Art. 214, $2b$ for b in the latter case, there will result by substitution in the formulæ of the respective problems,

$$P = \frac{3b^2 s}{2}, \text{ and } P' = 4bs^2$$

$$\therefore P : P' :: \frac{3b^2 s}{2} : 4b^2 s$$

$$:: \tfrac{3}{2} : 4$$

$$:: 3 : 8.$$

Ex. 1.—There is a triangular embankment of brick-work, each cubic foot of which weighs 117 lbs., and its depth B C is 14 feet; required its width at the base D C when it is just on the point of being overturned, the water standing at the brim. (See last figure.)

$$DC = b = a\sqrt{\frac{S}{2s}} = 14\sqrt{\frac{62 \cdot 5}{2 \cdot 117}} = \frac{35}{39}\sqrt{65} = 7\tfrac{1}{4} \text{ feet nearly.}$$

Hence the breadth of the base of the embankment must be at least 8 feet to ensure perfect stability.

Ex. 2.—A triangular embankment is 12 feet in depth, the weight of the material is 130 lbs. per cubic foot, required its width at the base when just on the point of being overturned by the pressure of the water, which is $10\frac{1}{2}$ feet deep.

Here put c depth of the water, then in this case $P = \frac{1}{2}c^2 S$ and $W = \frac{1}{2}abs$, as before, therefore $\frac{2}{3}b \times \frac{1}{2}abs = \frac{1}{3}c \times \frac{1}{2}c^2 S$, whence

$$b = c\sqrt{\frac{cS}{2as}} = 10\tfrac{1}{2}\sqrt{\frac{10\frac{1}{2} \times 62\frac{1}{2}}{2 \times 12 \times 130}} = 4 \cdot 816 \text{ feet} = 4 \text{ ft. } 9\tfrac{3}{4} \text{ in.}$$

NOTE.—The usual form of an embankment is that having a section in the form of a trapezoid with the longest side for its base, these embankments are usually formed of earth and clay, with or without a perpendicular or sloping face of brickwork against the water; the following proposition refers to embankments of this kind.

217. Prop.—*The section* A B C D *of an embankment is a prismoid, having a perpendicular face* B C, *required the conditions of equilibrium when the embankment is on the point of being overturned on* D *as a centre.*

Divide the embankment into parts by drawing A E perpendicular to D C; and let B C $= a$ as before, the top-breadth A B $=$ E C $= b$ and the bottom-width D E of the sloping part A E D $= c$; then the weights of the portions A C and A E D respectively for one foot in length, are $a\,b\,s$ and $\frac{1}{2}a\,c\,s$, these weights acting at the points N and I respectively. Now D N $=$ D I $+ \frac{1}{2}$ E C $= c + \frac{1}{2}b$, and D I $= \frac{2}{3}$ D E $= \frac{2}{3}c$; hence the sum of the moments of the embankment A B C D is

$$a\,b\,s\,(c+\tfrac{1}{2}b)+\tfrac{1}{2}\,a\,c\,s\times\tfrac{2}{3}\,c=\tfrac{1}{2}\,(b^2+2\,b\,c+\tfrac{2}{3}c^2)\,a\,s$$

which must be equal to the moment of the pressure of the water

$$\therefore \tfrac{1}{2}\,(b^2+2\,b\,c+\tfrac{2}{3}\,c^2)\,a\,s=\tfrac{1}{3}\,a\times\tfrac{1}{2}\,a^2\,\mathrm{S}$$

$$\text{or}\quad (b^2+2\,b\,c+\tfrac{2}{3}\,c^2)\,s=\tfrac{1}{3}\,a^2\,\mathrm{S}.$$

Hence, when the depth a of the embankment and its bottom-width $b+c$ are given, the breadth c or batter of the sloping part may be found, which is

$$c=\sqrt{\frac{3\,(b+c)^2\,s-a^2\,\mathrm{S}}{s}},$$

whence the width b of the top of the embankment becomes known.

Ex. 1.—A trapezoidal embankment is 12 feet deep, and the bottom-width 6 feet; required the top-width, when the embankment is on the point of being overturned, the weight of the material being 100 lbs. per cubic foot.

$$\mathrm{DE}=c=\sqrt{\frac{3(b+c)^2\,s-a^2\,\mathrm{S}}{s}}=\sqrt{\frac{3\times6^2\times100-12^2\times62{\cdot}5}{100}}=$$

$$\sqrt{18}=4\tfrac{1}{4}\text{ feet nearly,}$$

hence the top width A B $= 6 - 4\tfrac{1}{4} = 1\tfrac{3}{4}$ feet.

Here, as before, it will be proper to observe that the width of the embankment must be at least one foot greater both at top and bottom to secure its stability.

Ex. 2.—Required the top-width of the embankment when the depth is 14 feet and the bottom-width 7 feet, the weight of the material being as in the last example.

Ans. 2 feet nearly.

NOTE.—It very frequently happens the face of the embankment has also a slope or batter, in this case the section of the embankment must be divided into two triangle and a parallelogram, and the moments of the several parts added together, as in the last problem; but, after having already seen so much of like subjects, the student will have no difficulty in doing this.

REVETMENT WALLS.

218. DEF.—When a wall sustains the pressure of earth, sand, or any loose material, it is called a revetment wall.

219. The thrust of earth, &c., upon a wall is caused by a certain portion, in the shape of a wedge, tending to break away from the general mass. The pressure, thus caused, is similar to that of water, but here the weight of the material must be reduced by a particular ratio dependant upon the angle of natural slope, which is about 45° in earth of mean quality. Coulcomb has shewn that the angle which the line of rupture makes with the vertical is one half of the angle which the line of natural slope makes with the same vertical line. He has also further shewn, that when the earth is level at the top, the pressure of the earth may be found by considering it as a fluid, the weight of a cubic foot of which is equal to the weight of a cubic foot of the earth multiplied by the square of the tangent of half the angle included between the natural slope and the vertical. Therefore the square of the tangent of $\frac{1}{2}\,45^\circ = 22\frac{1}{2}^\circ = \cdot 1716$ is the multiplier which must be used in all ordinary practical cases to reduce a cubic foot of the material to a cubic foot of equivalent fluid which will have the same effect as the earth by its pressure upon the wall.

220. PROP.—*A perpendicular wall* A B C D *sustains the pressure of the earth* C B F (fig. to Art. 213), *required the conditions of equilibrium, when the wall is on the point of being overturned on* D *as a centre.*

Put $a =$ BC $=$ height of the wall, $b =$ AB $=$ its breadth; $s =$ the weight of one cubic foot of the wall, S $=$ that of one cubic foot of the earth, and $n = \cdot 1716$; then the weight of a cubic foot of the equivalent fluid is n S, and the pressure of the earth is

$$\frac{a^2}{2} \times n\,S,$$

whence the moment of the earth is

$$\frac{a}{3} \times \frac{a^2}{2} \times n\ S = \frac{a^3\,n\,S}{6},$$

and the moment of the wall is

$$\tfrac{1}{2}\,b \times a\,b \times s = \frac{a\,b^2\,s}{2},$$

and in case of equilibrium these moments must be equal,

$$\frac{a\,b^2\,s}{2} = \frac{a\,n\,S}{6},$$

$$\text{whence} \quad b = a\sqrt{\frac{n\,S}{3\,s}}.$$

Ex. 1.—A revetment wall is 40 feet in height, sustaining the pressure of earth of mean quality, which weighs 100 lbs. per cubic foot; it is required to determine the thickness of the wall, one cubic foot of which weighs 120 lbs.

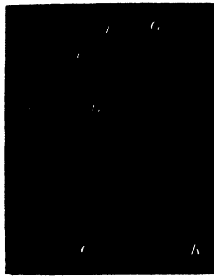
$$b = a\sqrt{\frac{n\,S}{3\,s}} = 40\sqrt{\frac{\cdot 1716 \times 100}{3 \times 120}} = 8 \text{ feet } 7\tfrac{3}{4} \text{ inches};$$

this thickness must be increased to about 10 feet, that the wall may have due stability.

Ex. 2.—Required the thickness of the wall at bottom when its height is 30 feet, its section trapezoidal, as in Art. 217, and its thickness at the top 2 feet, the weights of the wall and the earth being the same as in the preceding example.

SURCHARGED REVETMENTS.

221. When the earth stands above the wall A C with its natural slope A F, A C is called a *Surcharged Revetment*, C G being the line of rupture, and therefore A E G C B A is the part of the earth that presses upon the wall, which part must be taken into the calculation, with the exception of the portion A B F which rest upon the wall; *i. e.*, the calculation must be for the part C E F G, which must be reduced to its equivalent quantity of fluid by multiplying the weight of a cubic foot of it by the square of the tangent of the angle B C G = half the angle which the natural slope makes with the vertical, and then proceeding as in the last problem.



For complete investigations on the nature of revetments, see *Moseley's Mechanical Principles of Engineering and Architecture* and *Hann's Mechanics for Practical Men.*

ON ELASTIC FLUIDS.

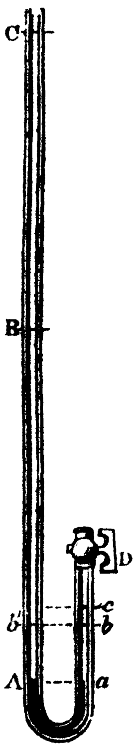
222. DEF.—Atmospheric air and other gases possess the property of contraction and expansion, and are, therefore, called *elastic fluids.* Atmospheric air is the best known of all elastic fluids, and shall, therefore, form the subject of the following investigations.

THE BAROMETER.

223. The annexed figure is a glass tube about 32 inches long, open at bottom and closed at the top. Let the tube be inverted and filled with mercury; then placing the finger on the open end, so as to prevent the mercury from escaping, reinvert it, and plunge the open end into a vessel of mercury; if the finger be now removed, it will be seen that the mercury will stand at the height of about 29 or 30 inches in the tube, above the level of the mercury in the vessel. That the mercury is supported in the tube by the pressure of the air on the surface of the mercury in the vessel, is evident from placing the barometer under the receiver of an air pump (to be hereafter described). As the air is exhausted the mercury sinks in the tube, and when the exhaustion is carried to its full extent, so very little pressure is produced on the surface of

the mercury in the vessel, that the mercury in the tube and in the vessel are nearly on the same level; and when the air is again admitted into the receiver, the mercury will rise in the tube to its previous height.

Since the pressure of the air on any portion of the surface of the mercury in the vessel is equal to the weight of the superincumbent column of air, the pressure of the mercury upwards against the lower end of the tube is the weight of a column of mercury, the base of which is the area of a section of the tube and height ab; and this pressure is balanced by the pressure of the air downwards on the surface of the mercury in the vessel.



224. Prop.—*The density of the air is proportional to the force that compresses it.*

Let C B A D be a bent cylindrical tube of glass, having the end C open and the end D closed; and let the communication between the two branches be stopped by pouring in a small quantity of mercury at C, till it fills the bent part A a; then by turning the cock at D, the air in a D will be of the same density as the air in AC. Now, close the cock at D, and pour in mercury at C, and it will force the mercury to rise in a D; continue this till the mercury stands at B, as high above b, to which point it has risen in a D, as the height of the mercury in the barometer; then the column of mercury Bb' is equal to the weight of a column of air resting on it at B, by the last article. Therefore the pressure against the air in a D, arising from the pressure of both the mercury and the air in B b', is twice as great as it was against the air in a D; and it is now found that $\text{D}b = \frac{1}{2}\text{D}a$, consequently the air being compressed into half its natural space, its density is doubled. Again, if another column of mercury be poured into CB, so that the height of the mercury in A C above that of the mercury in a D, shall be twice the height of the mercury in the barometer, the pressure against the air in a D will now be thrice as great as it was against it in a D, and the space C D now filled with air will be observed to be $= \frac{1}{3} a$ D, consequently the density in c D is equal to 3 times the density of the atmosphere. Similarly the density of air is found in all cases to be proportional to the compressing force.

