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ELEMENTARY ARITHMETIC

BY

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"A NEW GENERAL THEORY OF THE TEETH
OF WHEELS," ETC.

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P R E F A C E.

THE following Treatise on Elementary Arithmetic has been designed as the first of a continuous series of Treatises on those sciences which are usually comprehended under the somewhat indistinct name MATHEMATICS.

During a long experience as preceptor in the higher departments of mathematics, the Author of this work has observed that almost all the difficulties which the student encounters are traceable to an imperfect acquaintance with arithmetic. It seems as if this subject were never regarded as having in it anything intellectual. Arithmetic is considered as a kind of legerdemain, a talsamic contrivance, by means of which results are to be obtained in some occult manner, into the nature of which the student is forbidden to inquire ; hence many a one, as he advances in life, finds himself compelled to resume the study of the principles of arithmetic, and discovers that all along he has been working in the dark. Now, truly, there is hardly any branch of human knowledge which affords more scope for intellectual effort, or presents a more invigorating field for mental exercise than the science of number. It has therefore been the Author's aim to pre-

pare a text-book which should call mind, not memory, into exercise, and from which all mere dogmatism should be scrupulously excluded.

In arranging it, he has endeavoured to explain the reasons of the various operations by an examination of the nature of the questions which give rise to them, and in this manner has sought to prepare the student for understanding their applications to the higher branches of science. The gradual formation of systematic numeration has been traced, and the operations of palpable arithmetic have been taught in order to reflect a clearer light on the true nature of our figurate processes. The few lines that are devoted to the explanation of the ancient Greek and Arabic Notations may not be unacceptable to those who study the history and phenomena of the human mind. We are accustomed to designate the ordinary numerals as Arabic. The Arabs themselves call them *Rakam Hindi* in contradistinction to those described at page 23, which they call *Rakam Arabi*.

It is hoped that, independently of the arrangement of its parts, this treatise contains enough of new and original matter to prevent it from being regarded as an uncalled-for addition to a class of books already sufficiently numerous. The method of computing from the left hand, which has been practised and taught by the author for thirty years, is now published for the first time. Besides the merit of novelty, this method has the higher merit of great usefulness. A very slight acquaintance with it augments one's power over numbers in an unexpected degree, and the continued practice of it renders

computation a pastime. In the ordinary mode of computing we have never occasion to add more than nine to nine, or to take the product of more than nine times nine, and hence the limit of rapidity is soon reached. But when we begin to work from the left hand, every operation adds to our previous experience, and we soon become familiar with large numbers, so much so that the rapidity of mental calculation comes far to exceed the swiftness of the pen.

The multiplication of one large number by another by help of a movable slip of paper, though not to be recommended in actual business, is interesting, and the ability to perform it enables us to follow the operations for shortening the multiplication and division of long decimals.

The subject of prime and composite numbers has been introduced as preparatory to the theory of fractions ; and the doctrine of proportion has been reached by means of the method of continued fractions invented by Lord Brouncker. The immense power of this method, the almost unlimited range of its applications, as well as the beautiful simplicity of the idea from which it arose, recommend it to the close attention of every calculator. The doctrine of continued fractions is here divested of its technicalities, and presented in such a form as to be intelligible to beginners ; the formation of the successive approximating fractions being deduced without the aid of artificial symbols, perhaps more clearly than with that aid.

Among the minor improvements which have been in-

troduced, the mode of obtaining at once the continued product of several numbers, the plan for shortening division by a number of two or three places, and the arrangement of the work for finding the greatest common divisor of two numbers, may be mentioned ; Fourier's *Division Ordonnée* is also new to the English reader.

It was a matter of anxious deliberation whether the answers to the questions should be printed, and ultimately it was resolved to give these answers in a separate key. The Author, however, most earnestly entreats the student to trust to his own thorough comprehension of the matter, and to the care with which he works. Let him carry the feeling with him, that if his result differ from that given in the key, the key is likely to be wrong : above all things, cultivate self-reliance. In very many cases the manner in which the result is worked out is of more importance than the mere obtaining of it ; and in some cases, for obvious reasons, the answers ought not to be given at all.

The Author is in hopes of speedily laying before the public, in continuation, a treatise on the higher arithmetic, in which the doctrines of Powers, Roots, and Logarithms, are completely investigated.

EDINBURGH, *June* 1856.

INTRODUCTION.

LOGISTICS, or the science of number, is divided into several branches, according to the manner in which we represent numbers, or to the uses which we make of their symbols.

The most important of these branches is that which is known under the name of the Theory of Numbers. Its object is to investigate the laws which govern the relations of one number to another ; and, if we were to arrange a set of treatises in true logical order, one on this subject would necessarily be the first.

However, until we become familiar with numbers, it is impossible for us to comprehend any even of their simplest properties ; and it would be as vain to begin a set of mathematics with this science, as it would be to attempt to teach an infant its mother tongue by help of grammatical rules : in both cases, the language needed for conveying our instructions would be wanting.

Hence it is that we are forced to begin with that branch which is called ARITHMETIC, even although the simplest arithmetical operations presuppose a knowledge of recondite principles ; just as a child must talk, and talk well, before he can perceive those laws of grammar which have regulated his language.

Arithmetic teaches us how to represent numbers by definite characters, and how to employ these characters in various operations.

The science of Number is, strictly speaking, the science of existence. Geometry treats of the sizes and shapes of bodies ; mechanics of their weights and motions ; optics of their colours ; and other sciences treat of other qualities of external objects. All our knowledge of these qualities is obtained through our organs of sensation ; and it is clear that when the mind receives the idea of the quality of any object, the idea of the existence of that object must have been impressed upon it. Whether an object send us through the eye the information " I am green," or through the muscles the information " I am heavy," the primary idea communicated must be, " I am," " I exist."

Thus, with whatever senses we might have been gifted, whether some of those which we possess had been denied us, or whether others depending upon electricity, actinism, or some to us unknown quality of matter, had been bestowed upon us, the science of Existence would still have been the first of all the sciences. It forms the basis of all accurate knowledge ; it intermingles with and pervades all science ; and, whether we direct our attention to the most elementary or to the most exalted departments of study, we find the doctrine of numbers aiding us in our simplest lessons, or transcending our powers in the most complicated researches.

Our knowledge of the properties and relations of numbers, like that of the qualities of bodies, is derived altogether from experience ; and hence that branch of science which teaches us to represent and to operate upon numbers, is necessarily the foundation of all the other branches. It is true that the very outset in arithmetic introduces the doctrine of powers, and that even the ordinary nomenclature presupposes an acquaintance with a variety of general propositions strictly belonging to the more advanced portion of the subject ; and that arithmetic may thence be argued to be not the foundation of, but built upon other departments of logistics. But it must be kept in mind that science is not an aggregate of the properties of objects considered in themselves : it is the aspect of those properties as perceived by the human mind. That branch of science is to us

the most elementary which is the most easily understood, not that from which other branches might be deduced by the exercise of a matured logic.

In arranging a scientific treatise, we may follow the natural relation of the truths to be expounded, or we may be guided by the gradual development of the mind of the pupil. Each of these courses has its inconvenience. It is almost always the case, that the perception of those truths which are really elementary requires a thorough knowledge of the science and great experience in logical deduction: thus the truth that a times b gives the same product as b times a , is admitted, and used at the very outset of our arithmetical lessons; but Legendre, in his admirable *Théorie des Nombres*, considers it necessary to establish this proposition by a logical deduction from still more elementary truths: yet any one will concede that an attempt to place Legendre's demonstration at the beginning of a book on arithmetic, would be to place an extinguisher on the zeal and hopes of the pupil. We are thus forced to deviate from the strictly logical arrangement, and to suit our proceedings to the powers and aptitudes of the human mind. Yet it would be vain to make the gradually opening powers of the student our sole guide in arranging a set of lessons. The capabilities of the individual, the age at which he has begun, nay, even the ulterior views which have induced him to undertake the study, would need to be considered, and thus not one only, but many textbooks would be required: the arithmetic for an advanced student would be very different from the arithmetic for a beginner, and much repetition would ensue.

In the following work, I have endeavoured to combine both methods, by arranging the different divisions of the subject nearly in their natural order, and by marking with the successive letters of the alphabet those sections which a student should peruse in the various stages of his advancement. Thus a beginner may confine his attention to those paragraphs which are marked with the letter **A**, and which, by themselves, form a consistent and very elementary set of lessons. Having mastered

these, he may resume the study, taking along with him sections **A** and **B**, and then may again repossess the treatise, until he shall have thoroughly understood it all. Throughout the whole of the work, I have given full—perhaps, in the judgment of some, too full—explanations of the objects and reasons of the various processes, and have most carefully avoided teaching by rule ; and I would here most earnestly impress upon the attention of all who have charge of the education of the young, the danger of instructing by help of rules committed to memory. Perhaps no greater misfortune can befall a young person than that of being taught by rote ; for thereby the free exercise of his intellect is effectually cramped, and the most important object of education, the outleading of the judgment, entirely thwarted.

ELEMENTARY ARITHMETIC.

WHEN we contemplate several objects merely in reference to their existence, we obtain the idea of number. Thus, if we regard the fingers on one hand, we obtain the idea of the number *five*; or, if we put down as many apples, we get the same idea; or, even if we put before us objects of different kinds—as an apple, an orange, a nut, a shell, and a stone—we still, when we pay no attention to the nature of the things, but merely consider their being there, have the idea of the same number five. The idea of number is therefore said to be *abstract*, because all reference to the qualities of the objects is excluded from it.

Before we have proceeded a step in the study of this department of human knowledge, the necessity of assigning names to the various numbers is forced upon us, and so the first section of this science must be the *Nomenclature* and *Notation* of numbers. This section of logistics receives the name Arithmetic (from the Greek *αριθμος*, number). It treats of the various methods of representing numbers by words or by symbols, and of the uses to which these symbols may be applied.

When a great multitude of objects is presented to us, we cannot easily form an idea of the number. If the objects be few, as, for example, the fingers on the hand, we readily form an idea of the number, and can assign a name to it; but if a bag of nuts, or still worse, a sack of corn, were before us, it would

be an arduous task to count out the individuals, and assign a name to each number as we proceed. It is some time before a child learns to repeat the names of the smaller numbers, but no memory would ever be able to retain such a host of words as would be needed to tale out the grains in a sack of corn. In order to overcome this difficulty, recourse is had to a very simple and, at the same time, a very beautiful scheme. This scheme is to arrange numbers in groups, and these groups again in larger groups, and so on as far as may be needed.

Different nations, and even different classes in the same nation, have adopted different groups. Thus, some count by *dozens*, some by *scores*, others by *sixties*; but the plan of counting by *tens* is gradually displacing all others, and is adopted in every language; therefore I shall, in the first place, treat fully of this system, and afterwards notice the more important of the others.

CHAPTER I.

ON DECIMAL NUMERATION AND NOTATION.

A. In the ordinary system of counting, separate names are given to numbers as far as *ten*, which is the number of fingers on both hands. This is called the *decimal* system, from the Latin word *decem* (dekem), or the Greek *δέκα*, ten. The names of the higher numbers are compounded of these ten names, and of a few others.

B. This system is very ancient, as is evident from the construction of the Latin, Greek, Arab, and other languages. But our own language, and most of those in the north and west of Europe, bear traces of the modes of reckoning by *dozens* and *scores*. Remains of both of these modes are to be observed in the division of the foot into twelve inches, of the shilling into twelve pennies, and of the pound into twenty shillings.

A. The names of the earlier numbers in English are, *one, two, three, four, five, six, seven, eight, nine, ten*; after this we begin to count *ten and one, ten and two, ten and three, ten and four*; and so on. Instead, however, of *ten and one*, we use the old name *eleven*, and instead of *ten and two*, *twelve*; while the others are shortened into *thirteen, fourteen*, up to *nineteen*. After *nineteen*, or *ten and nine*, comes *ten and ten*, or two tens, and this name is contracted into *twenty*, probably from the word *twain*, a pair. We then go on, *twenty-one, twenty-two*, up to *twenty-nine*. The next number is *twenty and ten*, or three tens, which is contracted into *thirty*; after this we get to *forty, fifty, sixty, seventy, eighty*, and *ninety*.

B. Thus any number up to *ninety-nine* is named by grouping the objects in *tens*, naming the number of the groups and the number of single counters or units over: thus, *eighty-seven* means eight groups of ten each, and seven units more. The English names suit this system, with the exception of the words *eleven* and *twelve*, and with this irregularity that we say *fifteen* (that is, five and ten) when it would have been more in keeping to have said *ten and five*. In old books we often meet with such expressions as *five-and-forty*, but the custom is now to name the larger part of the number first. In the German language the old arrangement is still in use; thus we have *vier und zwanzig* (four-and-twenty), and so on all the way.

C. The French language retains more traces of the old ways of counting than the English; thus we have *onze, douze, treize, quatorze, quinze, seize*, and only then do we fall into the regular *dix-sept, dix-huit, dix-neuf*. In the provincial language the irregularities cease there; for afterwards we have *vingt, trente, quarante, cinquante, soixante, septante, octante, nonante*; corresponding exactly to our *twenty, thirty—ninety*. But, singularly enough, in the polite language we have such barbarisms as *soixante-seize* (literally sixty and seventeen), *quatre-vingt dix-neuf* (four score ten and nine), still in use in preference to the simple *septante-six, nonante neuf* of the country people. Our older authors clung long to the counting by scores; we find in our translation of the Bible *three score and ten* for seventy, *four score* for eighty.

The Arabs, in naming a number, invariably put the smaller part first, so that our words *thirteen, fourteen*, and the old *three-and-fifty*, are in the Arab style.