224.* Cor. 1.—Since the force compressing the air is balanced by its elasticity, the elastic force of the air is equal to the compressing force; hence, also, the air's elasticy is proportional to its density.

224.† Cor. 2.—Let P be the compressing force on a surface = 1, when the density = Δ, and p = compressing force, when the density = δ; then

$$P : p :: \Delta : \delta, \therefore P = \frac{p \Delta}{\delta}.$$

225. Prop.—*The density of any gas remaining the same, its elastic force increases in proportion to its increase of temperature.*

It appears from the experiments of Dalton, Gay, Lussac, and others, that all gases, under the same pressure, expand equally for equal increments of temperature, at least from the freezing to the boiling point of the thermometer; and the degree of expansion is the same in all. This expansion for each unit of bulk is $\frac{3}{8}$ of the bulk, from 32° to 212° of Fahrenheit's thermometer, that is, the expansion for 212°—32° = 180°, is $\frac{3}{8}$; and therefore the expansion for one degree $= \frac{1}{180} \times \frac{3}{8} = \frac{1}{480}$; hence, if V = volume, or solid content, of any gas at 32° temperature, and v = its volume at $t°$ temperature, then

$$v = V\left(1 + \frac{t-32}{480}\right), \text{ and } \frac{v}{V} = 1 + \frac{t-32}{480}.$$

Now, let P = pressure on a unit of surface of the gas, and p = pressure which would reduce the volume v at the temperature t to the volume V; then, by the preceding prop.,

$$P : p :: V : v;$$

hence, if $t-32 = \tau$, and $\alpha = \frac{1}{480}$, there will result

$$\frac{p}{P} = \frac{v}{V} = 1 + \alpha\tau,$$

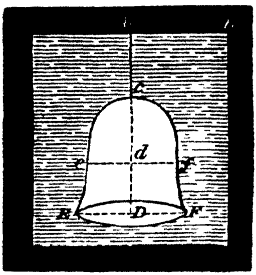
$$\therefore p = (1 + \alpha\tau)P,$$

$$\text{and} \quad p - P = a\,\tau\,P,$$

that is, the increase of elasticity is proportional to the increase of temperature.

THE DIVING BELL.

226. The *Diving Bell* is a vessel inverted in water, and let down to any depth by means of a rope, the air occupying the upper part of the vessel and diminishing in bulk as the vessel descends in the water. Let A B be the surface of the water, E C F the diving bell, D d the height to which the water rises in the bell; also let V be the content of E C F, v the content of e C f = space occupied by the condensed air, h = weight of a column of water, the pressure of which is equal to that of the atmosphere, A C = a, and C d = x. Now, when the air was in its natural state and occupied the whole space of the bell, its elasticity was measured by the height of the column of water h; but when it occupies the space e Cf, the pressure of the water is as the depth A d = $a + x$, and the pressure of the atmosphere is as h, therefore the whole pressure of the air in e C f is equal to the weight of a column of water of the height $h + a + x$; but the elastic force of the air is inversely as the space occupied; therefore,



$$h : h + a + x :: v : V.$$

227. Cor.—When the form of the bell is given, the relation of the above quantities may be determined in numbers. For instance, let the bell be in the form of a prism; then,

$$h : h+a+x :: v : V :: x : CD,$$

$$\therefore x^2 + (a + h)\,x = CD \times h,$$

whence the value of x may be found.

Ex.—Let the depth AC = a = 100 feet, C D = 10 feet; then since h = 32 feet, the above equation will become

$$x^2 + (100 + 32)\,x + 10 \times 32,$$

$$\text{or,} \quad x^2 + 132\,x = 320\,;$$

and by solving this quadratic, there results,

$$x = -66 + \sqrt{66^2 + 320} = 2{\cdot}41 \text{ ft.} = 2 \text{ ft. } 5 \text{ in. nearly} = \text{C}\,d,$$

the positive sign of the surd being used, the negative one being inadmissible.

ON THE EQUILIBRIUM OF FLOATING BODIES.

228. PROP.—*The centre of gravity of a body floating in a fluid, and the centre of gravity of the fluid displaced by the body, are in the same vertical line.*

The pressure of the body downward is its weight, which may be considered as collected at its centre of gravity; and the pressure of the fluid upward may also be considered as collected at its centre of gravity, and this pressure is the same as the weight of the body, acting in an opposite direction; also, since the body is at rest, the weight of the body downward, and the pressure of the fluid upward, must be opposite and equal; therefore, the two centres of gravity are in the same vertical line.

229. PROP.—*To determine when the equilibrium of a body, floating in a fluid, is stable, unstable, or indifferent.*

Let G be the centre of gravity of the body ACB, floating in a fluid, the surface of which is AB; let the centre of gravity of the fluid displaced be in the line MGm, when the body is at rest, and let G′ be the centre of the fluid displaced, when the body is moved through a small angle α; also let the vertical line G′M meet mGM in M, then M is called the *metacentre* of the floating body. Now, the weight of the body acts downward in the direction GP, and the pressure of the fluid acts upward in the direction G′M; and when M is situated above G, these two pressures obviously tend to bring the floating body back to its former position, and therefore, the equilibrium is *stable*. But if the metacentre M be below G, as at m, the weight of the body, and the pressure of the fluid in the opposite direction, tend to move the body farther from its former position, and therefore the equilibrium is *unstable*. Finally, if M coincide with G, the forces being equal, and acting on the same point in opposite directions, the body will be at rest in any position, and therefore the equilibrium is *indifferent*. Consequently the equilibrium of a floating body is *stable*, *unstable*, or *indifferent* respectively, as the *metacentre* falls above, below, or coincides with, the centre of gravity of the body.

230. Cor.—The moment of the force, tending to bring the body back to its former position, or to move the body farther from it, is the weight W of the body $\times$ FG $=$ W $\times$ MG $\times$ $\sin \alpha$; $\therefore$, when the weight W and the angle α are given, the stability varies as G M.

231. Prop.—*A body immersed in a fluid descends or ascends with a force equal to the difference between its own weight and the weight of an equal bulk of fluid; neglecting the resistance of the fluid.*

Let W $=$ wt. of the body, and w $=$ wt. of an equal bulk of the fluid; then the pressure downward is W, and that upward is w; therefore $W - w$ $=$ pressure or force, which causes the body to descend; also, W is the mass or weight moved, and g multiplied by the pressure or force divided by the mass moved, gives

$$\text{the accelerating force downward} = \frac{W - w}{W} g,$$

$$= \left(1 - \frac{w}{W}\right) g.$$

232. Cor.—When W is less than w, the body will ascend,

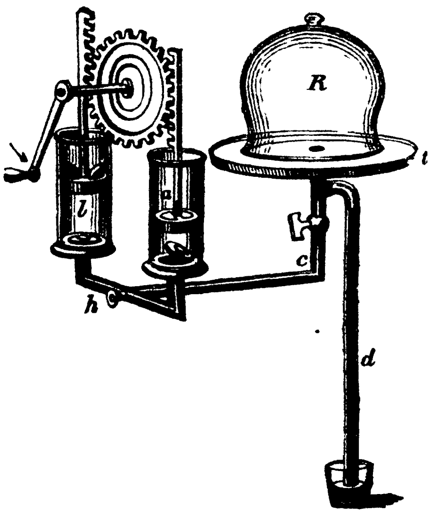
$$\text{and the accelerating force upward} = \left(\frac{w}{W} - 1\right) g.$$

ON HYDROSTATIC MACHINES.

THE AIR PUMP.

233. The *air pump* is a machine for exhausting the air from a close vessel, called a *receiver;* thus producing a near

approach to a perfect vacuum. The glass receiver R is fixed on a metal plate, and made perfectly air-tight. A pipe C communicates with the receiver and with two cylindrical barrels *a* and *b*, by means of two valves opening upwards, which are shewn at the bottoms of the barrels. In these barrels are two air-tight pistons with valves also opening upwards; these pistons are worked up and down by a rack wheel. Now, suppose the piston in *a* to be at the bottom, and that in *b* at the top of the barrel; then, as the piston in *a* ascends, a partial vacuum is formed below the piston, and the elastic force of the air in R and C, pressing upon the valve, opens it, and fills the barrel *a*. Next, let the wheel be turned back, and the piston *a* is now made to descend; the valve at the bottom of *a* is then closed by the pressure of the air upon it, and the valve in the piston within it



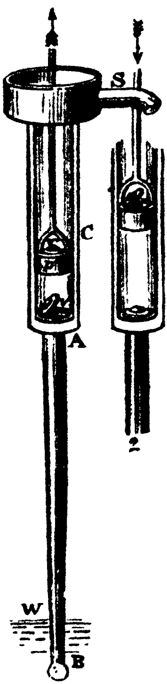
is opened, and the air in the barrel is forced out by reversing the motion of the wheel. The wheel acts in the same manner on the piston in the barrel *b*, thus expelling a barrel of air at every turn of the wheel, until the elastic force of the air in the receiver and pipe is not sufficient to open the valves at the bottoms of the barrels, and then the process of exhaustion must cease. The air is readmitted into the receiver by a cock at *h*. One end of a bent glass tube *d*, which is more than 30 inches in length, opens into the tube C, while the other end is immersed in a vessel of mercury; this tube acts as a guage, and shews by the ascent of the mercury within it, the amount of rarefaction in the receiver, because, as the rarefaction proceeds in the receiver, the elastic force of the air pressing on the mercury in the guage-tube is diminished.

234. Prop.—*To find the density of the air in the receiver of an air pump after any given number of turns of the wheel.*

Let R be the content of the receiver and pipe, and b the content of each barrel; then the air which filled the space R, when the piston in the barrel a was at the bottom, will fill the space $R + b$, when the piston in a ascends to the top of the barrel; therefore, if δ = density of the air before the stroke, δ_1 = its density afterwards, we shall have

$$R + b : R :: \delta : \delta_1,$$

$$\therefore \quad \delta_1 = \frac{R\delta}{R + b}.$$



Similarly it will be found that if δ_n be the density after n turns, that

$$\delta_n = \frac{R^n \delta}{(R + b)^n}.$$

Hence it appears that the density of the air in the receiver decreases in geometrical progression; and therefore can never be completely exhausted.

THE COMMON PUMP.

235. The common suction pump is usually thus constructed: AC is a cylindrical barrel, A B a pipe having its lower end in water, v is a fixed valve opening upwards, and p is an air tight piston, moveable by a handle or brake fixed to the rod, and having a valve v' opening also upwards. Now, let the piston p descend as low as it can, each valve being shut; then, when p ascends, there will be a vacuum in the barrel between A and C, and the valve v will be opened by the upward pressure of the air in the pipe AB, and the air will follow the piston and fill the empty space AC. The air in the pipe will thus be-

come rarefied, and hence the pressure of the air on the surface of the water at W will be greater than the pressure of the air in A B, and therefore the water will be forced a short distance up the pipe A B, till the equilibrium is restored. On again depressing the piston, the valve v is closed, and the valve v' forced open (as in fig. 2), through which the air in A C escapes. On raising the piston a second time, more air rushes from A B, and the water in the pipe rises still higher. Thus, by alternately raising and depressing the piston, all the air will be drawn out of A B, and the water will rise up to the valve v. The piston being now raised, water instead of air will open the valve v, and rush into the barrel, and, on lowering the piston, the water closes this valve v, thus preventing it from flowing back; at the same time the water forces open the valve v', and passes through it, so that the water is now both above and below the piston. This action being continued, the water will rise still higher above the piston, till it be discharged at the spout S.

NOTE 1.—In this pump the height of the valve v above the water must not greatly exceed 30 feet; because the pressure of the atmosphere, in its rarest state, will not raise the water in a vacuum above that altitude.

NOTE 2.—*The lifting and forcing pumps* are only modifications of that just described; they have, however, the advantage, if required, of raising water to the height of several hundred feet. See *Hydraulics, Weale's Series.*

236. PROP.—*To find the height to which the water will rise after any given stroke in the common pump.*

Let the water, after a given number of strokes, rise to P, in the pipe A B, and after the next stroke let it rise to p; (these points are not shewn in the fig.). Put h = height of a column of water equivalent to the pressure of the air, $AS = a$, $AP = b$, $c = h - PB$, and $Bp = x$; also put k = area of a section of the pipe A B, and mk = area of a section of the barrel AS. Now, let the piston be at A, then the elasticity of the air AP, together with the weight of the column of water B P, is equal to the pressure of the air, or is = column of water of the height h; hence

elasticity of air in AP = column of water above P = c;

let the water rise to p after the next stroke, then

elasticity of air in Ap = column of water above $p = c - x$.

Now, the air which filled the space A P, before the rise of the piston, will expand, after its ascent, and occupy the space $p\,S$; hence

density of air in AP : density of air in pS :: space pS : space AP.

$$:: (b - x)k + amk : bk,$$

$$:: b - x + am : b.$$

But the density of the air is proportional to its elastic force; hence

$$c - x : c :: b : b - x + ma; \quad \text{therefore}$$

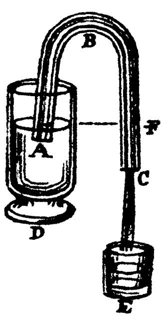
$$bc = (c - x)(am + b - x), \quad \text{whence}$$

$$x^2 - (am + b + c) + acm = 0;$$

whence the valve of x = rise of water due to one stroke, may be found.

THE SYPHON.

237. *The syphon* is a bent tube A B C. If its shorter leg A B be put into a vessel of water D, it will transfer the liquid to the vessel E, in the following manner. Draw the air out of the syphon by suction, or any other means, and the water will rise in it to B by the pressure of the air on the surface of the liquid above A, and then it will descend by its own gravity to C. The syphon being thus filled with liquid, the forces which act upon the liquid in the tube are the pressure of the air upon the surface above A, and the weight of the column of liquid BC, acting in the direction ABC; and the pressure of the air at C, and the weight of the column AB, acting in the opposite direction. But as the column BC is longer than the column AB by FC, the sum of the pressures in the direction A B C is greater than the sum of the pressures in the direction C B A, the liquid will, therefore, continue to flow in the direction A B C till the surface of the fluid falls to A.



NOTE.—The syphon will not act, if the length of the shorter leg be much greater than 30 feet; see *Note* 1, Art. 135.

238. Cor.—The action of intermitting springs that ebb and flow, as they are termed, depends on the principle of the syphon. Water being collected from various springs *a*, *b* into the cavern A B, and the only way by which it can be discharged is the channel B C D, which is bent like a syphon. When AB is so full of water, that it stands at the level A C, it will flow out, and continue to do so until it is either exhausted or on a level with the outlet B.

ON SPECIFIC GRAVITIES AND THE EQUILIBRIUM OF FLOATING BODIES.

239. Def.—The *specific gravity* of a body is its weight compared with the weight of some other body of the same magnitude. Thus, silver has about $10\frac{1}{2}$ times the specific gravity of water, because a cubic foot of silver contains about $10\frac{1}{2}$ times the quantity of matter that water contains, or is bulk for bulk $10\frac{1}{2}$ times heavier; the specific gravity of a body is, therefore, proportional to its density. The specific gravity of distilled water, at a temperature of 60°, is usually considered the unit of comparison, or 1, for all solids and liquids; and the specific gravity of air, at the same temperature, when the barometer is at 30 inches, is adopted as the unit of comparison for all gases and vapours.

240. Prop.—*If a body be either wholly or partly immersed in a fluid, it is pressed upwards by a force equal to the weight of the fluid displaced.*

Let A B be the horizontal surface of a fluid, and L M N a body suspended in it; draw the vertical lines *p r*, *q s* indefinitely near to each other; then the indefinitely small portion *m n* of the upper surface of the body is pressed downward by the weight of the column of fluid *m n s r*, and a like portion *p q* of the under surface is pressed upward by a force equal to the column of fluid *p q n m*; hence the difference of these forces, which presses upwards against *p q* is the weight of the column *mpqn*.

Similarly it may be shewn that the upward pressures against the whole body L M N exceed the downward pressures, by a quantity of fluid equal to the magnitude of the body, that is, the body is pressed upward by a force equal to the weight of the fluid displaced. Again, let A′B′ be the surface of the fluid, the body being now supposed to float in the fluid; then the pressure upward against *p q* is equal to the column *p r′ s′ q*; hence the sum of all the upward pressures is equal to the weight of fluid of the bulk M′ N′ L; that is, the body is pressed upward by a force equal to the weight of the fluid displaced.

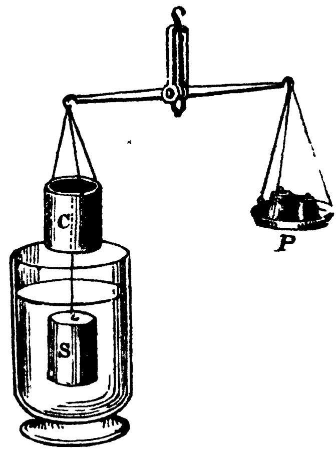
241. Cor. 1.—When a body *floats* in a fluid, it displaces a quantity of fluid equal in *weight* to itself; and when it *sinks* it displaces a quantity equal to its *bulk.*

242. Cor. 2.—The weight lost by a body, when wholly immersed in a fluid, is equal to the weight of an equal bulk of the fluid.

243. Cor. 3.—A solid placed in fluid will sink, if its specific gravity exceed that of the fluid; it will float on the surface, being at the same time partly immersed, if its specific gravity be less than that of the fluid; and it will remain wholly immersed, at any depth, if the specific gravities of the fluid and solid are equal.

244. Prop.—*To determine the specific gravities of bodies.*

(1.) *For a solid heavier than its bulk of water.*—The specific gravity of a solid body is found by the *hydrostatic* balance C P, which is a common pair of scales, with a fine silver thread attached to the under surface of the scale C. The substance S, the specific gravity of which is required, is first weighed in air, and then, being attached to the thread, is immersed in pure water, at the temperature of 60°, and again weighed. Let W be the weight of the body in air, and w its weight in water; then $W - w$ is the weight lost, which is equal to the weight of fluid displaced, by Cor. 2 of the preceding Prop.; hence $W - w$ is also the weight of water equal to the bulk of the body, the weight of which is W; therefore,



W : W — w : : wt. of body : wt. of an equal bulk of water
: : specif. grav. of body : specif. grav. of water.

Now, since the specific gravity of water is 1, by the Def. Art. 239; therefore

$$\text{specific gravity of the body} = \frac{W}{W - w}.$$

(2). *For a solid lighter than its bulk of water.*—Attach to the body another solid heavier than water, so that the compound body may sink in water. Put W' = weight of the heavy body in air, w' = its weight in water, C = weight of the compound body in air, and c = its weight in water; then

wt. of water = in bulk to comp. body = $C - c$,

ditto ———————— to heavy body = $W' - w'$,

∴ ditto ———————— to given body = $C - c - (W' - w')$;

hence W : $C - c - (W' - w')$: : spec. grav. of body : spec. grav. of water; and therefore

$$\text{spec. grav. of body} = \frac{W}{C - c - (W' - w')} = \frac{W'}{W + w' + c}.$$

(3). *For a liquid or powder.*—Weigh a vial, first when empty, secondly when filled with the liquid or powder, and thirdly when filled with pure water; then the weight of the liquid or powder divided by the weight of the water will be the specific gravity required.

(4). *For any kind of gas, &c.*—Extract the air, by means of an air-pump, from a flask, containing about a gallon, and formed of thin copper, with a narrow neck, which may be opened and closed at pleasure by a stop-cork; now, having weighed the flask, let it communicate with the vessel containing the gas, the specific gravity of which is required to be found. The flask, being filled, is again weighed; and the difference of these weights will be the weight of the gas, and, since the content of the flask is known, the specific gravity of the gas is found as before.

NOTE.—In the preceding methods of finding the specific gravities of bodies, it has been assumed that the weight of the body *in air* was the *true* weight of the body; but since air itself is a fluid, the body loses a portion of its weight in air, in the same manner as when weighed in water; a small correction is therefore required on this account, that the weight of the body *in a vacuum* may be obtained, which is the *true* weight.

245. PROP.—*The specific gravity* G *of a body is given, as determined by weighing it in air and water; to find its true specific gravity.*

Put $W =$ wt. of the body in air, $w =$ its wt. in water, as before, $x =$ its true wt. $=$ its wt. in a vacuum, and $\gamma =$ spec. grav. of air as compared with water; then the wt. of an equal bulk of water $= x - w$, and the wt. of an equal bulk of air $= x - W$; hence

$$x - w : x - W :: 1 : \gamma,$$

$$\therefore \quad x - W = \gamma(x - w), \quad \text{whence}$$

$$x = \frac{W - \gamma w}{1 - \gamma},$$

but the true specific gravity of the body is $\frac{x}{x - w}$, and by substitution,

$$\frac{x}{x - w} = \frac{W - \gamma w}{W - w}.$$

Now $\frac{W}{W - w} = G$, and $\frac{w}{W - w} = G - 1$; whence, by substitution, &c.,

$$\text{the true specific gravity} = G - \gamma(G - 1).$$

TABLE OF SPECIFIC GRAVITIES.

(*Weights in ounces per cubic foot.*)

METALS.

Platinum	19·500
Pure gold, cast	19·258
——— hammered	19·362
Mercury	13·568
Lead	11·352
Pure silver, cast	10·474
——— hammered	10·511
Bismuth, cast	9·823
Copper, do.	8·788
Cobalt, do.	7·812
Nickel, do.	7·807
Iron, do.	7·207
Bar iron	7·788
Steel, hard	7·816
—— soft	7·833
Tin, cast	7·291
Zinc, do.	7·191
Antimony, do.	6·702
Arsenic, do.	5·763

MINERAL PRODUCTIONS.

Ponderous spar	4·430
Oriental ruby	4·283
Oriental sapphire	3·994
Oriental topaz	4·011
Oriental beryl	3·549
Diamond	3·501 to 3·531
White Parian marble	2·838
Green marble	2·742
White marble of Carrara	2·724
Jasper	2·660 to 2·764
Granite	2·950
Pure rock crystal	2·653
Purbeck stone	2·601
White flint	2·594
Portland stone	2·580
Plumbago	1·860
Newcastle coal	1·270
Staffordshire coal	1·240
Pumice stone	·914

———

Flint glass	3·329
White glass	2·892
Green glass	2·642
Alabaster	2·000
Brick	2·000
Gunpowder, about	·937
Ice	·930

WOODS.