In the Tatar languages the system of numeration is perfectly regular throughout, there not existing in them a trace of the use of any other than the decimal system; but, singularly enough, the names for *twenty, thirty, forty*, and *fifty*, are not derived from those for two, three, four, five, although the names for *sixty, seventy, eighty*, and *ninety*, be modifications of those for *six, seven, eight*, and *nine*.

All these irregularities serve as indications of the gradual formation of numerical language.

A. Having now got up to ninety-nine, the next number would be ten-tens (or tenty, as little children will often have it). To this number a special name, *hundred*, is given; and we go on, one hundred and one, hundred and two, hundred and sixty-three, hundred and ninety-nine; then two hundred, three hundred, up to ten hundred, which receives the new name, *thousand*.

In this way, by help of a very few words, we are able to give distinct names to very large numbers, and to recollect their order. Young people should accustom themselves to count large numbers; it is only by actually counting that we can form even rough ideas of what is meant by one thousand, ten thousand, and such large numbers. Such exercises as the following should be done by beginners:—

1. How many nuts are in a pound?
2. How many grains of rice in a wine-glassful?
3. How many peas fill a quart measure?
4. Count out four hundred and seventy-three pebbles.
5. Tale out one thousand grains of wheat.

As it is very tiresome to count them out one by one, the learner soon finds it more convenient to reckon them in fives, or even in tens, at a time; and by exercising himself in this way, he becomes apt to understand other abbreviations.

B. The power of this mode of counting does not come from any peculiar virtue in the number *ten*, which is the root of the scale, but from the simplicity and uniformity of the plan.

The decimal system is followed in all European languages, and the higher names are formed according to a uniform system, until we come to one thousand thousands, which, in all languages that borrow from the Latin, is called a *million*; and even to one thousand millions the same nomenclature is used; but there

a divergence occurs, for the French call a thousand millions a *billion*, while the English give the name *billion* to a million of millions. This difference in the way of using the words may occasion mistake. The subjoined table exhibits the corresponding names in the two languages :—

ENGLISH.	FRENCH.
Thousand.	Mille.
Million.	Million.
Thousand Millions.	Billion.
Billion.	Trillion.
Thousand Billions.	Quadrillion.
Trillion.	Quintillion.
Thousand Trillions.	Sexillion.
Quadrillion.	Septillion.
&c.	&c.

But these are numbers so far out of the ordinary range of business, that very little inconvenience can ever be felt from the dissimilarity in the names.

In the course of commercial transactions we need to mark down, as well as to name numbers : and as we have often to deal with those who speak another language, it is desirable to have a system of marks which may be understood by all. When dealings are not very extensive, it is easy to make the required number of notches upon a stick, or to put the requisite number of pebbles into a bag. These methods are tedious when applied to extensive business, and improvements have been introduced which have led to the universal adoption of the Indian numerals. I shall endeavour to explain the nature of the successive steps, by following, not strictly their historical succession, but by showing how one step might easily have led to another.

A. Suppose that we have to reckon some enormous number, such as the number of grains on the floor of a granary, and that we are required to do it without the error of a single grain. At first we go on, *one, two, three, four*, but soon we get fatigued, lose count, and have to begin again : a few trials convince us that we must fall upon some other plan. The eye gets accustomed to a group of *five*, and two such groups make *ten* ; let us then begin to tale out the grains, ten at a time, and to count these groups *one, two, three*. In this way we get much more rapidly over our work ; but soon the number of tens even is so great that we lose count. We therefore heap up ten of these tens in a little parcel of one hundred each, intending to keep account of these hundreds. Soon, however, these heaps get mixed, and we see that, though we were to amass the grains in thousands, it would be difficult to keep the groups distinct.

Instead of trying to go on in this way, which, though very good for a small number, is quite useless for such a number as we are looking to, let us contrive some other plan.

Having procured a great quantity of tares, let us, as we go on, put aside a tare for each ten grains, and our work may be carried on for the present without keeping count ; only let us be careful to lay aside one tare for each ten grains. At the last, the enormous heap of wheat would be represented by a great quantity of tares, and probably a few grains over.

We have now to reckon the tares, and even this is an arduous task. We may render it easier by the very same plan. For each ten tares we put some other counter, say a *pea*, so that the pea may stand for ten tens, that is, for one hundred grains. The great quantity of tares will now be represented by several sackfuls of peas, with perhaps a few tares over.

Still the peas are very numerous : for each ten peas, then, let us put a bean, and tale them all out. The result will be a great number of beans, and not more than nine peas over. Proceeding in the same way, let us put an acorn for ten beans, a chesnut for ten acorns, a walnut for ten chesnuts, and then, perhaps, it may not be necessary to continue the system farther.

The fictitious or conventional value of the different seeds would then stand as under :—

Grain,		Unit,
Tare,		Ten,
Pea,		Hundred,
Bean,		Thousand,
Acorn,		Ten thousand,
Chesnut,		Hundred thousand,
Walnut,		Million ;

and if needed, we might continue the same plan, by using other counters.

When the whole of this tedious operation has been completed, we have perhaps *twenty-seven walnuts, two chesnuts, five acorns, no beans, eight peas, nine tares, and one grain*, in which case the number of grains on the floor must have been *twenty-seven million two hundred and fifty thousand eight hundred and ninety-one*.

EXERCISES.

1. Count out and represent by the above marks the grains in a tea-cupful of rice.
2. Count the number of grains in a quart of barley.
3. How many No. 10 shot go to a pound ?
4. How many grains of mustard-seed fill a pint measure ?
5. How many marbles are there in a hatful ?
6. How many apples are there in a peck ?

When we use such counters to stand for various numbers, we soon find that it is convenient to keep the different kinds apart from each other. For this purpose we may procure a shallow box divided into compartments, and assign to each counter its proper place ; and for farther convenience we may mark off a portion of each compartment as a store for those counters which are not in use.

The above-mentioned number would then be represented thus :—

the compartment assigned to chesnuts would be reckoned as a chesnut, and would stand for *one hundred thousand*. Thus we get over the temporary difficulty. We have also got a new idea, for if these borrowed counters can supply the place of others for a time, they may supply their place always. The values of the different counters are fictitious, and we may as well attach a fictitious value to the compartment in which the counter is placed as to the counter which is placed in it. This line of thought leads us to the next great step in the progress of improvement. Let us erase the names of the seeds from the tops of the compartments, and write those of the numbers for which the seeds stand, and then one kind of counters may do for all. After a little practice, even the names of the compartments may be omitted.

By help of a board with shallow grooves, and of a few counters of any kind, we are thus enabled to mark down large numbers, and to operate with them. The beginner in arithmetic should be provided with such an instrument, on which he should learn to perform the simpler operations. Such instruments are in use in China, where they are called *swan-pan*; the ancient Romans gave them the name Abacus ($\alpha\beta\alpha\zeta$), and used small marbles (calculi) as counters: hence comes the English word *to calculate*. Sometimes the counters are beads strung on parallel wires; this form prevents the loss of the counters, but it is not nearly so convenient as the open groove; it is in constant use throughout Russia.

A single example will sufficiently explain the use of the abacus in marking numbers. Let it be proposed to mark the number *eighty-five million seventy-three thousand four hundred and nine*:

○	○	○		○	○	○		○	○
○	○	○		○	○	○		○	○
○	○	○		○		○		○	○
○		○		○				○	○
○		○		○				○	

For this purpose we put *eight* counters in the groove for tens of millions to represent *eighty millions*; *five* counters in the groove for millions; *seven* in the groove for tens of thousands; *three* in that for thousands; *four* in the place for hundreds; and *nine* in the place for units. Here two grooves are empty.

EXERCISES.

A. By help of seeds mark the following numbers.

B. Mark the following numbers on the abacus.

1. Three hundred and sixty-five.
2. Two hundred and nine.
3. Two hundred and ninety.
4. Two hundred and nineteen.
5. Seven thousand five hundred and twenty-six.
6. Nine thousand two hundred and fifty-three.
7. Twenty-five thousand and ninety-one.
8. Five hundred and ninety thousand and three.
9. Nine million three hundred and nine thousand and eleven.
- B. 10.** Seven hundred and sixty-three million five hundred and two thousand one hundred and sixty-one.
11. Eight hundred and ninety-three million and nine hundred and one.
12. Nine hundred and three million.
13. Nine hundred million and three.
- A.** What numbers are expressed by the following seeds?
 1. Seven beans, nine peas, one tare, and three grains.
 2. Nine beans, eight peas, nineteen tares, and ten grains.
 3. Two acorns, three peas, one tare, and four grains.
 4. Five chesnuts, three acorns, nine beans, and three tares.
 5. Eight chesnuts, fifty beans, three peas, and twelve grains.
 6. One chesnut, three walnuts, nine beans, two tares, and seven grains.

7. Two walnuts, seven acorns, forty-four peas, sixteen tares, and ten grains.
8. Six walnuts, four beans, nine tares, one pea, and twelve acorns.
9. One grain, two beans, seven chesnuts, three tares, and nine peas.
10. Ninety-three acorns, five beans, three walnuts, fourteen tares, one grain, and eight acorns.
11. Seven tares, three beans, five chesnuts, four tares, one grain, six peas, three acorns, and one walnut.
12. Twenty-five chesnuts, sixty-four tares, three walnuts, one bean, seventy-seven grains, and four acorns.
13. One walnut, nine chesnuts, nine acorns, nine beans, nine peas, nine tares, and ten grains.

B. Mark the above numbers on the abacus.

C. This seems to have been the state of numerical notation just before the invention of the Indian system of written characters, a system which the Hindoos long possessed without being aware of its powers, but which became, in the hands of the Arabian muhendis, a most effective instrument of research, and which, farther improved and extended by the philosophers of Western Europe, has mainly contributed to the rapid progress of the exact sciences.

In order to mark the successive numbers upon a board, as when the cargo is put into a ship, or when goods are handed from one person to another, the common practice in all ages seems to have been, to make a score for each successive article—thus, /, one; //, two; ///, three; ////, four; sometimes proceeding in this way as far as nine, but more commonly stopping at four, and marking the five by a stroke diagonally across the others—thus, $\diagup$, five. Instead of marking the next group of five in the same line, it was written below—thus, $\diagup$ six; $\diagup$ seven; $\diagup$ eight; $\diagup$ nine; and $\diagup$ ten. This method of scoring

affords the first hint of the Roman numerals, which, notwithstanding their unwieldiness, are still used in titling chapters, and on clock faces. Though well adapted for the warehouse board, this method is inconvenient in the merchant's record-book. As the abacus enables him to compute with great readiness, he naturally seeks for some means of copying its results into his pages. This he could do, as we have done above, by ruling columns and inserting the proper number of marks in each column; by painting, in fact, the abacus upon paper. It is fatiguing, however, to make seven dots for seven, or nine dots for the number nine. The invention of the Indians was to use a particular mark for each of the first nine numbers; these marks were adopted by the Arabs, and from them by the Europeans. They are—

Arabic,	۱	۲	۳	۴	۵	۶	۷	۸	۹
European,	1	2	3	4	5	6	7	8	9

These marks, or *figures* as we call them, are written in the columns of the abacus, so that the last-mentioned number would be written thus—

or—	8	5		7	3	4		9	
	۸	۵		۷	۳	۴		۹	

where the blank spaces correspond to the empty columns of the swan-pan.

The Arabs write and read from right to left, the Europeans from left to right; but the Arabs, in naming a number, begin with the lowest denomination, while we begin with the highest; and hence the same arrangement of the figures suits both parties.

The marking of the bars is troublesome; let us try to avoid it. It is clear that if we were to leave out the bars in the above example, and to write ۸۵۷۳۴۹ 857349, quite another number would be put down—viz. *eight hundred and fifty-seven thousand three hundred and forty-nine*: the figures have no longer their proper ranks. We must then leave a space to stand for

each blank column—thus, 85 734 9; but this way of writing would lead to mistakes, since, if there happened to be two or three empty places together, it would be difficult to know how many were intended; and since, if the last place happened to be blank, we would have no means of knowing it. In order to avoid this cause of error the Arabs placed a simple dot, called in their language *نقطه*; *nokta*, in the blank space, and thus preserved the ranks of the figures. The above number would thus be written ٨٥.٧٣٤.٩ There is no punctuation in Arabic, but this dot or *nokta* might be confounded with our period or comma; and, again, the mere point might be easily converted, for fraudulent purposes, into any figure; so, European computers use 0, an empty ring, to mark the empty column, and this preserves in English its Arabic name, being called *nought*. The above number is then written 85073409, where the *nought* or *nulla* serves to show the position of the blank column, and thereby to preserve the ranks of the figures. The same mark 0, is sometimes called *zero*, from the Arab word *ذرة* *zerré*, an atom, and at other times *cipher*, from *صفر* *sifr*, empty. From this last word we make the verb to *cipher*, and the barbarous compound to *decipher*, neither of them a felicitous derivation.

EXERCISES.

1. Write all the numbers given in the preceding exercises in figures.
2. Write in figures the following numbers:—Twenty-one. Sixty-seven. Seventy-five. Fifteen. Fifty. Ninety-one. One hundred and thirty-six. Two hundred and two. Five hundred and ninety. One thousand six hundred and seventy. Five thousand and sixty. Seven thousand and five. Twelve thousand and eighty-three. Seventeen thousand and eleven. Nineteen thousand nine hundred and nine. Twenty thousand eight hundred and thirty-one. Thirty-four thousand and seventeen. Forty-nine thousand and sixty. Forty-four thousand. Forty-seven thousand

two hundred and sixty-one. Fifty-eight thousand nine hundred and nine. Eighty-three thousand six hundred and seventy-three. One hundred and one thousand and five. Three hundred and thirteen thousand four hundred and five. One hundred and ninety-one thousand nine hundred and nine. Two hundred and three thousand five hundred and seventy-two. Six hundred and eighty-one thousand and one. Three hundred and sixty-two thousand five hundred and forty-seven. Five hundred and seventy thousand. One million. Sixteen hundred thousand and thirteen. Three million five hundred and one thousand and thirty-eight. Fifty-three million forty-three thousand and thirty-three. Nine million nine hundred and ninety-nine thousand nine hundred and ninety-nine.