Lignum vitæ	1·333
Box, Dutch	1·328
—— French	·970
Heart of oak (60 years old)	1·170
Dry oak	·925
Mahogany	1·063
Beech	·850
Elm	·600
Fir	·570
Poplar	·383
Cork	·240

LIQUIDS.

Sulphuric acid	1·841
Nitric acid	1·217
Water from the Dead Sea	1·240
Human blood	1·053
Cow's milk	1·032
Cider	1·018
Sea water	1·026
Water at 60°	1·000
Wine	·994
Olive oil	·915
Pure alcohol	·792
Muriatic ether	·730
Naphtha	·708

GASES.

Atmospheric air	1·000
Do. compared with water	·001225
Oxygen	1·112
Chlorine	2·500
Hydrogen	0·069
Nitrous oxide	1·524
Carbonic acid	1·527
Coal gas	·450 to ·650

Ex. 1.—How many cubic feet are there in a ton of dry oak?

$$\frac{20 \times 112 \times 16}{925} = 38\tfrac{138}{185} \text{ cubic feet. } \textit{Ans.}$$

Ex. 2.—A piece of copper weighs 93 grains in air, and $82\frac{1}{2}$ grains in water; what is its specific gravity? *Ans.* 8857.

Ex. 3.—A piece of elm weighs 30 lbs. in air, and when a piece of copper, which weighs 32 lbs. in water, is connected with it, the compound weighs 6 lbs. in water; what is the specific gravity of the elm? *Ans.* 600.

Ex. 4.—A cast iron pipe is 6 inches diameter in the bore, and 1 inch in thickness; required the weight of a running foot. *Ans.* 67·45 lbs.

PART IV.

HYDRODYNAMICS.

246. *Hydrodynamics* treats of the motion of fluids, and of the forces which they exert upon bodies to which their action is applied.

247. Prop.—*The velocity of a fluid issuing from a small orifice at the bottom of a vessel, kept constantly full, is equal to that which a heavy body would acquire in falling through a space equal to the depth of the orifice below the surface of the fluid.*

Let AB be the surface of the fluid, D the small orifice.

Consider the fluid to be composed of an indefinite number of laminæ, which during their descent remain parallel; then, whatever moving force is lost by the descending fluid will be communicated to the fluid at the orifice. Let Dc be a small column of fluid discharged in the indefinitely small time t by the pressure of the column CD; and let Db be the which would have been discharged by its own weight, that is, by its gravity in the same time t. Let, also, V and v be the velocities of the fluid in the columns Dc, Db, by the pressures C D, Db; then, since the moving forces are as the quantities of motion produced in a given time,

$$c\,D : D\,b :: \text{column } D\,c \times V : \text{column } D\,b \times v; \quad \text{and}$$

because the spaces described by constant forces in equal times are as the velocities acquired

$$D\,c \times V : D\,b \times v :: \frac{V^2}{2\,g} : \frac{v^2}{2\,g};$$

and since v is the velocity in falling through Db, by the force of gravity,

$$F = \frac{v^2}{2\,g}, \quad \text{and similarly} \quad C\,D = \frac{V^2}{2\,g};$$

$$\therefore \quad V = \sqrt{2\,g\,.\,C\,D},$$

which is the velocity acquired in falling through C D by the force of gravity.

248. Cor.—Since fluids press equally in all directions, the preceding Proposition holds, when the orifice is in a side of the vessel, or when it is made to throw the fluid in a vertical or oblique direction; in the former case it will rise to the level of the fluid in the vessel.

249. Cor. 2.—If h = height of the vessel, a = area of orifice, and Q = quantity of fluid discharged in one second; then

$$Q = a\sqrt{2\,g\,h}; \quad \text{whence}$$

$$a = \frac{Q}{\sqrt{2\,g\,h}}, \quad \text{and} \quad h = \frac{2\,g\,a^2}{Q^2}.$$

Ex. 1.—Find the velocity with which water issues from a small orifice at the bottom of a vertical tube, filled with water to the height of 100 feet.

$$V = \sqrt{2 \cdot g \times 100} = \sqrt{64\tfrac{1}{3} \times 100} = 20\sqrt{16\tfrac{1}{12}} = 80\text{ft } 2\tfrac{1}{2}\text{in nearly.}$$

Ex. 2.—Find the same when the water issues into a vacuum, its upper surface being open to the air.

Here 32 feet must be added to the height of the water in the tube for the pressure of the atmosphere, after which the method of solution will be the same as in the last Example.

Ex. 3.—What quantity of water will be discharged from a vessel 10 feet high, in one second, through an orifice one inch in diameter in the bottom of the vessel?

NOTE 1.—The actual velocity has been found by D'Alembert to differ from the theoretical velocity considerably in many cases; but when the vessel is kept constantly full, and each stratum of the fluid is supposed to keep parallel to itself as it descends,

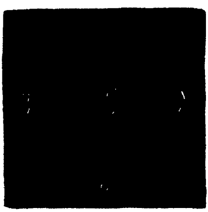
$$\text{the vel.} = \sqrt{\frac{2g.\text{CD}}{1-\frac{k^2}{\text{K}^2}}},$$

where K = surface of vessel, k = area of orifice; and when k is very small, the velocity becomes $\sqrt{2g.\text{CD}}$, which is the same velocity as that already found.

NOTE 2.—Experiments do not agree with this theory as to the quantity of water discharged; Bossut has shewn that the actual discharge : the theorectical discharge :: ·62 : 1, or nearly as 5 : 8.—The vein of water that issues through the orifice suffers a contraction, by which its section has been found to be diminished in the above ratio. This contraction has been called the *vena contracta*, and calling the area of the orifice 1, the area of the *vena contracta* will be $\frac{5}{8}$ = ·625 nearly. Hence the theoretical quantity of the fluid discharged must be multiplied by $\frac{5}{8}$ to obtain the true quantity.

251. PROP.—*To determine the time of emptying any vessel through a very small orifice.*

Let MN be the surface of the descending fluid in the vessel MON, and O the orifice; put PO = x, PN = PM = y, K = area of the descending surface, and k = area of the *vena contracta*, and t = time of discharge. Then the velocity of the fluid at the *vena contracta* = $\sqrt{2gx}$, and therefore the quantity of fluid discharged in the indefinitely small time dt is equal to $k\,dt\sqrt{2gx}$. Now, let the surface of the fluid in the vessel descend from MN *to mn* in the time dt; then Pp = dx, and the content MNnm = $-$ Kdx; but this is equal to the quantity of fluid discharged, therefore



$$k\,dt\sqrt{2gx} = -\text{K}dx; \quad \text{whence}$$

$$dt = \frac{-\text{K}dx}{k\sqrt{2gx}},$$

$$\text{or} \quad t = \int \frac{-k\,dx}{k\sqrt{2gx}}.$$

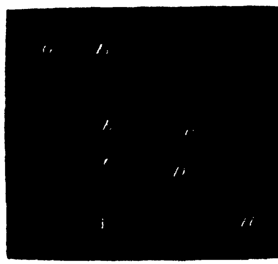
252. Cor.—When the vessel M O N is a solid of revolution round the vertical axis O P, then $K = \pi y^2$, and therefore

$$t = \int \frac{-\pi y^2\,dx}{k\sqrt{2gx}};$$

in which, if the value of y be substituted in terms of x, the integral may be readily found.

253. Prop.—*To find the distance to which water will spout, through a small orifice in the vertical side of a vessel placed on a horizontal plane.*

Let G A be a vessel filled with water, C a small orifice in the vertical side A B, and A H the horizontal plane. On A B describe the semicircle A F B, and draw the ordinate C D; then by Art. 249, Note 1, the velocity of the fluid will be very nearly equal to that which would be acquired by a body in falling down B C; this velocity must, therefore, be considered as that with which the fluid is projected. Now, the curve C H, described by the fluid, is a parabola, and $BC = \frac{1}{4}$ of its parameter at C, by Art. 161; and since the fluid evidently spouts horizontally, C is the vertex of the parabola, C A its axis, and A H an ordinate; therefore,



$$AH^2 = 4\,CB \times CA = 4\,CD^2 \text{ (by the nature of the circle)},$$

$$\therefore AH = 2\,CD.$$

254. Cor.—When the orifice bisects AB in E, the distance spouted by the fluid will be $= 2\,EF = AB =$ depth of the fluid, which is exidently the maximum distance that it can spout on the horizontal plane A H.

255. Prop.—*To find the velocity with which water is discharged from a reservoir of given height h, through a pipe of given length l, and diameter d.*

The experiments and investigations of M. Poncelet are considered strictly accurate, the limits of this work does not admit of their insertion here, the following is his formula for the velocity per second, all the dimensions being in feet.

$$v = 48 \sqrt{\frac{h\ d}{l + 54\ d}}.$$

Ex. 1.—Water is brought to supply Mentz from a reservoir $65\frac{3}{5}$ feet in height, by pipes 9843 feet in length, and $3\frac{3}{20}$ inches in diameter; required the velocity of the water per second.

First $3\frac{3}{20}$ in. = ·2625 of a foot, and $65\frac{3}{5}$ = 65·6 feet, then

$$v = 48 \sqrt{\frac{h\ d}{l + 54\ d}} = 48 \sqrt{\frac{65{\cdot}6 \times {\cdot}2625}{9843 + 54 \times {\cdot}2625}} = 2 \text{ feet per}$$

second nearly.

Ex. 2.—In the last example, how much water will be discharged in 24 hours?

The area of the section of the pipe = ·7854 × (·2625)² = ·0541 square feet, the quantity of water per second = 2 × ·0541 = ·1082 cubic feet, and 24 hours = 86400 seconds; ∴ the quantity of water brought by the pipe in 24 hours will be

$$86400 \times {\cdot}1082 = 9348\tfrac{1}{2} \text{ cubic feet.}$$

256. Prop.—*To determine the mean velocity with which water runs in rivers and open canals.*

The formula for this purpose is also derived from experiments, of which no less than 91 were made by Eytelwein on rivers and canals, the dimensions used by him are reduced to feet, and are the following:—

c = wet contour

s = area of a section of the fluid,

$\frac{s}{c}$ = hydraulic mean depth,

g = force of gravity,

α = angle of inclination of surface of stream,

and v = mean velocity; then

$$v = \sqrt{(\tfrac{50}{3})^2\, g \cdot \frac{s}{c} \sin\alpha + (\tfrac{1}{100})^2} - \tfrac{1}{100} = \text{the velocity in feet.}$$

Note.—It has been proved that the greatest velocity is at the surface in the middle of the stream; from which it diminishes towards the bottom and sides, where the velocity is least.

ON THE PERCUSSION AND RESISTANCE OF FLUIDS.

257. PROP.—*When a stream impels a plane perpendicular to its action, the force with which it strikes the plane is as* $A\,d\,V^2$; *where* A = *area of the plane,* d = *density of the fluid, and* V = *its velocity.*

The impulsive force of the stream is as the number of particles that strike against the plane in a given time, multiplied by the force of each; and the number of particles that strike the plane in a given time is evidently $= A\,d\,V$; also, the force of each particle is proportional to V; $\therefore$ the force of all the particles against the plane is as $A\,d\,V^2$.

NOTE.—It has been here supposed that after the particles strike the plane, their action immediately ceases; but in reality they rebound, and acting on those which are behind, retard their velocity; therefore a difference will result between theory and experiment.

258. COR.—If f = impulsive force of the stream against the plane A, k = a constant co-efficient to be determined by experiment, and h = height due to the velocity V, so that $V^2 = 2\,g\,h$; then

$$f = 2\,A\,k\,d\,g\,h.$$

259. PROP.—*If a stream strikes perpendicularly on a plane, which is itself in motion, the impulsive force is* $= A\,d(V-v)^2$; *where* v *is the velocity of the plane.*

It is obvious that both the number of particles which strike the plane, and the force of each particle, must be as the *relative* velocity, that is the difference of the absolute velocities the force will be as $A\,d\,(V-v)^2$, or $f = A\,k\,d(V-v)^2$.

260. COR. 1.—When v is opposed to V, that is, when the plane moves against the stream, then $f = A\,k\,d\,(V+v)^2$.

261. COR. 2.—When $V = 0$, $f = A\,k\,d\,v^2$; therefore a plane, moving against a fluid at rest, receives the same impulse as if the fluid were to move with the velocity v, and the plane to be at rest; that is, the resistance of a fluid to a body in motion is the same as the impulse of a fluid, which moves with the same velocity against a body at rest.

THE WATER WHEEL.

262. It has been found by experiment that a water wheel

performs the greatest quantity of work when the velocity of the water is $2\frac{1}{2}$ times that of the wheel, whence by Art. 259, the power of water (the velocity of which is given) striking the paddles or float-boards of wheel might be calculated; but the following method has been found in practice to be less complicated: for when a body descends from a given height, it is capable of raising a body of equal weight through the same height. Therefore, if water fall upon a wheel, the quantity of work which it is capable of performing, abating friction, is equal to the product of the weight of the water, and the height through which it descends; whether it falls upon the paddles of an undershot or a breast wheel, or into the buckets of an overshot wheel.

263. PROP.—*Given the breadth* a, *and depth* b, *of a stream, its mean velocity* v, *in feet per minute, the height* h, *of the fall, and* S = *specific gravity of water; it is required to determine the horse power of the water wheel, when the modulus of the machine is* n*th part of the work of the water, and* U = *units of work in a horse power.*

Water descending per minute...... $= a\,b\,v$ cubic feet.

Weight of water in the same time $= a\,b\,v\,S$ lbs.,

Hence work of water per minute $= a\,b\,h\,v\,S$,

And the work of the wheel......... $= n\,a\,b\,h\,v\,S$;

$\therefore$ HP = horse-powers.............. $= \dfrac{n\,a\,b\,h\,v\,S}{U}$.

Ex. 1.—The breadth of a stream is 5 feet, depth = 3 feet, mean velocity 20 feet per minute, and height of the fall 25 feet; required the HP of the water wheel which performs $\frac{4}{5}$ of the work of the water, that is, $\frac{1}{5}$ of the work of the wheel is lost by friction.

$$\text{HP} = \frac{n\,a\,b\,h\,v\,S}{U} = \frac{4\times5\times3\times20\times25\times62{\cdot}5}{5\times33000} = 11\tfrac{4}{11}.$$

Ex. 2.—The section of a stream is 4 feet by 3, the mean velocity of the water 20 feet per minute, and the fall 30 feet; what is the HP of the water wheel, its modulus being $\frac{4}{5}$; and how many bushels of corn will the wheel grind in a day of 14 hours, one HP being able to grind a bushel of corn per hour?

$$\text{HP} = \frac{n\,a\,b\,h\,v\,\text{S}}{\text{U}} = \frac{4 \times 4 \times 3 \times 20 \times 30 \times 62\cdot5}{5 \times 33000} = 10\tfrac{10}{11}.$$

$\therefore$ bushels ground per day $= 10\frac{10}{11} \times 14 = 152\frac{8}{11}$.

Ex. 3.—The section of a stream, the mean velocity, and fall of the water are the same as in the last example; how many cubic feet of water will the wheel raise to the height of 120 feet, the modulus of the machine being $\frac{2}{3}$ of the work of the water?

Put H = the height to which the water is pumped; then

Work of the wheel per minute........... $= n\,a\,b\,h\,v\,\text{S}$ units,

„ of pumping 1 cubic foot of water... $=$ H S „

$\therefore$ number of cubic feet pumped per min. $= \dfrac{n\,a\,b\,h\,v\,\text{S}}{\text{H}\,\text{S}} =$ $\dfrac{n\,a\,b\,h\,v}{\text{H}}$; which in number gives $\dfrac{\frac{2}{3} \times 4 \times 3 \times 20 \times 30}{120} =$ 40 cubic feet.

WORK PERFORMED BY THE SUN'S EVAPORATION.

264. The heat of the sun is continually raising the temperature of the atmosphere, thus making it capable of absorbing water from the immense surface of the oceans and seas that surround the earth. The water, thus raised, forms clouds at various elevations above the earth's surface. The sudden cooling of the atmosphere, either by cold currents or by meteoric changes, precipitates these clouds in the form of rain; while the dews of night descend by the gradual cooling of the atmosphere, through the absence of the sun. The water, therefore, which thus falls, may be considered as the measure of the sun's evaporating power. In the torrid zone the annual fall of rain and dew amounts, at a medium, to about 100 inches in depth, and at the northern border of the temperate zone, as at Archangel, the medium fall of water is about 20 inches in depth; the mean of these depths is 60 inches or 5 feet, which may be taken as the mean depth of water which descends upon the whole of the earth's surface. Now, if we take 900 feet as the mean height from which this water falls in the form of rain and dew, there will result,

The work of the water falling on one square mile of the earth's surface per minute, through the agency of the sun's evaporation in horse powers, that is,

$$\text{HP} = \frac{27878\cdot400 \times 5 \times 900 \times 62\cdot5}{365 \times 24 \times 60 \times 33000} = 452.$$

Hence, the work, thus done, on the whole surface of the globe, taking its diameter at 8000 miles will be

$$\text{HP} = 8000^2 \times 3\cdot1416 \times 452 = 90{,}880{,}000{,}000.$$

Now, taking the united powers of all the steam engines in the British Isles to be $2\frac{1}{2}$ millions of horse powers, and the united powers of all the steam engines in all the other states of the world to be $3\frac{1}{2}$ millions of horse powers, thus giving for the steam engines of the whole world 6 millions of horse powers, which, it is presumed, is not far from the truth, at the present time (1851), we shall have the work due to the sun's evaporation somewhat more than 15000 times the work of all the steam engines in the world, supposing them to work continuously day and night. This comparison shows how insignificant the most stupendous works of man are to those of his CREATOR. Though only a very trifling part of this vast power is available for the purposes of moving machinery, yet it serves a still more important purpose in watering and invigorating the vegetation on the surface of the earth, and in producing the countless small streams up to large rivers, which diversify and spread health throughout creation, as well as supply immense facilities for inland navigation. Such is the stupendous and magnificent scale by which we must measure the mechanism of creation, and such the boundless power and beneficence of the GREAT CREATOR.

One of the immense results of the power of evaporation may here be given in the

Work of the Great Fall or Cateract of the River Niagara.

This river, which discharges all the water issuing from the great central chain of lakes in North America, falls with astonishing grandeur over a perpendicular rock 133 feet in height, in one unbroken sheet; the rapids above this fall extend several miles, making an addition of 200 feet to the height of the fall; the whole height of the fall is therefore 333 feet. It is calculated that 33 millions of tons of water are discharged, at an average, per hour by this fall; hence the work of the water per minute may be readily determined in horse powers, that is

$$\text{HP} = \frac{33000000 \times 2240 \times 333}{60 \times 33000} = 12{,}432{,}000.$$

This river is, therefore, (see last Art.) capable of performing more work than twice the work of all the steam engines in the whole world.

WORK OR POWER OF THE TIDES.

Assuming that the average height of the rise of the tides in the Atlantic and Pacific oceans to be 20 feet, which is probably less than the true average, and the united length of the coasts of these two oceans (which may be said to extend from pole to pole) including their windings, to be 100,000 miles, we shall thus have a body of water 100,000 miles in length raised to the height of 20 feet, and of a breadth varying according to the widths of the respective oceans. This vast power is immensely greater than that which results from the sun's evaporation; (Art. 264) and is due to the joint attraction of the sun and moon. Although a very small portion of this immense power is used for mechanical purposes, on account of its being inconveniently situated for that purpose; because the shores of these oceans are exposed to tempests, which would in most cases greatly damage or entirely destroy any machinery, which might under other circumstances be conveniently moved by the tide. There are, however, a few ponds, which are filled by the tide in convenient situations, for moving the machinery of corn mills, &c. Yet the rise of the tide is of immense importance in aiding the purposes of navigation, by its repeated flow into numerous rivers, harbours, bays, creeks, &c., which would otherwise in many cases be almost useless for this purpose. Besides, the continued agitation of the ocean by the tide diffuses the saline matter, derived from some of the strata which forms part of its basin, equally throughout every part of its liquid mass; thus maintaining its waters in a perpetual state of salubrity, which would otherwise become stagnant, and in all probability so putrid as to be destructive to animal life. We may hence perceive another grand purpose of the GREAT CREATOR carried ont by the agency of the tide for the continued renovation of nature, and of far greater importance than its use as a moving power for machinery, which the ingenuity of man by the agency of steam can produce in localities more convenient for his several requirements.

MISCELLANEOUS QUESTIONS IN HYDROSTATICS AND HYDRODYNAMICS.

Ex. 1.—The depth of water pressing against an embankment is 9 feet, required the pressure upon 40 feet of its length. *Ans.* $45\frac{1}{5}$ tons.

Ex. 2.—An empty vessel is sunk 600 feet in sea-water, required the pressure on a square inch of its surface. *Ans.* $209\frac{3}{8}$ lbs. nearly.

Ex. 3.—A flood-gate 10 feet deep and 5 feet wide, is placed vertically in water; required the pressure on the upper and lower halves of the gate, the water standing at the top. *Ans.* $3906\frac{1}{4}$ and $11718\frac{3}{4}$ lbs.

Ex. 4.—A cubical iceberg swims, with its sides vertical, 100 feet above the level of the sea, required a side of the cube. *Ans.* 326 yards.

Ex. 5.—Find the thickness of a wall at the bottom, the section of which is a right angled triangle, sustaining a body of water against its vertical side, the height of the wall being 12 feet, the depth of the water 10 feet, and the specific gravity of the wall to that of the water as 11 to 7. *Ans.* 5 feet $1\frac{3}{4}$ inch.

Ex. 6.—The concave surface of a cylindrical glass bottle, filled with fluid, is divided into 4 annuli, so that the pressure on each annulus is equal to the pressure on the base; required the height of the cylinder and the breadth of each annulus, the radius of the cylinder being given.

Ex. 7.—Required the thickness of the wall, in Ex. 5, when its section is a vertical rectangle. *Ans.* 4 feet $2\frac{1}{2}$ inches, nearly.

Ex. 8.—A diving bell in the form of a cone is let down into the sea to the depths of m and n fathoms; required the heights to which the water will rise within it, its axis and diameter of its base being respectively a and b feet, and the barometer standing at 30 inches.

Ex. 9.—How deep will a globe of oak sink in common water, its radius being one foot? *Ans.* 1 foot $8\frac{1}{3}$ inches.

Ex. 10.—Each of the 8 pontoons, used in floating the parts of the Britannia tubular bridge was a parallelopiped 100 feet in length, 25 feet in breadth, and 12 feet in depth; required the weight of their united power of buoyancy, supposing them all to sink till even with the surface of the water, and that the weight of each pontoon was 200 tons.