Write out in words the following numbers :—

11, 15, 23, 27, 56, 99, 101, 273, 300, 542, 768, 1011, 1101, 1110, 1935, 2740, 3091, 9990, 12001, 15625, 27020, 56401, 89053, 167584, 501327, 990811, 1533386, 3010677, 7037002, 9000009.

D. For the convenience of being read off, it is usual to write the figures of large numbers in groups of threes, fours, or fives, according to the taste of the writer : but for reading the names of the numbers in English, it is best to write the figures in groups of six, slightly dividing each group in the middle, thus—58 736 896 134. The advantage of this mode of spacing may be best seen from an extended example, as—

547	032	810	426	341	207	580	018
$\left. \begin{array}{c} \text{Hun.} \\ \text{Ten.} \\ \text{Unit.} \end{array} \right\}$	$\left. \begin{array}{c} \text{Hun.} \\ \text{Ten.} \\ \text{Unit.} \end{array} \right\}$	$\left. \begin{array}{c} \text{Hun.} \\ \text{Ten.} \\ \text{Unit.} \end{array} \right\}$	$\left. \begin{array}{c} \text{Hun.} \\ \text{Ten.} \\ \text{Unit.} \end{array} \right\}$	$\left. \begin{array}{c} \text{Hun.} \\ \text{Ten.} \\ \text{Unit.} \end{array} \right\}$	$\left. \begin{array}{c} \text{Hun.} \\ \text{Ten.} \\ \text{Unit.} \end{array} \right\}$	$\left. \begin{array}{c} \text{Hun.} \\ \text{Ten.} \\ \text{Unit.} \end{array} \right\}$	$\left. \begin{array}{c} \text{Hun.} \\ \text{Ten.} \\ \text{Unit.} \end{array} \right\}$
Thous.	Unit.	Thous.	Unit.	Thous.	Unit.	Thous.	Unit.
Trillions.		Billions.		Millions.		Units.	

which is thus read :—

Five hundred and forty-seven thousand and thirty-two *trillions*, eight hundred and ten thousand four hundred and twenty-six *billions*, three hundred and forty-one thousand

two hundred and seven *millions*, five hundred and eighty thousand and eighteen.

Read the following numbers :—

57260190; 700896701; 9520100191; 1998700322;
23057010698; 101057142377; 51932017005;
1002964720500; 1790520300120; 729401378702300;
570000010326031; 200103477511992485;
541560398642972111; 310562341523622001055763897;
56990909103654227389223943202513200.

Write the following numbers in figures :—

Two hundred and nine million, nine hundred and eighteen thousand and one.

Seventeen thousand and five million, eight thousand and two.

Four hundred and ten thousand three hundred and eighty-six million, seven thousand five hundred and ninety-six.

Fifty-three billion, seven hundred and four million, three hundred and eleven thousand one hundred and one.

Two hundred and three billion, five hundred and sixty-seven thousand and twelve.

One thousand seven hundred and twelve billion, five hundred and six thousand million, seven thousand and one.

Ninety-nine thousand five hundred and eight billion, five hundred and seventy-six thousand and one million, four hundred and seventy-eight thousand three hundred.

Seven hundred and six thousand five hundred and forty-two billion, eleven thousand two hundred and eighty-eight million, three hundred and one thousand and five.

Four hundred and seventy-one trillion, six hundred and thirteen thousand one hundred and five billion, nine hundred and three thousand four hundred and seventy-eight million, four hundred and sixty thousand two hundred and three.

Three hundred and ninety-six thousand five hundred and three trillion, eight hundred and eleven billion, sixty-nine thousand one hundred and nine million, two hundred and seventy-three.

E. These examples exhibit well the enormous power of the Indian notation. By help of a few figures, we can indicate numbers far beyond what any human being could count out, although his whole life were devoted to the task ; and it can be shown that the entire number of ordinary grains of sand which would go to make up the whole mass of the earth does not reach what would be written with some thirty-five figures : that is to say, that there is not a sextillion of grains of sand in the whole world. Yet, by help of this notation, we can write down and operate upon such numbers as if they were quite familiar to us, and that without the fear of mistaking even a single unit. It is important to remark that all this power is derived from the simplicity and uniformity of the system. It is not to be wondered at that the decimal notation has been everywhere welcomed and adopted : it is truly a subject for greater wonder that a single trace of any other system remains in use. Still, some of the old methods retain their hold, and serve, by the contrast, to set out the advantages of the new. This chapter would not be complete without a short notice of the most remarkable of these.

In the Roman notation, a stroke *l* is used for one, an *x* for ten, a *c* or *Ꝣ* for one hundred, and an *m* for one thousand ; and each of these symbols is halved, thus, *v* for five, *l* for fifty, and *d* for five hundred. To indicate two, two lines *ll* are used ; for twenty *xx*, and for two thousand *mm*. This method, however, would become tedious. Thus *llllllllll*, for nine, or *cccccccc* for nine hundred, would be tiresome to write or to reckon. To remedy this tediousness, the marks *v*, *l*, and *d*, are employed ; thus eight is marked *viii*, eighty *lxxx*, eight hundred *dccc* ; and, still farther to abbreviate, the nine is marked *ix* (one before ten), the ninety *xc* (ten before one hundred), and the nine hundred *cm* (one hundred before a thousand). From this it is seen that the numeral notation of the Romans was but a step removed from utter barbarism : even the abbreviations evince the lowest state of inventive power. Such symbols could hardly be used in computation, being much less convenient than the counters on the abacus.

The Greeks made considerable progress in the art of notation, and advanced, indeed, within a single step of the simplicity of our present numerals. Their alphabet consisted of twenty-four letters; to these they added three marks, so as to make up the number to twenty-seven, or three times nine. The first nine they used to represent *units*, the second nine *tens*, and the third nine *hundreds*, thus :—

α	1	ι	10	ρ	100
β	2	κ	20	σ	200
γ	3	λ	30	τ	300
δ	4	μ	40	υ	400
ϵ	5	ν	50	ϕ	500
ζ	6	ξ	60	χ	600
η	7	$\omicron$	70	ψ	700
θ	8	π	80	ω	800
	9	ζ	90	ϑ	900

By help of these, it was easy to mark any number below a thousand. Thus, 789 was written $\psi\pi\theta$, or 804, $\omega\delta$. To mark larger numbers they put an μ ($\mu\upsilon\sigma\iota\alpha$, a thousand) under the letter, to make it have one thousand times its usual signification, and latterly this μ was contracted to a small mark. Thus five hundred and forty-six thousand seven hundred and forty-three was written :

$\phi \mu \zeta \psi \mu \gamma$
M M M

To have discarded the two second sets of numerals, and introduced the mark for an empty space, would have converted this system into that now in use. We find these numerals employed to mark the chapters of our Greek New Testaments.

The Arabs are said to have copied this mode of notation from the Greeks, using the ancient Arab letters; and when the Perso-Arab letters came into use, they changed the form merely of the numeral letters, without disturbing the order of the old alphabet. The subjoined order of the Arab numeral signs agrees closely with that of the Hebrew letters :—

ا ب ج د هـ و ز ح طي كلمن س ع ف ص ق ر ش ت ث خ ذ ض ظ

The first nine of these represent *units*, the second nine *tens*, and

the third nine *hundreds*; while the twenty-eighth letter stands for one thousand, thus :—

1000 غ	100 ق	10 ي	1 ا
	200 ر	20 ك	2 ب
	300 ش	30 ل	3 ج
	400 ت	40 م	4 د
	500 ث	50 ن	5 هـ
	600 خ	60 س	6 و
	700 ذ	70 ع	7 ز
	800 ض	80 ف	8 ح
	900 ظ	90 ص	9 ط

The order of writing these is from left to right : thus 859 would be written طانض or joined طنض and this circumstance has been regarded as showing that the Arabs must have borrowed this method from the Greeks. However, when we consider that the compound Arabic numeral names begin with the lowest denomination, we see that the above characters follow the natural order of Arabic writing ; and so far as this argument goes, it might be adduced to show that the Greeks borrowed from the Arabs, and the necessity for the three additional marks ϵ epise-mon, ζ koppa (originally almost identic, in form as in name, with the long Arab kef), and $\wp$ sanpi, would seem to turn the evidence against the Greeks. The priority must be established by historical and other evidence. The particular order of the Arab letters would almost indicate that the Jews had first used this method.

From these three examples of numeral notation, we see that the course of improvement has been by slow steps. It is rather startling to think that the perfecting of such a simple thing as our ordinary numerical scale should have occupied mankind for many thousand years. It is not much more than two hundred years since John Neper gave the last improvement to its fractional or descending part.

CHAPTER II.

ON THE ADDITION OF NUMBERS.

A. A merchant has already in his store a number of articles of a certain description, and receives a new supply of the same, how many has he now in all? The answer to this, and to a multitude of similar questions, is obtained by *adding* the number of the new supply to that of the old stock. The total amount is called the *sum*, from *summum*, Latin. Our verb, to add, is from the Latin; but it is worth notice that the word عدد (*aded*) in Arabic means number.

In order to obtain the sum of two numbers, we take as many counters as represent each of them, and, having put these together, count how many there are in all. Thus, if a farmer who had five horses purchase seven horses more, he may ascertain the entire number of his horses by putting *five* counters and *seven* counters together, and reckoning them all up. In this way it is found that *five* and *seven* make *twelve*.

In the course of business the same numbers, especially the smaller ones, occur over and over again, so that a great deal of labour is saved by keeping in mind those sums that have been found already. The sums of all numbers up to *ten* and *ten* should be firmly impressed in the memory. This is best accomplished by practice, that is, by the learner counting them up for himself. The use of printed addition-tables should be most carefully avoided, as tending seriously to repress mental activity and self-reliance.

Add together the following numbers :—

Two and three. Four and two. Three and four. Six and three. Nine and four. Two and seven. Five and four. Ten and seven. Eight and two. Eleven and five. Thirteen and eight. One and eight. Seven and nine. Sixteen and five. Three and eight. Four and eighteen. Nine and nine. Three and five. Twenty-one and nine. Six and twenty-two. Fourteen and eleven. Seven and eight. Twelve and seven. Thirty-two and thirteen. Two and forty-three. Nine and fifteen. Five and eighteen. Eight and fourteen. Thirteen and twenty-three. Six and six. Eighteen and twenty-seven. Two and eleven. Four and seven. Nineteen and thirty-four. Twenty-seven and nine. Seventeen and nineteen. Twenty-five and twenty-three.

This very simple and obvious process is the only one known for obtaining the sum of two numbers. We may abridge the operation by calling in the aid of our former experience, or by a careful arrangement of the counters ; but in all cases the counters representing the two numbers have virtually to be reckoned up.

One shortening of the labour has most probably by this time occurred to the mind of every beginner. For example, in adding fifteen to twenty-three, we may leave out the twenty-three counters, and having put down only fifteen, reckon these twenty-four, twenty-five, up to thirty-eight, which is the sum required. Every one also sees that seventeen added to five must give the same result as five added to seventeen. .

Add together the following numbers :—

Thirteen and forty-five. Twenty-three and twenty-seven. Five and forty-eight. Thirty-six and nineteen. Eleven and fifty-one. Thirty and twenty-nine. Thirty-nine and thirty-nine. Eight and forty-seven. Fifty-six and eighteen. Thirteen and sixty-nine. Sixty-three and ten. Sixty-four and thirteen. Fourteen and thirty-seven. Seventy-three and twenty-eight. Thirty-five and sixty-two. Seven and fifty-nine. Eighty-one and eighteen.

When we get among large numbers, this way of proceeding becomes tedious. Now, we have seen that very large numbers may be represented by a few counters : let us try if such counters will help us to add.

If the sum of forty-seven and eighty-nine were wanted, we might, instead of counting out these large numbers of grains, put *four tares and seven grains* for the one, *eight tares and nine grains* for the other, and heap these together, thus getting in all *twelve tares and sixteen grains*. Now, for ten grains we may put one tare, and then we shall have *thirteen tares and six grains*; lastly, for ten tares we may put one pea, so as to represent the sum of the two numbers by *one pea, three tares, and six grains*; and thus we discover that the sum of *forty-seven and eighty-nine* is *one hundred and thirty-six*.

Again, if we have to add *three thousand seven hundred and eighty-six* to *five thousand six hundred and ninety-three*, we would represent the first number by three beans, seven peas, eight tares, and six grains; the other, by five beans, six peas, nine tares, and three grains; and, putting these together, would have in all eight beans, thirteen peas, seventeen tares, and nine grains. But we need never have more than nine counters of one kind, since for ten of them we may put one counter of the next higher class. The above counters, then, may be changed into nine beans, four peas, seven tares, and nine grains, representing the number *nine thousand four hundred and seventy-nine*.

How much shorter than to have taled out the whole of this immense number of counters! Such is an example of the advantages of order and regularity.

By help of seeds, add :—

1. Two hundred and ninety-five, to three hundred and seventy-four.
2. Five hundred and sixty-three, to sixty-nine.
3. Four hundred and thirty-seven, to five hundred and sixty-three.
4. Seven hundred and one, to two hundred and twenty-four.