PART V.

CENTRAL FORCES.

DEFINITIONS.

265. *Centripetal force* is a force which continually tends to draw or impel a body towards a certain fixed point or centre.

266. *Centrifugal force* is that which impels the body to recede from such a centre, if it were not prevented by the centripetal force; this force, according to the first law of motion, impels the body to move uniformly in a straight line.

267. The centripetal and centrifugal forces are called *central forces*, because, by their combined action on the same body, they cause it to describe a curve round a centre.

268. *The radius vector* is a line drawn from the centre of force to the moving body.

269. Prop. 1.—*When a body, acted upon by a force tending to a fixed point or centre, has also a projectile motion in a direction not passing through that centre, it will move in a curve line situated in one plane; and the radius vector will describe equal areas in equal times, or areas proportional to the times.*

270. Let the body move along A B uniformly in a unit of time by a projectile or centrifugal force; then by Newton's first law of motion if no other force were to act on the body, it would move in the same straight line to a, in the next unit of time, making the distance B a = A B. Now, when the body is at B, let a centripetal force, or a force tending to the centre S, act upon it, and by a single impulse draw it along B p in a unit of time, in the same manner as if this force alone acted upon it. Complete the parallelogram B a C p, and join S C, S a. Then, since the body would move along B a in consequence of the original force, and along B p in consequence of the centripetal force at S, by the composition of motion (Art. 10), the body will move along B C, the diagonal of the parallelogram. Also, because C a is parallel to S B, the area S B C = area S B a (since A B = B a) area S A B. In like manner, if the two forces act on the body at C, the centrifugal in the direction C b, and the centripetal in the direction C S, these forces will cause the body to describe the diagonal C D of the parallelogram C b D q, and the radius vector S D will describe the area S C D = area S B C = area S A B, in the next unit of time; and so on continually. Now, suppose the unit of time to be diminished, and the number of units to be increased, both indefinitely; then the areas described by the radius vector in these units will still be equal to each other, and consequently the polygon A B C D.....E will ultimately be a curve line, and the centripetal force which was assumed to act by impulses at B, C, D.....E, will be a continuous force acting at every point of the curve; and, since it has been proved that equal areas are described by the radius vector in equal times, it is evident that in different times the areas described will be proportional to these times.

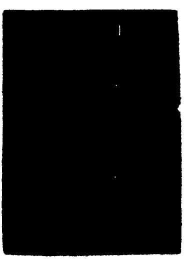
NOTE.—This is *Kepler's first law of Planetary Motion*, which he discovered, by the aid of observation alone, and which was first confirmed by *Mathematical Demonstration by Sir J. Newton*, as shewn in this proposition.

271. Cor.—If a body move in a curve, so that the radius vector, drawn from the body to a fixed point, passes over areas proportional to the times, the body is acted upon by a centripetal force tending to that fixed point.

272. Prop. 2.—*The velocity of body, moving in a curve* A D E *at any point* D, *is inversely as the perpendicular* S Y *drawn from the centre of force* S *upon the tangent* D Y *to the curve at* D. (See last figure.)

Put $SY = p$, τ = time of describing C D, and v = velocity at D; then $CD = \tau v$, and the area $SCD = \frac{1}{2} CD \cdot SY = \frac{1}{2} \tau v \times p$. Now, if A = area described by the radius vector in a unit of time, $A\tau$ will be the area described in the time τ = area SCD; $\therefore A\tau = \frac{1}{2} r v p$, and $v = \frac{2A}{p}$, that is, the velocity varies as $\frac{1}{p}$, or inversely as the perpendicular upon the tangent, since the area A described in a unit of time is constant.

273. Prop. 3.—*If one body be drawn in a straight line* A S *towards the centre of force* S, *and another body revolve in a curve line* A M B *about the same centre* S; *then if the force at* S *be equal at all equal distances, and the velocities of the bodies be equal in any one case, when they are at equal distances from* S, *their velocities will always be equal at equal distances from* S.

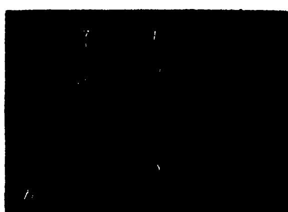


Let the velocities of the two bodies at the equal distances S N, S M be equal. Take Mm an indefinitely small arc, which may be considered as a straight line, and describe the circular arcs M N, mn, from the centre S, and draw the radii vectores M S, m S; from p, the intersection of M S, mn, draw pq perpendicular to Mm. Let f be the accelerating force at N or M towards the centre of force S, and let Nn or Mp represent this force. Now, the force Mp may be resolved into the two, Mq, pq: of these two forces Mq alone is efficient in accelerating the body's motion at M. Put ϕ = angle SMm; then the actual force $Mq = f \cos \phi$, and, because M m is indefinitely small, the increase of velocity from M to m will also be indefinitely small; therefore M m may be ultimately supposed to be described with a uniform velocity; and since the velocities at N and M are equal, if $\tau =$ time of describing N n uniformly, then $\frac{\text{M}m}{\text{M}p}\tau = \frac{\tau}{\cos\phi} =$ time of describing Mm uniformly with the same velocity. Now, the increment of the velocity from N to n is equal to the product of the accelerating force and time $= f\tau$; and the increment of the velocity from M to m $= f\cos\phi\,\frac{\tau}{\cos\phi} = f\tau$, which is the same as in the former case. Therefore, since the velocities at N and M are equal, and the increments of the velocities from N to n and from M to m are also equal, the velocities at n and m must also be equal; and similarly it may be shewn that the velocities are equal at all other equal distances.

274. Cor.—When the bodies pass through the centre of force S, or recede from it, the same proposition holds good with respect to their velocities. In the former case, though theoretically correct, it is practically impossible.

275. Prop. IV.—*If a body describe the circumference of a circle* A a B *uniformly in consequence of a projectile and an attractive force, the latter being situated at the centre* S; *the accelerating force acting upon the body is measured by the square of the velocity divided by the radius of the circle.*

Let the body describe the arc A a uniformly in the time τ, with the velocity V, and let A S = R; then A $a = \tau$ V. Now, the body would describe the tangent A T uniformly with the same velocity, and in the same time, if it were not acted upon by the central force at S; but as it describes the arc A a, it is evident that the force at S, upon the body at A, would make it describe An or Ta in the time τ; aT, an being respectively perpendicular to A T, A S. Let $f =$ accelerating force at A; which force



may be considered constant through the indefinitely small space A n or T a, therefore, by Art. 134, A n = T a = $\frac{1}{2} f \tau^2$.

But A $n \times 2$ A S = (chord A $a)^2$

$$= (\text{arc A } a)^2 \text{ ultimately} = V^2 \tau^2,$$

$$\text{and } \therefore \tfrac{1}{2} f \tau^2 \times 2 R = V^2 \tau^2.$$

$$\text{Hence } f = \frac{V^2}{R} \cdot$$

275A. COR. 1.—If T be the time of one entire revolution of the body round the centre of force S; then, since the whole circumference A a B = $2 \pi R$, there will result

$$T V = 2 \pi R, \text{ or } V = \frac{2 \pi R}{T} \cdot$$

$$\text{Hence } f = \frac{V^2}{R} = \frac{4 \pi^2 R}{T^2} \cdot$$

276. COR. 2.—If a body describe the circumference of a circle with a uniform velocity, *the centripetal and centrifugal forces will be equal*, because the distance of the body from the centre of force is always the same, the two forces are in equilibrium; hence the centrifugal force f' is $= f$, and is measured by $\frac{V^2}{R}$; therefore, generally, when the centre of force is the centre of the circle,

$$f = f' = \frac{V}{R} \cdot$$

277. COR. 3.—If a body be retained in a circle of radius = R, by a rigid rod joining the body and the centre of the circle, or if the body be retained in a circular curve, as a railway train is retained by the rails and the flanges of the wheels, and if a given angular velocity = V be communicated to the body; then the force f will be evidently compounded of the value $\frac{V^2}{R}$ and mass M of the body, that is,

$$f = M \cdot \frac{V^2}{R} \cdot$$

But M (by the definition) = $\frac{W}{g}$, W being the weight of the body, and g the force of gravity at the earth's surface,

$$\therefore f = \frac{W . V^2}{g . R},$$

which is called by Mathematicians the VIS VIVA, or living force, half of which, namely,

$$f = \frac{V^2 . W}{2g},$$

is the force which tends to produce motion in machines; and since $h = \frac{V^2}{2g}$, h being the height due to the vel. V, there results,

$$f = h . W = \frac{V^2 . W}{2g},$$

which is the Formula used in the article on water wheels, and in various other parts of this work. See *Moseley's Principles of Engineering* and *Hann's Mechanics.*

The following are given to illustrate the preceding propositions and their corrolaries.

284. PROB. 1.—Taking the radius of the earth to be 4000 miles, the mean distance of the moon from the centre of the earth to be 60 of the earth's radii, to determine the attractive force exerted on the moon, which causes it to revolve round the earth in $27\frac{1}{3}$ days, the earth being assumed to be at rest.

By Art. 275,

$$f = \frac{4\pi^2 R}{T^2} = \frac{4 \times 3{\cdot}1416^2 \times 4000 \times 5280}{27\frac{1}{3} \times 24 \times 60 \times 60} = {\cdot}0089 \text{ feet.}$$

And since the force of gravity at the earth's surface $= 32\frac{1}{6}$ feet $= g$, we shall have

$$f : g :: 1^2 : 60^2, \text{ that is,}$$

$${\cdot}0089 : 32\tfrac{1}{6} :: 1 : 60^2 \text{ nearly;}$$

therefore the attractive force of the earth varies inversely as the square of the distance from its centre.

285. PROB. 2.—The radius of gyration of a grindstone is 2 feet, its weight $\frac{1}{3}$ of a ton, and it makes 360 revolutions in a minute; required its centrifugal force, or tendency to burst.

Here $V = \frac{4 \times 3{\cdot}1416 \times 360}{60} = 75{\cdot}4$ feet per second nearly.

and by Art. 277,

$$f = \frac{W \, . \, V^2}{g \cdot R} = \frac{\frac{1}{4} \times (75{\cdot}4)^2}{32\frac{1}{6} \times 2} = 44\tfrac{1}{3} \text{ tons nearly.}$$

286. Prob. 3.—The radius of a grindstone is r, its weight W, and the velocity of its circumference v feet per second; required the centrifugal force.

The radius of gyration, in this case, is $\frac{1}{2} r \sqrt{2} = R$.

Hence the velocity V of the centre of gyration $= \frac{\frac{1}{2} R \sqrt{2} \times v}{R} = \frac{1}{2} \sqrt{2}\, v$.

$$\therefore \quad f = \frac{W \, . \, V^2}{g \, . \, R} = \frac{W \, . \, v^2}{g \, . \, r \sqrt{2}}.$$

Note.—This formula gives generally the amount of centrifugal force which tends to tear asunder a circular wheel or disc of uniform thickness, when it is whirled round with a great velocity. The great amount of centrifugal force, as shewn in Art. 285, is the cause of the frequent violent ruptures of grindstones, and the serious accidents thence resulting.

287. Prob. 4.—Required the centrifugal force, in the last Prob., when $W = 16$ cwts., $r = 1\frac{1}{2}$ feet, and $v = 80$ feet per second.

$$f = \frac{W \, . \, v^2}{g \, . \, r \sqrt{2}} = \frac{16 \times 80^2}{32\frac{1}{6} \times \frac{3}{2} \sqrt{2}} \text{ cwt.} = \frac{16 \times 80^2}{32\frac{1}{6} \times \frac{3}{2} \sqrt{2} \times 20} = 75 \text{ tons.}$$

288. Prob. 5.—A circular disc, the weight of which is W, is whirled round so as to make S revolutions in a minute; the radius of the disc is R, the radius of its axle r, and the friction upon it $\frac{1}{n}$ of the whole weight; required the number of revolutions the disc will make before its stops.

Radius of gyration $= \frac{1}{2} R \sqrt{2}$ feet.

Velocity of wt. per second $= \frac{2\pi \times \frac{1}{2} R \sqrt{2} \times S}{60} = \frac{\pi R S \sqrt{2}}{60}$ ft.

$\therefore$ units of work in the disc $= \frac{W}{2g} \times (\text{vel.})^2 = \frac{\pi^2 S^2 R^2 \, . \, W}{g \times 60^2}$,

being the work due to the height fallen through to acquire the given velocity.

Circumference of the axis $= \pi r$, and

units of work destroyed by friction in 1 revolu. $= \pi r \times \frac{W}{n}$.

Put N = No. of revolutions before the disc stops; then

$\pi r \times \frac{W}{n} \times N =$ units of work destroyed by friction.

$$\therefore \frac{\pi^2 S^2 R^2 W}{60^2 g} = \pi r \times \frac{W}{n} \times N, \text{ whence } N = \frac{r \pi S^2 R^2}{60^2 g r}.$$

Also, if s = given number of revolutions per second, then $S = 60\,s$, and by substitution

$$N = \frac{n \pi s^2 R^2}{g r}.$$

NOTE.—These formulæ are independant of the weight of the disc, as they obviously ought to be.

Ex.—A disc of metal is whirled round with a velocity of 88 feet per second, being the speed of the wheels of a railway train moving at the rate of 60 miles per hour, the radius of the disc is 5 feet, the radius of its axle 2 inches, and the coefficient of friction $\frac{1}{10}$, or $n = 10$; required the number of revolutions which the disc will make before it stops.

By Art. 288,

$$\text{No. of revo.} = \frac{n \pi s^2 R^2}{g r} = \frac{10 \times 3{\cdot}1416 \times 88^2 \times 5^2}{32\frac{1}{6} \times \frac{2}{12}} = 1134515.$$

THE FLY WHEEL.

290. When a moving power is supplied irregularly, as by the piston of a steam engine, the action of which is intermitting or by impulses, while various machines moved by this important power require a regular force, the method of regulating the motion of such machines is by means of a fly wheel, in which a ponderous mass of metal, revolving freely on an axis, is connected with the machinery, and by its inertia produces a resevoir as well as a regulator of force; since a small surplus of force acting for a short time will accumulate a considerable power in the fly wheel, and this power being applied suddenly for a short time is capable of supplying the short intermissions of the moving power, and producing a near approximation to perfect regularity in the motion of the machinery.

291. **Prob. 7.**—The weight of a fly-wheel = W lbs., the external and internal radii of the rim are R and r feet; the wheel makes s revolutions per second, the diameter of the axis is d inches, and the friction upon $\frac{1}{n}$ of the whole weight of the wheel; required the units of work in the wheel, and the number of revolutions which it will make before it stops, the inertia of the axle and spokes of the wheel being neglected as not materially affecting the result.

Then

$$\text{The radius of gyration } k = \sqrt{\frac{R^2 + r^2}{2}}.$$

$$\text{Units of work in the wheel} = s^2\left(2\pi\sqrt{\frac{R^2+r^2}{2}}\right)^2 \times \frac{W}{2g} = \frac{\pi^2 s^2 W}{g}\left(R^2 + r^2\right).$$

$$\text{Circum. of axis} = \frac{\pi d}{12}.$$

$$\text{Work destroyed by friction in 1 revo.} = \frac{\pi d}{12} \times \frac{W}{n} = \frac{\pi g W}{12 n}.$$

Put N = number of revolutions made by the wheel before it stops; then

$$\text{Whole work destroyed by friction} = \frac{\pi d W}{12 n} \times N,$$

$$\therefore \frac{\pi d W}{12 n} \times N = \frac{\pi^2 s^2 W (R^2 + r^2)}{g};$$

$$\text{whence} \quad N = \frac{12 n \pi s^2 (R + r^2)}{g d}.$$

Note.—This result is also independent of the weight of wheel.

Ex.—The external and internal radii of the rim of a fly wheel are 5 and 3 feet; it makes 3 revolutions per second; the diameter of the axle is 2 inches, and the friction upon it $\frac{1}{10}$ of the weight of the wheel, or $n = 10$; how many revolutions will it make before it stops?

$$\text{The No. of revo. N} = \frac{12 \times 10 \times 3{\cdot}1416 \times 3^2 \times (5^2+3^2)}{32\frac{1}{6} \times 2} = 1797.$$

292. Prob. 8.—*When the force acts in one direction only, to find the limits within which the angular velocity of the fly-wheel varies.*

Let A R Q S be the fly wheel, O its centre, O R a crank, on which the rod P R acts in directions parallel to the diameter E F; and let P be a constant force acting on the rod PR; also, let Q be a weight equivalent to the resistance to the motion of the machine, and acting perpendicularly at Q the extremity of the radius OQ. Draw R H, rh perpendicular to E F and indefinitely near to each other; also, put RO $= r$, QO $=$ R, and $\pi =$ circumference to rad. $= 1$. Now let the point R move through the indefinitely small space Rr; then the force of P is measured by the product of the resolved part Rn of the force P and the small space Rr, which product is evidently $= \mathrm{P} \times \mathrm{R}r = \mathrm{P} \times \mathrm{H}h$, since the force nr is wholly ineffectual. Let the force P act from E to F, then Hh becomes $=$ the diameter E F, and the whole force from E to F will be $= \mathrm{P} \times \mathrm{EF} = \mathrm{P} \times 2r$; and the whole dynamical effect of the resistance of the machine in an entire revolution, which is represented by Q, will be equal $\mathrm{Q} \times 2\pi\mathrm{Q}$; but, since the whole effect of the force P is consumed by the useful and useless resistances of the machinery taken together, there results.

$$2\mathrm{P}r = 2\mathrm{Q}\pi\mathrm{R},$$

$$\text{or} \quad \mathrm{P} = \frac{\mathrm{Q}\pi\mathrm{R}}{r}.$$

Now, let the resistance Q just balance the force P, when the crank is in the two positions O R, O S; and put the angle R O A $= \frac{1}{2}$ the angle R O S $= \phi$; then,

$$\mathrm{P}r \cos\phi = \mathrm{QR};$$

and by substituting the value of P in this equation, there results, after reduction,

$$\cos \phi = \frac{1}{\pi} = \frac{1}{3 \cdot 1416} = \cdot 3183;$$

$$\therefore \phi = 71^\circ 26',$$

$$\text{and arc } R A S = 2 \phi = 142^\circ 52'.$$

293. For the double acting engine we find in a similar manner.

$$\cos \phi = \frac{2}{\pi} = \frac{2}{3 \cdot 1416} = \cdot 6366.$$

$$\therefore \text{arc } R S = 2 \phi = 100^\circ 54'.$$

294. **Note.—As further investigation on this subject would necessarily involve a detailed exposition of the theory of the steam engine, which is apart from the object of this work; therefore, the reader, who desires efficient information on this subject, may consult *Tredgold on the Steam Engine, Moseley's Engineering, Hann's Treatise on the Steam Engine.***

THE GOVERNOR.

295. Prob. 9.—*To explain the use and principle of the governor of the steam-engine, and to find the position which its balls assume in consequence of the centrifugal force, the angular velocity being given.*

A B is a vertical shaft turning freely on the sole B by its connection with the machinery of the steam engine; CP, CQ are two bars moving freely on the centre C, and carrying the two weights P, Q; F D, F E are two rods connected to the bars at D, E, and attached to a collar I, which is capable of sliding freely up and down the shaft AB. This collar is united with a lever which opens or closes the throttle valve, which supplies the cylinder with steam. When

AB revolves too fast, the balls by their centrifugal force fly outward, raising the slide I, and partially closing the throttle valve; and when the shaft moves too slowly the balls collapse, and the slide consequently descending, admits a more

full supply of steam, thus regulating the motion of the engine to almost complete uniformity.

The weight P is acted on by two forces, *i. e.*, the centrifugal force in the direction P *n*, and gravity in the direction P *o*; and taking P *n*, P *o* to represent these two forces, complete the parallelogram P *n m o*, we shall have the triangles P *m o*, A P O similar, and, if *f* represent the centrifugal force, and W the weight of P, then

$$\frac{f}{W} = \frac{PO}{CO}.$$

Also, by Art. 277,

$$f = \frac{W.V^2}{g.PO},$$

and by substituting this value in the preceding equation, there results after reduction

$$CO = \frac{g.PO^2}{V^2},$$

or, if v = the angular velocity per second of the governor at an unit's distance from the shaft A B, then $V = PO.v$, and by substitution,

$$CO = \frac{g}{v^2}.$$

Now, if n = number of revolutions per minute, then $\frac{n}{60}$ = number per second.

$$\therefore\ v = \frac{2\pi n}{60} = \frac{\pi n}{30}.$$

$$\text{whence}\quad CO = \frac{32^2 . g}{\pi^2 n^2}\ \text{in feet.}$$

$$= \frac{30^2 \times 32\frac{1}{6} \times 12}{(3{\cdot}1416)^2 n^2}\ \text{in inches.}$$

$$= \frac{35200}{n^2}\ \text{inches nearly.}$$

296. The throttle valve of the steam engine cannot be opened without an adequate force exerted by the governor, which may be measured by finding what weight will produce

that force. Let p be the required weight, W being the weight of one of the balls of the governor, as in the last article; then *Hann* has shewn in his valuable works on the steam engine, referred to in the Note Art. 294, that

$$\frac{p}{\mathrm{W}} = \frac{21}{100} \cdot \frac{\mathrm{C\,P}}{\mathrm{C\,D}},$$

and that, if $\frac{\mathrm{C\,P}}{\mathrm{C\,D}} = \frac{3}{2}$, which is the usual proportion in the governor,

$$\frac{p}{\mathrm{W}} = \frac{21}{100} \cdot \frac{3}{2} = \frac{63}{200};$$

$$\therefore\ \mathrm{W} = \frac{200\,p}{68} = 3{\cdot}174\,p,$$

and if $p = 10$ lbs., then $\mathrm{P} = 31\frac{3}{4}$ lbs. nearly.

THE SUPER-ELEVATION OF THE EXTERIOR RAIL IN RAILWAY CURVES.

297. The super-elevation of the exterior rail, or the rail on the convex side of the line, in railway curves, the radii of which are within certain limits, is rendered absolutely necessary to counteract the centrifugal force produced by the velocity of the train, since all moving bodies have a tendency to continue their motion in a direct line. From this cause the railway train is impelled towards the exterior rail, and would finally leave the rails, were it not prevented by the conical inclination of the tire and the flanges of the wheels.

298. PROP.—To determine the centrifugal force of a railway train, or that portion of the weight of the train, which makes it tend to leave the curve.