5. Nine hundred and sixty-five, to three hundred and seventy-nine.
6. One thousand two hundred and one, to nine hundred and sixty-nine.
7. One thousand seven hundred and ninety-seven, to seven hundred and sixty-nine.
8. Three thousand seven hundred and thirty-five, to two thousand nine hundred and eighty-seven.
9. Five thousand seven hundred and twenty-nine, to one thousand eight hundred and ninety-three.
10. Nine thousand five hundred and seventy-five, to eight thousand six hundred and ninety-six.
11. Eleven thousand eight hundred and fourteen, to nine thousand one hundred and eighty-six.
12. Twenty-three thousand and eighty, to nineteen thousand seven hundred and one.
13. One hundred and thirty-nine thousand six hundred and thirty-three, to ninety thousand three hundred and sixty-seven.
14. Seven hundred and forty-four thousand five hundred and seventy-six, to five hundred and sixty-four thousand three hundred and seventy-two.
15. Five million six hundred and four thousand eight hundred and one, to two million eight hundred thousand nine hundred and thirty-one.

B. As we found the representation of numbers on the abacus easier than that by diversified counters, so we may expect it to afford greater facilities for adding.

Let us try, by help of the abacus, to add the number *eighty-two thousand and ninety-seven* to *forty-nine thousand three hundred and seventy-six*.

We have only to represent these on the abacus, keeping the two numbers separate by means of a thin stick until they be arranged. On removing the stick so as to allow all the counters in each groove to be collected, we have at once a representation

of the required sum; however, if there be ten counters in any one groove, we may remove them, taking care to put one

	○ ○ ○ ○ ○ ○ ○ ○	○ ○		○ ○ ○ ○ ○ ○ ○ ○ ○	○ ○ ○ ○ ○ ○ ○
	○ ○ ○ ○	○ ○ ○ ○ ○ ○ ○ ○ ○	○ ○ ○	○ ○ ○ ○ ○ ○ ○	○ ○ ○ ○ ○ ○
○	○ ○ ○	○	○ ○ ○ ○	○ ○ ○ ○ ○ ○ ○	○ ○ ○

counter in the next higher place. Having done this, we obtain a representation of the desired sum in a state fit to be read off. In our present example the sum is *one hundred and thirty-one thousand four hundred and seventy-three*.

By help of the abacus add the numbers given in the preceding examples, and also the following :—

1. Three hundred and sixty-seven, to three hundred and twenty-one.
2. Four hundred and seventy-five, to three hundred and seven.
3. Three hundred and ninety; to four hundred and eighty-eight.
4. Seven hundred and thirty-two, to six hundred and seventy-one.
5. Nine hundred and ninety-three, to eight hundred and seven.
6. Eight hundred and sixty-four, to ten hundred and eleven.
7. Two thousand one hundred and three, to five thousand seven hundred and forty-nine.
8. Six thousand nine hundred and thirty-five, to eight thousand four hundred and two.
9. Twelve thousand and eighty-one, to fourteen thousand nine hundred and sixty-six.

10. Seventeen thousand five hundred and ten, to nineteen thousand four hundred and ninety-nine.
11. Twenty-one thousand six hundred and twenty-seven, to twenty-three thousand and fifty-eight.
12. Fifty-six thousand six hundred and thirty-two, to fifty-four thousand four hundred and seventy-nine.
13. Eighty-three thousand seven hundred and sixty-three, to ninety-one thousand five hundred and three.
14. One million three hundred and one thousand, to two million three hundred and fifty-six thousand.
15. Three million one hundred and seventy thousand nine hundred and eighty-two, to four million six hundred and twenty-one thousand four hundred and fifty-six.
16. Sixteen million three hundred and fifty-nine thousand seven hundred and thirty-one, to twenty-three million five hundred and one thousand two hundred and seven.
17. Two hundred and ninety-three million seven hundred and thirty-four thousand nine hundred and sixty-eight, to four hundred and thirty-two million four hundred and two thousand three hundred and fifty-eight.

C. Having learned to add the numbers by help of the abacus, we may proceed to try addition with the Indian numerals.

While using the seeds, we had to collect those of one kind ; thus the peas and tares were not counted pell-mell, but the peas separately, and the tares separately ; also, while using the abacus, we added the counters in each groove—that is, the tens were added separately, and the hundreds separately. So, when adding by help of numeral figures, we must take care to collect those of one rank ; it will not do to add the figure which is in the place of tens to that which is in the place of thousands. For the purpose of regulating this, it is usual (though not necessary) to write the one number below the other, placing units under units, tens under tens, and so on. In this way we prevent the mixing of the different ranks.

Thus, if to *seven thousand five hundred and thirty-one* we wish to add *four thousand nine hundred and seventy-five*, we would write them thus—

$$\begin{array}{r} 7531 \\ 4975 \end{array}$$

And, beginning with the units, would add the numbers in each column. It is usual to draw a line under the numbers which are to be added, and to write the sum below it—

$$\begin{array}{r} 7531 \\ 4975 \\ \hline 12506 \end{array}$$

The operation is thus managed : Five units and one unit make six units. Three tens and seven tens make ten tens; but we have no figure for ten, nor do we need one, for ten tens make one hundred. We therefore write a blank in the place of tens, and carry one to the place of hundreds. One hundred, nine hundreds, and five hundreds, make fifteen hundreds, or five hundreds and one thousand; so we write five in the place of hundreds, and carry one to the thousands. One thousand, four thousands, and seven thousands, make twelve thousands, so that the entire sum is *twelve thousand five hundred and six*.

While performing such operations, we soon come to see that it is not necessary to keep in mind the rank of the figures on which we are operating. We may drop the names of the ranks and go on, 5 and 1 make 6; 7 and 3 make 10—write 0 and carry 1; 1, 9, and 5 make 15—write 5 and carry 1; 1, 4, and 7 make 12. But, although not expressed, the values of the figures are always understood.

Perform the following additions :—

176	394	357	562	575	726
325	273	498	618	833	1274

869	762	1846	2198	1873
922	1471	1577	1952	2565
1999	2736	2059	5309	4754
2673	3527	3823	3270	3961
5290	7982	9721	17329	9982
6713	6134	8015	8956	14706
243025	380571	67 831 627		
701986	537069	18 079 429		

As also all the examples preceding.

D. The addition of one number to another is by far the most frequent arithmetical process, and every one who desires to become an expert computer should give great attention to it. There are several little things worth notice ; thus, on adding 9 the sum is always one less than the figure to which 9 is added : thus 8 and 9 make 7, 7 and 9 make 6 in their own rank, with 1 carried, of course, to the higher rank.

Similarly, if we have to add 99, we may add 100 and subtract 1 ; or if we have to add such a number as 497, we might prefer to add 500 and deduct 3 ; and so in multitudes of analogous cases.

But by far the best means of acquiring facility in calculation is to practise the addition of two numbers, beginning at the left or highest rank. In attempting addition in this way, our first and only difficulty is to know whether the figures in the rank to which we have come must or must not be augmented by unit brought up from the rank below. This difficulty vanishes on being confronted. Let us try to add the two following numbers, beginning at the left hand :—

$$\begin{array}{r}
 418\ 732\ 620\ 384 \\
 235\ 287\ 748\ 635 \\
 \hline
 654\ 020\ 369\ 019
 \end{array}$$

Here the sum of the digits in the highest rank is 6, and we see at a glance that the succeeding figures fall far short of ten, and cannot send anything up; so, without any hesitation, we write the 6. The sum of the second pair of digits is 4, but the succeeding ones clearly make more than 10, so our 4 becomes a 5. The next pair of digits alone would give 3, but the following pair make 9, which may become 10 if anything be sent up to it; and on glancing still forward we find 11, hence our 3 becomes a 4, and the following figure a 0. In this way we easily proceed along the whole line. If several 9's occur in succession, we must look forward until we arrive at a sum which is either more or less than 9, in order to determine whether unit be or be not to be brought up, as in this example:—

$$\begin{array}{r}
 230731648025519832 \\
 569268377374480157
 \end{array}$$

I earnestly recommend the somewhat advanced student to practise addition in this way until he feel it quite easy. He may write two numbers at the top of a page, add these together, then the sum to the last, the new sum to the preceding, and so on; by this means he is likely to obtain every variety of case. The subjoined example may serve to show the process.

$$\begin{array}{r}
 48132760913 \\
 69250194761 \\
 117382955674 \\
 186633150435 \\
 304016106109
 \end{array}$$

And so on.

Expertness in this operation prepares the way for facilities in other branches of arithmetic.

A. When we have to find the sum of several numbers, we may proceed by adding the second to the first, the third to the sum of the preceding, and so on ; but it is more convenient, when we use counters, to collect them all at once. Thus, if we be required to add the numbers *seven hundred and eighty-three, two thousand four hundred and ninety-one, one thousand seven hundred and eight, four thousand and seventy-seven, and nine hundred and sixty-four*, we first exhibit each of them by means of seeds, and then put all those seeds together ; after which we find *seven beans, twenty-seven peas, thirty tares, and twenty-three grains*. Since ten grains are of the same value with one tare, twenty of the grains may be removed, and two tares put for them. In the same way we may take away thirty of the tares, putting three peas for them ; for the thirty peas we may put three beans, and for the ten beans one acorn ; after all which changes we find our counters to be *one acorn, two tares, and three grains*, which stand for the number *ten thousand and twenty-three*.

By help of seeds, perform the following additions :—

1. Two hundred and thirty-seven. Five hundred and eleven. Ninety-three, and four hundred and seventy-one.
2. Five hundred and sixty-three. One hundred and ninety-eight. Seven hundred and thirty. Fifty-one. Eight hundred and nine. Eleven hundred and eighty-six. One thousand and forty-seven. Six hundred and twenty-nine ; and two thousand three hundred and sixty-five.
3. One thousand seven hundred and fifty-one. Two hundred and three. Nine hundred and thirty-four. Two thousand two hundred and thirty-six. Nine. Three thousand seven hundred and fifty-four. Four thousand five hundred and seventy-nine. Five hundred and three. One thousand eight hundred and eighty-seven ; and three thousand two hundred and ten.

B. The addition of several numbers may be performed on the abacus by placing the counters to represent the numbers, massing

those in each compartment together, and then taking away the redundant tens, replacing each ten by one counter in the higher rank. The preceding example would be done thus—

			o o o o o	o o o o o	o o o
		o o	o o	o o o	
			o o o o	o o o o o	o
		o		o o o o	
			o o o o o		o o o o o
		o o o o	o o		o o o
				o o o o o	o o o o o
				o o	o o
			o o o o o	o o o o o	o o o o
			o o o o	o	
	o			o o	o o o

As exercises, the student may take the preceding examples.

C. When numbers are represented by figures, we experience little difficulty in adding several of them together. Having written them in order, units under units, tens under tens, and so on, we begin with the figures in the units column, and, having summed them up mentally, we carry the tens of the sum to the next column. Passing to the column of tens, we proceed in the same way; and thus we go on until the whole have been collected. The principle of the process cannot be better seen than by comparing it with the corresponding process on the abacus. The preceding example would stand thus:—

$$\begin{array}{r}
 783 \\
 2491 \\
 1708 \\
 4077 \\
 \underline{964} \\
 10023
 \end{array}$$

Add together the following numbers:—

7527	17808	34877	67331
5891	2799	15320	75615
<u>8597</u>	<u>52341</u>	<u>7855</u>	<u>97753</u>

197 300	576 397	895 737
201 799	750 395	127 885
99 678	1 912 732	977 321
<u>325 774</u>	<u>17 864</u>	<u>235 717</u>

369 152	7 326 967	22 654 381
929 318	652 034	53 271 729
2 325 967	892 363	15 347 197
97 324	5 791 625	3 765 119
<u>870 592</u>	<u>9 153 794</u>	<u>84 135 928</u>

375 964 852	936 573 211
592 384 671	759 131 573
435 624 765	875 398 700
26 427 351	997 107 317
117 330 759	725 363 097
271 957 580	967 351 463

795 776 301	137 502 996
1 199 638 525	5 739 941 023
983 700 032	7 360 542 099
752 110 359	128 357 921
815 673 278	5 327 172 910
198 351 576	22 757 009 645
592 679 322	715 136 799
99 763 185	4 365 413 297

CHAPTER III.

ON SUBTRACTION.

A. WHEN a merchant has sold some of his goods, and wishes to know what quantity he has still in his store, he takes, or *subtracts*, as we say, the number of the articles which he has sold from the number of those which he possessed before the sale. The word *subtract* comes from two Latin words, *sub*, under, and *traho*, I draw : and contains the idea to draw from under, as when we dig away from a heap of charcoal.

The obvious way of taking one number from another is to place as many counters as represent the larger number, and to take out the smaller number from among them : to imitate, in fact, the dealing of the merchant. Thus, if we wish to take *eight* from *thirteen*, we lay out thirteen counters, and strike off eight, thus— $\circ \circ \circ \circ \circ \circ \circ \circ / \circ \circ \circ \circ$ —when it is seen that five remain. With small numbers, this is our only process ; but when the numbers are large, it becomes very tedious. It may be abridged by the use of classified counters. Thus, if it were proposed, from the number

Eight million three hundred and four thousand two hundred and ninety eight,

to subtract this one,

Three million one hundred and thirty-seven thousand four hundred and thirty-five,

we might represent them by seeds, thus :—

Eight walnuts, three chesnuts, no acorns, four beans, two peas, nine tares, and eight grains.

Three walnuts, one chesnut, three acorns, seven beans, four peas, three tares, and five grains.