Let V = velocity of the train per second, R = radius of the curve, F = centrifugal force, and g force of gravity at the earth's surface, also let W = weight of the train; then by Art. 277,

$$\mathrm{F} = \frac{\mathrm{W\,V}^2}{g\,\mathrm{R}}.$$

Ex. 1.—When $\mathrm{R} = \frac{1}{2}$ a mile $= 2640$ feet, V = velocity = 30 miles per hour = 44 feet per second, and $g = 32\frac{1}{6}$ feet

= velocity of a body falling from rest, at the end of a second; then

$$F = \frac{W \times 44^2}{32\frac{1}{6} \times 2640} = \frac{22}{9651} W = \text{nearly } \tfrac{1}{44} W,$$

that is, the force that urges the train to quit the curve is $\frac{1}{44}$ of its whole weight, in this case.

Ex. 2.—When V = 60 miles per hour = 88 feet per second, and R the same as in Example 1; then

$$F = \frac{W \times 88^2}{32\frac{1}{6} \times 2640} = \text{nearly } \tfrac{1}{11} W;$$

that is the force, in this case, is $\frac{1}{11}$ of the weight of the train. Hence it may be perceived how extremely dangerous high velocities are in curves of small radius.

299. NOTE.—This great amount of centrifugal force, in curves of small radius, would be very much increased by the high velocities, which some are sanguine enough to expect as likely to be attained on railways; since this force varies as

$$\frac{V^2}{R} \text{ or as } V^2$$

for the same curve: thus for a velocity of 120 miles per hour, on a curve of $\frac{1}{2}$ of a mile radius, we shall have

$$f = \frac{W \times 176^2}{32\frac{1}{6} \times 2640} = \tfrac{4}{11} W,$$

that is, the centrifugal force is, in this case, more than $\frac{1}{3}$ of the whole weight of the train; while for curves of 1 mile radius, which are very common in railways, $f = \frac{2}{11}$ W, or nearly $\frac{1}{5}$ of the weight of the train. It must, therefore, be evident that a velocity of 120 miles per hour, or even one of 90 miles per hour, must be extremely dangerous, especially on an embanked curve, should any accident throw the train off the line, which is often the case with the present velocities. Moreover, the resistance of the air, which varies as V^2, must be considerably augmented by high winds opposed to the direction of a train of these great velocities; while its engine would require a power greatly superior to those now in use.

300. This force, except in curves of very small radius, is counteracted by the conical inclination of the tire of the wheels, each pair of which is firmly fixed on the axle which turns with them; the inclination of the tire is commonly about $\frac{1}{2}$ an inch in the whole breadth of the wheel, which is $3\frac{1}{2}$ inches. This inclination of the tire with the lateral play of the flanges of the two wheels of $\frac{1}{2}$ an inch on each side, and the centrifugal force urging the train towards the exterior

rail, when moving in a curve, increase the diameter of the outer wheel, and diminish that of the inner one, which causes the train to roll on conical surfaces, thus necessarily producing a centripetal force to counteract the tendency of the train to leave the curve. However, in curves of very small radius, the centrifugal force is not sufficiently counteracted by the centripetal force thus generated, the centre of which last named force is the vertex of the cone, of which the increased and diminished diameters of the wheels are sections. The amount, therefore, of this centripetal force shall be determined in the following—

301. Prop.—*The velocity of the train, the guage of the rails, the radius of the wheels, and the inclination of their tire being given, to determine the centripetal force generated by the conical inclination of the tire of the wheels of the train, and by the centrifugal force impelling the train outwards.*

Let d = mean diameter of the wheels of the train, δ = increment and consequently the decrement which the diameters of the exterior and interior wheels respectively receive, through the conjoined action of the centrifugal force and the inclination of the tire; then under these circumstances the respective diameters of the exterior and interior wheels will be

$$d + \delta \text{ and } d - \delta;$$

also, if R' = radius of a circle which the centre of a carriage would describe in consequence of the inclination of the tire of the wheels, and b = breadth of the road or guage of the rails; then $R' + \frac{1}{2} b$, and $R - \frac{1}{2} b$ are radii which would be described respectively by the exterior and interior wheels; and by similar triangles,

$$d + \delta : d - \delta :: R' + \tfrac{1}{2} b : R' - \tfrac{1}{2} b,$$

whence $d : \delta :: 2R' : b$, and

$$R' = \frac{b\,d}{2\,\delta}.$$

Or, if $\frac{1}{n}$ = inclination of the tire, and Δ = deviation of the wheels, then

$$\delta = \frac{2\Delta}{n},$$

and, by substitution,

$$R' = \frac{b\,d\,n}{4\,\Delta}.$$

Now V and W representing the velocity and weight of the train, as in Art. 298, the centripetal force corresponding to the radius R′ will be

$$F' = \frac{W\,V^2}{g\,R'},$$

or, by substituting the value of R′,

$$F' = \frac{4\,W\,V^2\,\Delta}{b\,d\,g\,n}.$$

302. Prop.—*To determine the deviation of the wheels, and the radius of the curve, when the centrifugal and centripetal forces, in* Art. 298 and 301, *just balance each other.*

Because the forces F and F′ act in contrary directions, they will hold each other in equilibrium when they become equal, and the train will cease to have a tendency to quit the curve; this will take place when

$$\frac{W\,V^2}{g\,R} = \frac{W\,V^2}{g\,R'},$$

$$\text{or} \quad R = R'.$$

Also, by Art. 298 and 301.

$$\frac{W\,V^2}{g\,R} = \frac{4\,W\,V^2\,\Delta}{b\,d\,g\,n};$$

$$\text{whence } \Delta = \frac{b\,d\,n}{4\,R};$$

which is the deviation requisite to produce an equilibrium between the centripetal and centrifugal forces of the train. And, since $R = R'$, the vertex of the imaginary cone, of which the increased and diminished diameters of the wheels are sections, will coincide with the centre of the curve, there will consequently be no dragging on the wheel on either of the rails.

If, in $R' = \frac{b\,d\,n}{4\,\Delta}$, $d = 3$ feet, $b = 4$ feet $8\frac{1}{2}$ inches $=$

4·7 feet = breadth of the narrow guage, $\frac{1}{n} = \frac{1}{7}$, and $\Delta = \frac{1}{3}$ of an inch, the radius of curvature corresponding to this deviation, when the two forces are in equilibrium, will be

$$R' = \frac{b\,d\,n}{4\,\Delta} = 4.7 \times 3 \times 7 \div 4 \times \frac{1}{3} \times \frac{1}{12} = 888 \text{ feet.}$$

But, since an accidental depression of the exterior rail might cause the flange of the wheel to rub the rail on that side; it would be advisable, for the sake of greater safety, to limit the value of R′ to not less than 1200 or 1500 feet. Moreover, in curves of less than 1500 feet radius, it will at once appear that a super-elevation of the exterior rail will be absolutely necessary to counteract the excess of the centrifugal above the centripetal force.

303. PROP.—*To determine the super-elevation of the exterior rail in railway curves of less than* 1200 *or* 1500 *feet radius; the same things being given as in the preceding proposition.*

Let x = super-elevation of the exterior rail; then, since b = breadth of the way, the inclination of the plane on which the train moves $= \frac{x}{b}$ to rad. = 1, and hence the gravity of the train will impel it to the interior rail with the force

$$F'' = \frac{W\,x}{b}.$$

This force, together with the centrifugal force, resulting from the deviation of the train to exterior rail of the curve, must hold the centrifugal force in equilibrium; therefore, from Articles 298 and 301, there will result

$$\frac{W\,x}{b} + \frac{W\,V^2}{g\,R'} = \frac{W\,V^2}{g\,R};$$

$$\text{whence} \quad x = \frac{b\,V^2}{g}\left(\frac{1}{R} - \frac{1}{R'}\right),$$

which is the formula for the super-elevation of the exterior rail, and due to *Pambour;* who, by solving it for some of the usual cases, produces the following

TABLE OF THE SUPER-ELEVATION TO BE GIVEN TO THE EXTERIOR RAIL IN CURVES.

Designation of the Waggons and the Way.	Radius of the Curve in Feet.	Super-elevation to be given to the Rail in Inches, the Velocity of the motion in Miles per hour being:—		
		10 Miles.	20 Miles.	30 Miles.
Waggon with wheels 3 feet in diameter. Guage of way, 4·7 feet. Play of the waggons on the way, 1 inch. Inclination of the tire of the wheels, 1 in 7.	250	1·14	5·60	12·90
	500	0·57	2·83	6·56
	1000	0·29	1·43	3·30
	2000	0·15	0·71	1·65
	3000	0·10	0·47	1·10
	4000	0 07	0·36	0·83
	5000	0·06	0·28	0·66

The corectness of the above results is pretty generally conceded. It must, however, be considered, that it is extremely difficult, if not impossible, to realize in practice, the precise conditions and proportions determined by these important formulæ; as accidental depressions and enlargements of guage of part of the rails, as well as many other matters that cannot be subjected to calculation, will unavoidably derange these results.

The reader, who wishes for further information on these subjects, may consult *Tredgold on the Steam Engine; also, Baker's Railway Engineering, and his Land and Engineering; in which approved and Practical Systems of laying out the works of Railways, &c., &c., will be found.*

MISCELLANEOUS EXERCISES.

(I.) If a body move in a curve by means of a projectile and centripetal force, the latter acting in the direction of right ordinates, and varying inversely as the nth power of the distance from the obscissa or axis of the curve; prove that the velocity V of the body in the curve is equal to

$$\frac{d\,z\,(m\,y^{-n})^{\frac{1}{2}}}{d^2\,y};$$

in which y is the ordinate indicating the position of the body, z the curve, and m the force of gravity at a unit's distance from the axis.

NOTE.—This question and its solution was published by the Author in the

Gentleman's Math. Companion for 1823–4.—Its solution is on a new principle, as he has not seen a similar method of solution adopted by any other author; and the method may be obviously extended to the motions of bodies in curves, when acted upon by central forces as well as by parallel forces, as in the Question. The solution is also given below.

Let AP be the curve described by the body, A M its axis, PM, RQN two ordinates indeffinitely near to each other, and P R a tangent to the curve at P. The body at P would describe P R if the centripetal force did not act, in the same time in which it now describes P Q; therefore RQ represents the space through which it is drawn by the centripetal force. Now $PR = dz$, $RQ = \frac{1}{2} d^2 y$ (*Dealtry's Fluxional Calculus*, Art. 165.), and the force of gravity at $P = m y^{-n}$ ultimately = force of gravity at Q. The motion through the indefinitely small space $RQ = \frac{1}{2} d^2 y$ may be considered to be uniformly accelerated by the force $m y^{-n}$; therefore the time of describing R Q = time of describing $PR = \left(\frac{d^2 y}{m y^{-n}}\right)^{\frac{1}{2}}$; hence, because the motion in in the direction P R is uniform, the velocity V of the body in the curve = P R ÷ time of describing P R, that is,

$$V = dz \div \left(\frac{d^2 y}{m y^{-n}}\right) = \frac{dz}{d^2 y} (m y^{-n})^{\frac{1}{2}}.$$

(II.) The equal and uniform bars A B, B C are moveable about each other on an axis passing through B, which is perpendicular to the plane A B C, while a pin at the end A, of the bar B A, if moveable in a vertical groove A C; and the end C, of BC, is moveable about an axis passing through C, and perpendicular to the plane A B C; it is required to find the velocity of A, and the action on the groove when A arrives at a given position in consequence of gravity and its original motion.

NOTE.—This question was proposed and answered by *B. Gompertz, Esq., F.R.S., &c.* in the *Gentleman's Math. Companion, for* 1821–2.

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