Having laid out the larger number, let us begin the subtraction with the grains. Five grains from eight grains leave *three grains*: three tares from nine tares leave *six tares*: but now we meet with a difficulty. We cannot take four peas from two peas: how, then, are we to proceed? Let us recollect. A bean is worth ten peas; so if we remove one of the four beans, and put ten peas instead, we shall have three beans and twelve peas. Now we can manage the subtraction: four peas from twelve peas leave *eight peas*. Again the same difficulty occurs. We cannot take seven beans from three beans, but we may borrow an acorn. Alas! there are no acorns to be borrowed. We then go a step farther, change a chesnut for ten acorns, and one of these acorns for ten beans; in this way we have two chesnuts, nine acorns, and thirteen beans. Seven beans from thirteen beans leave *six beans*; three acorns from nine acorns leave *six acorns*; one chesnut from two chesnuts leaves *one chesnut*, and three walnuts from eight walnuts leave *five walnuts*, so that we have left—

Five walnuts, one chesnut, six acorns, six beans, eight peas, six tares, and three grains,
which represent the number—

Five million one hundred and sixty-six thousand eight hundred and sixty-three.

Perform the following subtractions by help of seeds:—

From thirty-seven, take twenty-five.

From sixty-two, take thirty-three.

From fifty-eight, take sixteen.

From seventy-six, take fifty-seven.

From forty-one, take thirty-one.

From one hundred and twenty-eight, take forty-three.

From two hundred and seventy-one, take ninety-seven.

From three hundred and eighty-five, take one hundred and seventy-nine.

From five hundred and eleven, take three hundred and seventy-six.

From seven hundred and thirty-five, take four hundred and eight.

From eight hundred and forty-nine, take seven hundred and thirty-seven.

From one thousand seven hundred and ninety-four, take seven hundred and ninety-five.

From five thousand six hundred and forty-one, take three thousand eight hundred and seven.

From nine thousand and seventy-three, take eight thousand nine hundred and ten.

From eleven thousand five hundred and sixty-four, take seven thousand three hundred and ninety-two.

From fifteen thousand seven hundred and seventy-eight, take ten thousand three hundred and twelve.

From eighteen thousand seven hundred and six, take three thousand five hundred and sixty-three.

From twenty thousand and one, take three thousand nine hundred and ninety-six.

From twenty-five thousand seven hundred and forty-three, take seventeen thousand nine hundred and thirty-four.

From thirty-three thousand six hundred and seventy-two, take thirty-two thousand nine hundred and fifty-seven.

From seventy-nine thousand and thirty-three, take seventy-three thousand six hundred and fifty-eight.

From one hundred and thirty-one thousand seven hundred and fifty-four, take ninety-nine thousand and seventy-nine.

From two hundred and seventy thousand, take one hundred and seventy-five thousand three hundred and ninety-one.

From five million and thirty-four thousand one hundred and seven, take three million seven hundred and twenty-five thousand four hundred and forty-two.

From eleven million one hundred and eleven thousand one hundred and ten, take nine million nine hundred and ninety-nine thousand nine hundred and ninety-nine.

B. The principles of this process are so clear, that any explanatory remarks might only tend to mystify a simple matter. The application of these principles to the abacus offers

no difficulty. Thus the above operation may be performed as below :—

	0000 0000	000		0000	00	0000 0000 0	0000 0000
	000	0	000	0000 000	0000	000	0000 0
	00000	0	0000 00	0000 00	0000 0000	0000 00	000

Perform the previous and the following subtractions on the abacus :—

From twelve million five hundred and forty-nine thousand seven hundred and sixty-two, take eight million three hundred and seventy-four thousand eight hundred and eighty-five.

From one hundred and fifty-two million three hundred and twenty-five thousand two hundred and ninety-six, take sixty-two million three hundred and twenty-five thousand two hundred and ninety-seven.

From one hundred and ninety-four million seventy thousand and five hundred and ten, take one hundred and thirty-four million seven hundred and sixty-seven thousand nine hundred and eleven.

C. To perform subtraction by help of figures is also a very easy operation. When the number to be diminished, or the *minuend* in any rank, is less than the number to be subtracted, or the *subtrahend*, we have to borrow unit from the next higher rank, reckoning it as ten in the rank on which we are operating ; and should there be a nought in the higher rank, we must go on to the next higher, as is seen in the above example performed on the abacus, or by help of graduated counters. Performed in figures, it stands thus :—

8 304 298
 3 137 435
 5 166 863

Perform the following subtractions :—

2 501 379	3 972 347	7 359 114
1 916 288	1 895 639	3 286 533

5 972 341	12 174 993	24 753 948
<u>6 729 815</u>	<u>9 355 746</u>	<u>85 274 933</u>

197 376 549	357 010 008	573 105 394
<u>172 854 210</u>	<u>293 752 184</u>	<u>701 500 300</u>

5 340 123 017	921 053 721	37 562 379 355
<u>4 932 007 396</u>	<u>999 078 615</u>	<u>41 176 884 279</u>

D. It is quite obvious that subtraction cannot be performed unless the *minuend*, or number to be lessened, be greater than the *subtrahend*, or number to be taken away. Certainly a merchant cannot take out of his store more goods than the store contains; so that, before trying to subtract, we must examine whether the subtraction be possible. For the purpose of discriminating the larger of two numbers, we observe that unit in any one place is of more value than all the figures that can be written in the lower places. Thus, 1 000 000 is more than 999 999, since by adding unit to this latter we obtain the former. Unit, then, in the seventh place, must be greater than any six place-number, since 999 999 is clearly the greatest of all these. From this we at once see, that of two numbers, that one is the larger which has the higher rank; also, if the highest figures of two numbers be of the same rank, that one of which the first figure is the greater must be the larger number: thus, of the two numbers 78827 and 81462, the latter is the larger, since the figure 8, in its highest place, is greater than the 7 in the highest place of the other, both being of the same rank; for although the figures 8827 be greater than 1462, the difference

is, and must always be, more than made up by the difference in the higher figures. Again, should it happen that the highest figures are alike, as in this case, 437682, 448653, the larger number may be recognised by fancying these figures left off, and examining only the residues, 37682, 48653. There is then no difficulty in distinguishing the greater of the two numbers.

The advanced student should not rest contented with being able to subtract readily, beginning with the units; he ought to be able easily to perform the operation, beginning at the left hand. This, indeed, will cost him little trouble, since he can see at a glance whether the subsequent figures of the minuend need assistance or not. Thus, in performing the subtraction

$$\begin{array}{r} 4\ 723\ 904 \\ 2\ 581\ 921 \\ \hline \end{array}$$

we take 2 from 4, but not 5 from 7, since the succeeding figure 2 needs to be helped against the 8 of the subtrahend. We therefore take 5 from 6 and 8 from 12. Again, we do not take 1 from 3, since the 90 of the minuend is less than the 92 of the subtrahend; therefore, it is 1 from 2, 9 from 18, 2 from 10, and 1 from 4.

It is usual to place the subtrahend under the minuend; but this is not necessary, neither is it always convenient. The expert computer should subtract upwards or downwards, just as it happens. It is a good way of practising to write two long numbers at the top of a page, to take their difference, then the difference of the two last, and so on, subtracting the under from the upper, or the upper from the under, as the case may be, and always beginning at the left hand: thus,

$$\begin{array}{r} 732014\ 497682\ 316049 \\ 519732\ 025864\ 190437 \\ 212282\ 471818\ 125612 \\ 307449\ 554046\ 064825 \\ 95167\ 082227\ 939213 \\ \hline \&c. \quad \&c. \quad \&c. \end{array}$$

C. When a merchant's transactions are numerous, he may not have time to balance his accounts after each one. Perhaps, at the end of the week, he may wish to know what remains in his store after sundry purchases and sales. In such a case we have additions and subtractions all mixed together. Let us make up an example. Say that last Saturday there was left in a corn-merchant's store 5723 quarters of wheat, and that his transactions during the week were—

Monday—Arrived from Odessa 1329 qrs., from Taganroc 853 qrs.; sold to A. B. & Co. 4870 qrs.

Tuesday—Received from Dantzic 2793 qrs.; sold to C. N. & Co. 3500 qrs., and to P. Q. 900 qrs.

Wednesday—Sold to A. B. & Co. 500 qrs.; received from Kertch 3253, per “Mary and Ann,” 4735 per “Swallow.”

Thursday—*Nil*.

Friday—Sold to C. N. & Co. 7500 qrs.

Saturday—From Memel 1739 qrs.,
and that we wish to know how many quarters ought to be in the granary on Saturday evening.

Here it will readily occur to the learner that it would be convenient to have marks whereby to distinguish the additions from the subtractions. Such marks are constantly used; they are +, called *plus*, for addition, and −, called *minus*, for subtraction; and by their help, the whole of the above transactions would be written thus: 5723 + 1329 + 853 − 4870 + 2793 − 3500 − 900 − 500 + 3253 + 4735 − 7500 + 1739.

To find the result of all these operations, we may take them in the order in which they occurred, thus:—

Saturday...Stock in hand, . . .	5723
Monday....From Odessa, . . .	+ 1329
... From Taganroc, . . .	+ 853
	7905
Tuesday....Sold to A. B. & Co., . . .	− 4870
	3035
Tuesday ...From Dantzic, . . .	+ 2793
Carry forward,	5828

	Brought forward,	5828
Tuesday.....	Sold to C. N. & Co., .	- 3500
		<u>2328</u>
..	Sold to P. Q., . . .	- 900
		<u>1428</u>
Wednesday...	Sold to A. B. & Co., .	- 500
		<u>928</u>
...	From Kertch, . . .	+ 3523
		<u>+ 4735</u>
		<u>9186</u>
Friday.....	Sold to C. N. & Co., .	- 7500
		<u>1686</u>
Saturday.....	From Memel, . . .	+ 1739
	Stock in hand, . . .	<u>3425</u>

And this process is very satisfactory, since it shows the state of matters after each separate transaction.

It is at once apparent that the result at the week's end would have been the same, although the order of the transactions had been varied, provided they had all taken place; so we may fancy that all the arrivals had been on the Monday, and all the sales on the Saturday. We may even, for all that it matters to the result, have made the whole of the sales in slump; that is to say, we may gather together all the additive quantities in one sum, all the subtractive quantities in another, and then take the latter sum from the former: thus—

5723	- 4870
+ 1329	- 3500
+ 853	- 900
+ 2793	- 500
+ 3523	- 7500
+ 4735	- 17270
+ 1739	
20695	
- <u>17270</u>	
3425	

This plan has the advantage of showing the amount of the sales effected. The learner may exercise himself in changing the order of the operations in this and in the following examples.

Collect the following numbers together:—

+	3 701	—	7 234	+	2 764	+	157 214
+	27 538	—	42 705	—	5 720 376	—	7 253 855
—	7 990	+	187	—	918 455	—	128 661
+	156 327	—	7 601	+	273 171	+	23 542 972
+	73 465	+	5 987	+	6 310 042	—	4 626 754
+	7 739	+	172 763	+	522 314	+	7 798 932
—	94 630	—	9 977	—	986	—	14 317 217
—	243 705	+	15 762	+	36 107	—	9 315 677
+	7 659	—	27 184	—	48 783	—	54 112 539
—	111	—	3 976	—	1 124 277	+	8 300 375
—	30 537	—	199 472	+	79 100	+	13 762 538
+	673 500	+	56 705	—	372 566	—	73 130
+	3 570	+	99 810	+	54 372	+	513 939
—	35 497	+	8 754	—	112 738	+	4 212 647
—	385 181	—	346 643	—	926 549	—	1 733
—	5 779	+	1 325	—	3 154 970	—	27 042 009
+	120 030	—	73 644	+	39 954	+	99 232 014
—	7 643	+	398 277	—	3 142	+	375 311
+	39 013	+	7 560	+	4 231 772	—	31 327 802
—	197 745	+	64 043	+	972 435	+	23 770
—	39 096	—	7 341	—	4 563	—	9 736 054
+	531 054	+	43 732	+	9 865 371	+	15 719

+ 35 760 581 662
 + 939 070 375
 — 1 241 852 674
 + 794 386
 — 73 260 130
 — 974 262 569
 + 20 156 044 384
 + 625 976 042
 — 7 483 574
 + 5 305 762 011
 — 53 120 352 942

+ 935 760 312
 — 67 948 540
 — 876 531 949 003
 — 4 272 355 739
 + 11 673 291 571
 + 944 030
 — 720 310 684
 — 49 562 476
 + 574 807 438 117
 + 213 635 782 438
 + 180 588 909 974

+	153 823 176 532	+	94 134 763 005 342
—	4 968 325 710	+	391 457 576 134
—	931 720 583 612	+	734 207 581
+	857 261 938	—	5 487 910 652 730
+	3 771 826 584	—	853 766 451 943
—	12 429 365 107	+	74 371 040 632 117
+	2 756 318 942 753	—	18 369 583 472 898
—	835 274 680 240	—	57 068 513 676
+	134 681 004 765	+	34 714 397 055 329
—	272 857 126	—	973 604 321 715
—	5 107 399 741	+	8 740 327 643 107
+	24 635 782 611	—	371 162 142 706

D. It often happens, in the course of extensive calculations, that the numbers to be operated with are already written in their places, some of them being additive and some subtractive. To proceed in either of the ways explained in the last article would, in most cases, involve the rewriting of the whole. This can be avoided by help of a little reflection.

If only one number should be subtractive, it is not at all difficult to carry on the addition and subtraction at once: thus, if we have

$$\begin{array}{r}
 38\ 769 \\
 +\ 42\ 305 \\
 -\ 21\ 634 \\
 +\ 15\ 723 \\
 \hline
 75\ 163
 \end{array}$$

we have only to subtract each digit of the subtrahend as we go on. Thus 9 and 5 make 14, less four; 10 and 3, 13. But when it happens that the figure in the subtrahend is greater than the sum of the additive numbers, we must borrow from the additives in the next column, or, what comes to the same thing, carry one to the subtrahend. With a little practice we soon become expert at this work. If there be more than one of the numbers subtractive, the operation becomes more troublesome, and also more liable to error. In order to prevent mistakes, it is convenient to draw a pencil line through those numbers that are to

be subtracted. For the sake of acquiring expertness, the student ought to practise this ; but in actual business, the method about to be described is much to be preferred.

Collect the following numbers together as they stand :—

+	375 168	—	53 477 098	+	861 542 707
—	754 931	+	6 210 435	—	54 938 465
+	269 710	+	73 541 649	—	763 019 186
+	135 427	—	21 365 472	+	34 732 657
+	865 142 397	+	527 356 217	—	4 347 542 768
+	790 856 105	—	356 185 732	+	5 421 379 639
—	539 774 538	+	607 938 477	—	4 736 241 078
—	903 268 174	—	126 357 251	+	2 990 637 543
+	215 379 006	—	652 761 711	+	3 681 766 664
+	237 928 614	—	846 327 511	—	147 120 675
—	531 772 856	—	699 532 079	+	523 795 326
—	407 349 725	+	535 946 327	+	237 694 211
+	762 583 169	+	904 310 665	+	993 572 341
—	194 700 534	—	576 342 112	+	642 074 503
+	336 891 752	+	793 633 579	—	240 375 769
—	755 604 917	+	950 631 779	—	365 147 689
+	842 871 254	—	372 547 354	+	867 340 124
+	623 942 571	—	635 498 160	+	120 156 947
—	936 510 769	+	586 705 914	—	715 642 939
—	374 162 933	—	376 541 337	+	547 318 455
+	494 573 216	+	865 399 527	—	312 517 963
+	171 945 364	—	131 546 874	+	654 391 447

The mixed subtractions are, however, generally performed by help of what are called *Complementary* numbers (from *con* and *pleo*, I fill).

When we have to subtract 9 from any number, we readily observe that we may subtract unit from the higher place, and add unit to that with which we have to do. Or if we have to subtract such a number as 3995, we may subtract 4000 and add 5 : the principle being this, that having subtracted 4000,

we have subtracted 5 too much, and must correct the result by adding the five.

It is then possible, in a kind of a way, to convert subtraction into addition. Thus, to subtract 372 481 is the same thing as to add 627 519, deducting one million from the amount. For example—

$$\begin{array}{r} 860\,935 \\ - 372\,481 \\ \hline 488\,454 \end{array} \quad \text{or} \quad \begin{array}{r} 860\,935 \\ 627\,519 \\ \hline 488\,454 \end{array}$$

the reason being, that the sum of these two numbers is 1000 000. These numbers, and all pairs of numbers of which the sum is unit followed by nought, are said to be complementary to each other, because together they fill up the scale ; and the addition of the one gives the same result as the subtraction of the other, with the error of unit in the next higher place.

E. In trigonometric calculations, and in all cases where logarithms are used, we employ these complements, very rarely making any subtraction, and we thus become expert in reading the arithmetical complement. This indeed is a very easy matter, since each figure has to be taken from 9, with the exception of the last effective figure, which is to be taken from 10. Thus the arithmetical complement of 3798246 is 6201754 ; that of 8179230, 1820770, where the 3 is regarded as the last effective figure ; and so on. After a little practice, it is quite as easy to read off the complement as the number itself from a table, and we obtain the advantage of having all our operations represented by addition. Thus, if we had to perform the operations, $\log. 237 + \log. 379 - \log. 238 - \log. 541 + \log. 39$, we would proceed as under :—

$$\begin{array}{rcl} \text{Log. } 237 & = & 2.374\,7483 \\ \text{Log. } 379 & = & 2.578\,6392 \\ \text{Com. Log. } 238 & = & 7.623\,4230 \\ \text{Com. Log. } 541 & = & 7.266\,8027 \\ \text{Log. } 39 & = & 1.591\,0646 \\ & & \hline & & 1.434\,6778 \end{array}$$

having rejected 20 from the result, that is 10 for each complement.

F. Sometimes it happens that the same number may be additive for one part of a calculation and subtractive for another. In such a case, to avoid writing the number over again, we (if it be written in ink) draw a dark pencil line through it when it is subtractive, and efface that line when it is additive: the lined numbers are then read as their complements.

As an example of the convenience of this method, let us suppose that the three sides of a trigon being given, we wish to compute its area, the radii of the interior and three exterior circles of contact, and also the three angles. Here we have eight calculations, the performance of which in the usual way, or even in the way described in the last paragraph, would require four logarithms to be written eight times over: by help of a black lead pencil we shall make one writing suffice. The three sides being supposed 437, 563, 608, we have the following calculation:—

608	196 log. =	2.292 2561	+	+	+	+	-	+	+	-
563	241 log. =	2.382 0170	+	+	+	-	+	+	-	+
437	367 log. =	2.564 6661	+	+	-	+	+	-	+	+
1608	804 log. =	2.905 2560	+	-	+	+	+	-	-	-
		10.144 1952	5.072 0976	= log. area						
		4.333 6832	2.166 8416	= log. int. radius						
		5.014 8630	2.507 4315	} log. ext. radii						
		5.380 1612	2.690 0806							
		5.559 6830	2.779 8415							
		9.204 3510	9.602 1755	} log. tan. $\frac{1}{2}$ angles,						
		9.569 6492	9.784 8246							
		9.749 1710	9.874 5855							

from which we find that the area of the trigon is 118058.6, the length of the radius of the inscribing circle 146.839; those of the radii of external contact 321.68474, 489.8798, and 602.3397; while the three angles are $43^{\circ}36'48''$, $62^{\circ}42'25''.7$, and $73^{\circ}40'46''.3$, respectively.

From this, the advanced computer will readily perceive the advantage of a little mechanical artifice.

CHAPTER IV.

ON MULTIPLICATION.

A. The next arithmetical operation is called multiplication, from *multum*, much, and *plico*, I fold. It is the process of repeating a number over and over again.

Thus, if a merchant have fifty casks of sugar, and in each cask if there be thirty-seven loaves, he will find how many loaves he has in all by repeating the number thirty-seven fifty times over; that is, as we say, by multiplying thirty-seven by fifty. In order to do this we may take thirty-seven grains, and another thirty-seven grains, and another, and another, until we have in all fifty times thirty-seven grains, and then may count these out. But this, in general, would be a very tedious process, and we must look for some way to shorten it. In reality, the repetition of the counters must always be made; only we may contrive to make the repetition easy.

When the numbers are not large, it is quite easy to find the result of the multiplication, or the *product* as it is called, from *pro*, for, and *duco*, I lead. Thus, if we

	*	*	*	*	*	*	*
be asked what <i>five</i> times <i>seven</i> come to,	*	*	*	*	*	*	*
we have only to make a row of seven	*	*	*	*	*	*	*
dots, repeat that row in all five times,	*	*	*	*	*	*	*
and count the entire number of dots,	*	*	*	*	*	*	*

which is thirty-five; and in this way we discover that five times seven make thirty-five; nor is there any better way.

Let the learner do over and over the following examples in this way until he be well acquainted with the results, but ON NO ACCOUNT let him have recourse to a printed multiplication table, or

be set to get such a table by ROTE. It is a thousand times better to see a child counting up his sums or products on his finger ends than to see him turn up an addition or multiplication table; in the one way, he is industriously and surely acquiring knowledge for himself, in the other, he is only using at second hand that which has been obtained by others.

Multiply the following numbers together :—

Three times two.	Seven times four.	Nine times nine.
Twice two.	Six times five.	Seven times eight.
Three times four.	Five times seven.	Five times nine.
Four times two.	Six times four.	Eight times three.
Four times five.	Eight times two.	Nine times six.
Three times five.	Three times eight.	Four times nine.
Four times three.	Nine times two.	Twice ten.
Three times six.	Seven times five.	Seven times seven.
Four times seven.	Five times six.	Eight times six.
Twice four.	Four times eight.	Nine times eight.
Five times three.	Once two.	Ten times four.
Twice five.	Nine times three.	Six times seven.
Four times four.	Five times eight.	Eleven times three.
Twice three.	Six times six.	Eight times four.
Three times three.	Twice nine.	Five times ten.
Three times one.	Seven times six.	Eight times eight.
Six times two.	Eight times nine.	Seven times nine.
Five times two.	Nine times five.	Eleven times seven.
Twice seven.	Six times eight.	Ten times five.
Four times one.	Three times nine.	Eight times eleven.
Seven times three.	Ten times two.	Four times ten.
Five times four.	Eight times five.	Eight times seven.
Four times six.	Three times ten.	Nine times ten.
Five times five.	Six times nine.	Ten times eleven.
Three times seven.	Nine times four.	Three times eleven.
Six times three.	Twice eleven.	Ten times ten.
Seven times two.	Nine times seven.	Seven times eleven,
Twice eight.	Six times ten.	&c.

While performing multiplication in this way, we can hardly help noticing that the product of one number by another is the same with the product of that other by the former. Thus *seven times five* gives the same product with *five times seven*; and it is easy to see that it must always be so; for while we have five horizontal rows with seven dots in each, we have seven vertical rows with five dots in each.

When the numbers grow large, this way of performing multiplication becomes tedious: let us try whether we may not get some help from our graduated counters. For example, let it be proposed to multiply the number

Seven hundred and twenty-three thousand five hundred and nine,
by the number *seven*.

We might represent the above *multiplicand*, or number to be multiplied, by

Seven chesnuts, two acorns, three beans, five peas, and nine grains,
and then repeat these seven times, so as to get

Forty-nine chesnuts, fourteen acorns, twenty-one beans, thirty-five peas, and sixty-three grains,
which would represent, but rather clumsily, the required product.

However, when there are more than nine counters of any one kind, we may change ten of them for one counter of the next higher class: thus for *sixty* of the grains we may put *six* tares; for *thirty* of the peas we may put *three* beans; and so on until there be not so many as ten counters of any one kind: and then we shall have

Five walnuts, six acorns, four beans, five peas, six tares, and three grains,
which represent the number

Five million sixty-four thousand five hundred and sixty-three.

Multiply the following numbers together:—

Twenty-seven thousand nine hundred and thirty-two, by four.

Thirty-one thousand eight hundred and fifty-six, by three.

Fifty-six thousand seven hundred and nineteen, by five.

Sixty-one thousand three hundred and seventy nine, by six.

by laying a rule across, and then place seven times these counters in the spaces below, as shown in the figure: lastly, whenever we find ten counters in any groove, we remove them, taking care to place one counter in the next higher groove, and continue to do this as long as we can. The result gives us the counters which represent the product in a state fit to be read off.

Perform the following multiplications on the abacus:—

Seventeen, by three.

Twenty-six, by five.

Fifty-three, by two.

Sixty-nine, by six.

Ninety-five, by four.

One hundred and forty-one, by six.

One hundred and fifty-seven, by ten.

One hundred and eighty-six, by five.

Two hundred and fifty, by eight.

Three hundred and ninety-nine, by four.

Four hundred and seventeen, by six.

Four hundred and sixty-three, by five.

Five hundred and seventy-three, by nine.

Six hundred and twenty, by three.

Eight hundred and forty-nine, by seven.

One thousand one hundred and eleven, by ten.

One thousand three hundred and fifty-six, by five.

One thousand nine hundred and eighty, by two.

Two thousand and thirty-two, by eight.

Five thousand nine hundred and twenty-seven, by six.

Eleven thousand five hundred and ninety-four, by seven.

Eighty-one thousand six hundred and fourteen, by eight.

One hundred and nine thousand seven hundred and fifty-six, by seven.

Two hundred and twenty-six thousand three hundred and forty, by four.

Two million nine hundred and twenty-eight thousand six hundred and eleven, by nine.

Three million and seven thousand five hundred and seventy-one, by eleven.

Two million seven hundred and ninety-nine, by seven.

Two million eight hundred and fifty-three thousand four hundred and sixty-one, by four.

Four million and seventy-two thousand nine hundred and thirty, by eight.

Eleven million six hundred and forty-eight thousand seven hundred and ninety-eight, by eleven.

Twenty-seven million six hundred and twenty-five thousand one hundred and thirteen, by six.

Fifty-five million and ninety-three thousand four hundred and thirty-one, by twelve.

Fifteen million eight hundred and seventy-three thousand and seventeen, by seven.

Three hundred and sixty-five millions eight hundred and twenty thousand seven hundred and ninety-four, by thirteen.

Five hundred and one million nine hundred and sixty-two thousand seven hundred and twelve, by nine.

Nine hundred and eighty-seven million six hundred and fifty-four thousand three hundred and twenty-one, by twelve.

C. This multiplication may also be readily performed by help of the Indian numerals, if we be acquainted with the smaller products. Thus, having written down the multiplicand as in the margin, and also the multiplier, 723509 in any convenient place, that we may not forget it, we begin the operation with observing that 7 times 9 are 63; but we cannot write the 63 in the place of units. We must write only the 3 there, and carry the 6 to the place of tens. We then take 7 times 0 tens, that is 0 tens, and bring up the 6 tens, so as to make 6 tens. Again, 7 times 5 hundreds make 35 hundreds, that is, 3 thousand 5 hundred; so we write the 5 in the place of hundreds, and carry the 3 to the place of thousands. 7 times 3 thousand make 21 thousand; add to this the 3 thousand brought up, and we have

$$\begin{array}{r} 723509 \\ \times 7 \\ \hline 5064563 \end{array}$$

24 thousand, that is 4 thousands and 2 ten thousands to be taken to the next place. 7 times 2 ten thousands make 14 ten thousands, or, with the 2, 16 ten thousands, that is 6 in the place for ten thousands, and 1 to be carried up to that for hundred thousands. Lastly, 7 times 7 hundred thousand make 49 hundred thousands, which, with the 1 brought up, makes 50 hundred thousand, or 5 millions. However, in working out the product, we do not need to attend to the name of the rank on which we are operating; it is enough to get the right product, to bring up what is to come from the lower rank, and to carry what is to go to the higher.

The learner ought sedulously to practise multiplication by small numbers, until he feel it to be no trouble to him. A very good way of forming examples is to take any number and multiply it, again the product, then the new product, and so on by some small number, thus :—

723509 by 7.
5064563 by 7.
35451941 by 7.
248163587, &c.

Perform the following multiplications :—

285 926 by 3.	7 468 327 by 7.	35 643 510 by 8.
537 042 by 4.	12 973 065 by 4.	46 834 759 by 7.
2 640 918 by 2.	18 426 344 by 6.	76 219 543 by 9.
5 209 758 by 5.	23 780 946 by 5.	98 158 976 by 3.
279 348 563 by 6.	523 600 476,	5 times by 6.
593 472 815 by 8.	712 352 014,	3 times by 9.
94 638, 8 times by 4.	1120 760 328,	4 times by 8.
156 972, 9 times by 3.	1543 746 893,	6 times by 7.
185 799, 6 times by 5.	2792 354 096,	7 times by 5.
327 055, 6 times by 7.	3 644 9737 85,	4 times by 6.
4 309 672, 10 times by 6.	4 536 792 854,	6 times by 8.
9 325 723, 5 times by 2.	5 753 647 396,	5 times by 9.
34 563 119, 7 times by 4.	9 341 785 601,	10 times by 7.
78 546 849, 11 times by 6.	12 865 974 087,	11 times by 4.

84 347 092, 9 times by 4.	15 799 872 356, 9 times by 6.
165 032 794, 6 times by 8.	24 978 563 286, 7 times by 8.
270 936 485, 4 times by 7.	79 841 657 943, 12 times by 9.
398 751 816, 10 times by 5.	96 127 862 359, 15 times by 2.

The product of two numbers is represented by the sign $\times$, which may be read *times*, or *multiplied by*; thus, 7×5 means the product of 7 by 5, and may be read 7 times 5, or 7 multiplied by 5.

A. We soon get acquainted with the product of small numbers; but when the multipliers are large, the process, even by help of the classified counters, is tedious. Thus, if we had to multiply some number by eighty-three, it would require a great many counters of each kind. There is, however, a method of shortening the process, and I proceed to explain it gradually.

If we had to multiply a number, say *five thousand seven hundred and twenty-nine* by *ten*, and if, to prepare for the work, we had represented the number by five beans, seven peas, two tares, and nine grains, we might see at once that an *acorn* stands for *ten beans*, and that, therefore, *five acorns* may take the place of the fifty beans, and similarly of the other counters; so that, in order to multiply a number by ten, we have only to put for each counter a counter of the next higher rank; thus we shall have five acorns, seven beans, two peas, nine tares, representing the number fifty-seven thousand two hundred and ninety. Of this unlooked-for facility we shall make great use afterwards.

EXAMPLES.

Multiply the following numbers by ten :—
 One hundred and seventy-eight.
 Seven hundred and ninety-two.
 One thousand six hundred and thirty-one.
 Twelve thousand eight hundred and ninety-five.
 Thirty-four thousand and fifty-seven.

One hundred and twenty-nine thousand six hundred and eight.
Six hundred and forty-three thousand nine hundred and sixty-two.
Seven hundred and fifteen thousand nine hundred and fifty-eight.

B. Multiplication by ten is still more easily performed by help of the abacus ; for we have only to remove the counters into the next higher grooves. Thus, the above multiplication done on the abacus is—

				00000	00000	00	00000
					00		0000
			00000	00000	00	00000	
				00		0000	

Multiply the following numbers by ten on the abacus :—
Sixty-three thousand two hundred and seventy-nine.
Two hundred and seventy-five thousand eight hundred and forty-three.
Nine hundred and twenty-six thousand seven hundred and twelve.
Three million one hundred and fifty thousand eight hundred and seventy-two.

C. To multiply by ten, using the Indian numerals, is quite as easy ; for, by removing any figure, or *digit*, as it is sometimes called (from *digitus*, the forefinger or pointer), one step to the left, we increase its value ten times. Thus, ten times 5729 is 57290, where the nothing is written at the end for the purpose of raising the rank of the other figures.

Multiply the following numbers by ten :—

3 731.	908 701.	13 754 309 110 000.
9 365.	12 761 050.	767 593 268 173 594.
15 124.	397 077 532.	897 359 000 000.
56 384.	5 364 017 000.	937 625 103 810.
297 600.	394 285 764 323.	1 303 690 010 070 000.

A. Having now learned how to multiply by ten, we can easily multiply by any number of tens, as by *twenty, thirty, forty, &c.*; for if we first multiply by *ten*, and then double the result, we shall have multiplied by twenty. In order to multiply a number by fifty, we have to raise the counters one rank, and multiply by *five*, and so on. Thus, if I wish to find what *seventy* times *three thousand five hundred and seventy-three* come to, I put, to represent the number, *three beans, five peas, seven tares, and three grains*, and raise these one rank; that is, I put *three acorns, five beans, seven peas, and three tares*, which stand for ten times the multiplicand; lastly, I take these counters *seven* times, and obtain *two chesnuts, five acorns, one pea, and one tare*, which stand for *two hundred and fifty thousand one hundred and ten*, the product wanted.

Perform the following multiplications:—

Three hundred and forty-seven, by twenty.

Eight hundred and twenty-nine, by forty.

One thousand five hundred and sixty-one, by thirty.

Five thousand eight hundred and twenty-four by fifty.

Nine thousand three hundred and seventy-eight, by twenty.

Fifteen thousand eight hundred and thirty-five by forty.

Twenty-three thousand and fifty-six, by thirty.

Sixty-three thousand nine hundred and seventy-eight, by sixty.

One hundred and seventy-four thousand two hundred and sixty-nine, by fifty.

Three hundred and eighty-two thousand and thirty-three, by eighty.

Six hundred and fifty-seven thousand four hundred and eighty-four, by seventy.

B. This multiplication can be still more quickly done on the abacus, for we have not the trouble of exchanging each pea for a bean, each bean for an acorn, and so on; it is quite enough to

Forty-nine thousand five hundred and ten, by sixty.

Fifty-six thousand eight hundred and seventy-three, by thirty.

Ninety thousand seven hundred and fifty-eight, by seventy.

One hundred and thirty-six thousand and eighty-one, by sixty.

Two hundred and nine thousand six hundred and thirty-eight, by seventy.

Five hundred and seventeen thousand two hundred and three, by eighty.

Seven hundred and ninety-two thousand six hundred and fifty-four, by ninety.

One million eight hundred and thirty-four thousand six hundred and seventy-three, by sixty.

Three million six hundred and forty-two thousand nine hundred and eleven, by ninety.

Fifty-six million four hundred and eighty-five thousand seven hundred and one, by seventy.

Nine hundred and sixty-one million five hundred and twelve thousand eight hundred and forty-four, by eighty.

When numbers are represented by figures, this operation is readily performed; we have only to proceed as if we were multiplying by seven, writing the figures one place to the left. Thus—

$$\begin{array}{r} 3573 \\ 70 \\ \hline 250110 \end{array}$$

Perform the following multiplications:—

361 529 by 30.	5 928 013 by 60.	23 750 468 by 80.
789 106 by 20.	9 302 763 by 50.	57 198 346 by 70.
2 397 564 by 40.	15 624 783 by 60.	79 385 416 by 90.

A. As we have now learned to multiply by any number of tens, we may proceed to multiply by any number whatever under

one hundred. For if we have to repeat some number, say forty-seven times, we may take it first *forty* times, then *seven* times, and add the two products together. Let it be proposed to take *fifty-four thousand seven hundred and sixty-three, forty-seven times*.

We put down for the multiplicand *five acorns, four beans, seven peas, six tares, and three grains*, and take this forty times, obtaining *two walnuts, one chesnut, nine acorns, five peas, and two tares*; also we take it seven times, obtaining *three chesnuts, eight acorns, three beans, three peas, four tares, and one grain*: then collecting these two results together, and making the usual exchanges, we get *two walnuts, five chesnuts, seven acorns, three beans, eight peas, six tares, and one grain*, which mark the number *two million five hundred and seventy-three thousand eight hundred and sixty-one*.

Perform the following multiplications:—

Thirty-seven, by seventeen.

Ninety-four, by twenty-three.

One hundred and fifty-eight, by forty-five.

Two hundred and sixteen, by nineteen.

Two hundred and eighty-five, by twenty-eight.

Four hundred and seventy-nine, by eleven.

Eight hundred and forty-six, by fifty-one.

One thousand three hundred and nineteen, by thirty-six.

One thousand nine hundred and fifty-eight, by forty-nine.

Two thousand six hundred and seventy-three, by fifty-seven.

Three thousand four hundred and eighty-two, by thirty-four.

Five thousand six hundred and ninety-seven, by sixty-seven.

Seven thousand five hundred and ninety-three, by fifty-four.

Nine thousand eight hundred and seven, by sixty-eight.

Three hundred and four thousand one hundred and sixty-five,
by twenty-eight.

B. The same operation would be performed on the abacus thus—

- Two hundred and fifty-four, by twelve.
Three hundred and seventy-nine, by seventeen.
Five hundred and forty-eight, by fourteen.
Seven hundred and nineteen, by eighteen.
Nine hundred and nine-eight, by twenty-three.
One thousand two hundred and fifty-one, by thirty-nine.
Two thousand seven hundred and sixteen, by fifty-seven.
Five thousand seven hundred and sixty-four, by nineteen.
Six thousand one hundred and fifty-five, by twenty-five.
Seven thousand three hundred and forty-seven, by thirty-six.
Nine thousand seven hundred and twenty-one, by forty-seven.
Twelve thousand five hundred and sixty-nine, by twenty-one.
Fifteen thousand and seventy-two, by forty-three.
Twenty-five thousand seven hundred and ninety-one, by fifty-four.
Eighty-six thousand four hundred and fifteen, by sixty-nine.
Two hundred and seventy-nine thousand three hundred and twenty-six, by forty-five.
Five hundred and twenty-seven thousand one hundred and ninety-eight, by seventy-two.
Eight hundred and fifty-one thousand six hundred and forty-two, by sixty-five.
One million seven hundred and thirty-eight thousand two hundred and ninety-one, by eighty-six.
Two million nine hundred and ten thousand three hundred and twenty-five, by ninety-four.
Six hundred and ninety-seven thousand two hundred and thirty-one, by seventy-three.
One million six hundred and ninety-five, by sixty-nine.
One million eight hundred and twelve thousand and forty-seven, by seventy-two.
Two million seven hundred and nine thousand eight hundred and thirty-four, by eighty-five.
Five million eleven hundred and ninety-nine, by eighty-one.
Nine million two hundred and eighteen thousand six hundred and fifty-five, by ninety-eight.

Thirteen million seven hundred and two thousand eight hundred and forty-seven, by eighty-six.

Twenty-nine million one hundred and six thousand eight hundred and ninety-one, by fifty-seven.

Seventy-two million sixty-eight thousand five hundred and thirty-nine, by seventy-nine.

One hundred and sixty-five million nine hundred and eighty-one thousand seven hundred and ninety-three, by ninety-five.

C. When the numbers are written in figures, this multiplication is easily done. Having written the multiplier under the multiplicand for convenience of

	54763
	47
	219052
	383341
	2573861

reference, as in the margin, we multiply by 4, putting the figures one step to the left (this is equivalent to multiplying by 40); then we multiply by 7, and add the two products; or we may begin by taking seven times, and then take forty times the multiplicand, for the order is quite immaterial; only, whichever way we do, we must observe to put the figures in their proper places. Neither is it necessary to write the multiplier under the multiplicand, although in general it be convenient to do so.

Perform the following and some of the preceding multiplications by help of figures :—

57 by 17.	693 by 77.	26530 by 47.
93 by 32.	2116 by 95.	789 028 by 67.
87 by 24.	3876 by 57.	3 952 161 by 53.
159 by 26.	5792 by 63.	10 270 396 by 85.
231 by 42.	9827 by 84.	18 700 954 by 98.

A. As we have seen that, in order to multiply a number by ten, it is enough to change each counter of that number into a counter of the next higher rank, so we may also see that, in order to multiply by one hundred, we have only to raise the rank two steps. Thus an acorn stands instead of one hundred peas, and therefore if for every pea we place one acorn, we shall

augment the number one hundred fold. For example, if we have to take the number *five hundred and seventy-three* one hundred times, we represent it by *five peas, seven tares, and three grains*; and then for each pea put an acorn, for each tare a bean, and for each grain a pea, so as to get *five acorns, seven beans, and three peas*, which represent the required product, *fifty-seven thousand three hundred*.

Perform the following multiplications by means of seeds :—
Multiply thirty-seven, by one hundred.
Sixty-eight, by one hundred.
One hundred and fifty-four, by one hundred.
Five hundred and nine, by one hundred.
Two thousand nine hundred and eighty-one, by one hundred.

It is quite as easy to multiply any number by one thousand ; for one bean represents a thousand grains, one acorn a thousand tares, and so on ; so that, by raising the rank of the counters three steps, we may obtain the product of the multiplication by one thousand. Thus, one thousand times *seven hundred and forty-eight* will be represented by *seven chesnuts, four acorns, and eight beans*, so that it must be *seven hundred and forty-eight thousand*.

Perform the following multiplications by means of seeds :—
Multiply twenty-six, by one thousand.
Thirty-nine, by one thousand.
Fifty-two, by one thousand.
Seventy-nine, by one thousand.
One hundred and fifteen, by one thousand.
Eight hundred and forty-five, by one thousand.
Three thousand five hundred and seven, by one thousand.
Eleven thousand and eighty-two, by one thousand.
Twenty-six thousand nine hundred and forty, by one thousand.
One hundred and one thousand two hundred and six, by one thousand.

B. So also, we can multiply any number by one hundred, by one thousand, by ten thousand, &c., by removing its counters on the abacus, two places, three places, four places, &c. higher up on the scale, that is, toward the left hand.

Perform the following multiplications by means of the abacus :—

Multiply fifty-nine, by one hundred.

Seventy-two, by one hundred.

One hundred and six, by one thousand.

One hundred and eighty-one, by one hundred.

Two hundred and fifty-three, by one hundred.

Two hundred and thirteen, by one thousand.

Five hundred and eight, by ten thousand.

Eighty-two thousand seven hundred and twelve, by one hundred.

One hundred and seventeen thousand six hundred and fifty-three,
by one thousand.

Two hundred and five thousand and ninety-eight, by ten thousand.

Six hundred and twenty-seven thousand four hundred and thirty,
by one hundred thousand.

C. And again, when a number is written in figures, we multiply it by one hundred when we remove its figures two steps, by one thousand when we remove its figures three steps to the left, and so on for all multipliers which are represented by unit followed by nought.

Perform the following multiplications by help of figures :—

376 by	100.	637 500 by	10 000.
569 by	100.	5 020 000 by	10 000.
1200 by	1000.	1 276 391 by	100 000.
9361 by	100.	10 000 000 by	100 000.
15701 by	1000.	25 361 000 by	10 000.
70100 by	10 000.	76 905 761 by	100 000.
235 816 by	1000.	100 013 070 by	100 000.

A. Having now learned to multiply by one hundred and one thousand, we are able to multiply by such a number as *three hundred and forty-six*. Suppose that we had to multiply *four*

thousand six hundred and ninety-seven by the above number, we would mark it by the counters, *four beans, six peas, nine tares, and seven grains*; and observing that, in order to multiply by one hundred, we must raise the rank two steps, we take these raised counters three times, making *twelve chestnuts, eighteen acorns, twenty-seven beans, and twenty-one peas*, which stand for three hundred times our number; again, forty times the number will be *sixteen acorns, twenty-four beans, thirty-six peas, and twenty-eight tares*; while six times the number is *twenty-four beans, thirty-six peas, fifty-four tares, and forty-two grains*; so that three hundred and forty-six times the proposed number will be expressed by *twelve chestnuts, thirty-four acorns, seventy-five beans, ninety-three peas, eighty-two tares, and forty-two grains*; which come to the same thing as *one walnut, six chestnuts, two acorns, five beans, one pea, six tares, and two grains*, and stand for the number *one million six hundred and twenty-five thousand one hundred and sixty-two*.

Or we might have shortened each product separately, and thus have had to use fewer counters.

Perform the following multiplications by help of seeds :—
Multiply one hundred and fifty-seven, by one hundred and seventeen.

One hundred and seventy-two, by one hundred and three.

Two hundred and sixty-eight, by one hundred and ninety.

Four hundred and three, by two hundred and fifteen.

Eight hundred and forty-seven, by two hundred and thirty-nine.

One thousand two hundred and sixty-one, by two hundred and sixty.

One thousand five hundred and seven, by three hundred and eight.

Two thousand one hundred, by three hundred and forty-five.

Two thousand nine hundred and seventy-one, by four hundred and twelve.

Four thousand one hundred and sixty-two, by four hundred and ninety-seven.

Six thousand five hundred and forty-eight, by five hundred and thirty-six.

Eight thousand three hundred and forty-seven, by five hundred and twenty-nine.

Nine thousand and sixty-four, by six hundred and eighty-one.

Eleven thousand and thirteen, by eight hundred and forty-five.

B. Such a multiplication is done on the abacus thus :

					0000	00000	00000	00000
						0	0000	00
						000	0000	00000
								0
		0	0000		000000	0		
			0	00000	00000	00000	00000	
				000	00	000	000	
				00	000000	0	000000	00
					000		000	
		0	000000	00	000000	0	000000	00
			0				0	

Perform some of the preceding and the following multiplications by means of the abacus :—

Multiply five hundred and thirty-seven, by two hundred and ninety-five.

Eleven hundred and eighteen, by three hundred and twenty-nine.

Two thousand one hundred and sixty-eight, by three hundred and sixty.

Seven thousand eight hundred and ten, by four hundred and sixty-three.

Twelve thousand nine hundred and fifty-two, by six hundred and eighty-eight.

Eighteen thousand six hundred, by nine hundred and seventy-four.
Thirty-four thousand seven hundred and fifty-two, by one thousand seven hundred and eighty.

C. Again, the above example of multiplication may be done in figures, as in the margin. Here we have begun with the multiplier 6 units, which gives the product 28182. Then we have used the multiplier 40, which gives 187880; and lastly, the multiplier 300, which gives 1409100. However, we have not written the 0 and 00, because the rank of each figure is determined sufficiently by the place in which it is written.

$$\begin{array}{r} 4697 \\ 346 \\ \hline 28182 \\ 18788 \\ 14091 \\ \hline 1625162 \end{array}$$

Now, in examining the actual work, we see at once that we need not trouble ourselves about the three hundred as a multiplier; it is quite enough to attend to the 3, and proceed as if we were multiplying by 3, only we must write the result two steps up on the scale.

In order, then, to multiply by a large number, we multiply by each digit of that number in succession, beginning with the units and removing the product one step to the left; by this means the lowest figure of each product comes exactly under the figure by which we are multiplying.

$$\begin{array}{r} 3287 \\ 2053 \\ \hline 9861 \\ 16435 \\ 6574 \\ \hline 6748211 \end{array}$$

If there should happen to be a blank in the multiplier, we must of course pass two steps to the left, as in the adjoining example.

Again, if the unit place of the multiplier be empty, we must write the 0, in order to preserve the rank of the product, as in this example; and similarly if more of the lower places be blank.

$$\begin{array}{r} 37962 \\ 5270 \\ \hline 2657340 \\ 75924 \\ 189810 \\ \hline 200059740 \end{array}$$

Perform the following multiplications by help of figures:—

1257 by 268.	5460 by 548.	12700 by 815.
1934 by 371.	7892 by 607.	16204 by 930.
2673 by 440.	9052 by 692.	25316 by 1471.

124963 by	1764.	845327004 by	381579.
306720 by	2600.	897652934 by	497316.
519604 by	2955.	927632085 by	512768.
580276 by	3484.	900315427 by	628700.
645300 by	4271.	1285000691 by	631503.
896274 by	4683.	1370937328 by	690008.
900184 by	5207.	1592464017 by	870645.
1307852 by	6348.	1865412793 by	935174.
2735960 by	7006.	2631592706 by	1983549.
3472906 by	7629.	7692053688 by	3794169.
4009925 by	8094.	9270035146 by	5236127.
5693800 by	8739.	23271593260 by	5960900.
9501684 by	19352.	50165473298 by	6373000.
13754062 by	25000.	74532617326 by	7438193.
37862139 by	37516.	8604952743 by	9405821.
59211470 by	54529.	157069389751 by	17729372.
256791438 by	89276.	230058417398 by	24615328.
480519763 by	135791.	498631006357 by	42395621.
690541375 by	179346.	724315988246 by	54932874.
765427192 by	305670.	2980436517385 by	75609358.
730695128 by	369248.	5609378462537 by	84726432.

If the multiplier be the product of two numbers, we may multiply first by the one of these factors and then the product by the other. Thus, instead of multiplying at once by 91, we may first multiply by 13 and then the product by 7; or first by 7, and then the product by 13. Or, if the multiplier be the product of several numbers, we may multiply by these numbers in succession.

Perform the following multiplications:—

5218 by	27.	25701 by	330.	350892 by	1587.
7930 by	54.	26147 by	538.	579673 by	1922.
2785 by	82.	69230 by	678.	872509 by	1998.
8764 by	121.	71489 by	738.	7620568 by	8968.
16321 by	176.	95065 by	939.	11564280 by	9735.
25078 by	256.	127658 by	1243.	29367854 by	84684.

D. We need very often to multiply one number by another, and any one who wishes to become expert in arithmetic ought to practise this operation until he find it easy. It is not enough to do the exercises that are set down ; the learner should not rest satisfied until he cease to experience any hesitation ; after that, he may proceed to consider the following remarks, which are intended to lead him to the acquisition of rapidity and accuracy. Unless, however, he feel himself quite at home in ordinary multiplication, he need not attempt to learn the way to shorten and facilitate the process.

The first kind of shortening which we may notice consists in taking advantage of peculiarities in the numbers with which we have to deal. Thus, in the last example,

2053
3287
14371
57484
6159
6748211

we may make 3287 the multiplier, and, having multiplied by 7, we observe that 28 is just four times seven ; so that instead of multiplying by the 8 and by the 2 separately, we may multiply the last found product by 4. Having thus disposed of the 28, we proceed to multiply by 3, observing to pass two places to the left.

It is very rarely that we are unable to find some such circumstance ; the same example furnishes another instance. Thus, having multiplied first by 7

2053
3287
14371
16424
65696
6748211

and then by 8, in the usual way, we come to 32, which is four times eight, and thus we have only to multiply the last product by 4. In either of these ways we save a line ; the former is the easier of the two.

The student may go over all the preceding examples, seeking for such shortenings.

To take advantage of such circumstances as these, we often break the order of the multiplication. Thus, if the product of 654378 by 567392 were required, we observe that 56 is eight times 7, while 392 is just 7 times 56. On this account we begin with the 7, then multiply this product by 8, placing it a step to the left ; lastly, we multiply this by 7, placing the result four steps to the right. Or, again making 654378 the

654378
567392
4580646
36645168
256516176
371288842176

multiplier, we notice that 9 times 6 is 54, and that 7 times 54 is 378.

Keeping this plan in view, the student may find exercise in again going over the preceding examples.

Again, we sometimes find it convenient to split a figure into two parts: thus, if the multiplier were 39767, we might take first the three, its triple 9, its double 6, and then the two 7's: or, dividing the middle 7 into two parts, 2 and 5, we might imagine our multiplier to be composed of 39200, 560, and 7.

Perform the following multiplications, using the methods for shortening:—

7065159 by	543646.	47596201 by	2651863.
7319642 by	572819.	52172835 by	3765083.
8340517 by	712546.	63549287 by	4593672.
9705483 by	813649.	81368270 by	5679267.
12709862 by	917495.	256198435 by	6430857.
15371968 by	962460.	543691863 by	7159319.
17536207 by	1053579.	837205684 by	9726327.
19476285 by	1335462.	437486546 by	72285369.
24710694 by	1440624.	9537054658 by	15367953.
26095837 by	1763958.	27655737942 by	738193681.
38054726 by	2107328.		

E. After having practised these methods for shortening the process, the student may proceed to try to obtain the result without writing down the intermediate steps. When the numbers are under 100, this is not difficult; thus, if we have to multiply 47 by 64, we observe that 4 times 7 units make 28 units; we therefore write down the 8, keeping the 2 tens in hand: next, we see that we have 4 times 4, 16 tens, and also 7 times 6, 42 tens, in all 58 tens, which, with the two brought up, make 60 tens; we then write the 0, and keep the 600 in hand: lastly, we have 6 times 4, 24 hundreds, which, with the 6 in hand, make 30 hundred, and thus our product is 3008.

The student ought to practise this operation: in general, he will find it easier to take the two cross products first.

Perform the following multiplications at once :—

23 by 35.	69 by 45.	95 by 83.	78 by 61.	89 by 98.
27 by 28.	81 by 47.	89 by 96.	56 by 98.	73 by 81.
34 by 26.	48 by 87.	75 by 54.	48 by 79.	90 by 56.
49 by 35.	76 by 63.	97 by 88.	89 by 66.	96 by 87.
53 by 74.	85 by 57.	75 by 93.	74 by 93.	85 by 97.
89 by 25.	92 by 64.	46 by 54.	99 by 84.	99 by 99.
73 by 55.	57 by 68.	86 by 59.	73 by 86.	

When the multiplicand is a large number, and the multiplier of two places, the product can be written down at once without much trouble. Yet, in truth, it is more useful to know how this can be done than to be expert in doing it, since the chance of error is considerable.

Let it be required to multiply 57839, by 37. We take first the product of the units, which is 63, giving 6 in hand ; this, with 7 times 3, makes 27, and with 3 times 9 (other 27), 54 ; 5 in hand, with 7 times 8, 61, and 3 times 3, 70 ; 7 in hand with 7 times 7, 56, and 3 times 8, 80 ; 8 in hand with 7 times 5, 43 and 3 times 7, 64 ; 6 in hand with 3 times 5, 21.

$$\begin{array}{r}
 57839 \\
 \times 37 \\
 \hline
 2140043
 \end{array}$$

Perform the following multiplications at once :—

351 by 15.	5173 by 29.	39158 by 68.
376 by 23.	6782 by 65.	559382 by 76.
427 by 27.	15961 by 54.	621493 by 58.
578 by 36.	27025 by 46.	737615 by 64.
591 by 38.	35892 by 67.	891467 by 79.
709 by 43.	47165 by 49.	947652 by 83.
795 by 59.	68573 by 56.	1173826 by 84.
934 by 62.	84907 by 73.	1359760 by 93.
1255 by 67.	96430 by 65.	2756394 by 89.
1798 by 72.	127691 by 59.	4751308 by 95.
3516 by 37.	156382 by 71.	7759864 by 98.

Even when both factors are large numbers, the product can be written down at once : this way of working, however, is to

be regarded as a matter of curiosity, on account of the great risk of mistake. It is merely an extension of what has been explained ; thus, in the adjoining example the only product which can give units is that of 6 units by nine units. But tens result both from the 6 times 2 tens and from the 9 times 8 tens ; altogether, then, we have 89 tens ; hundreds result from the three products, 6 times 7, 8 times 2, and 3 times 9 ; altogether, then, we have 93 hundreds : thousands are obtained from 6 times 4, 8 times 7, 3 times 2, and 7 times 9, making in all 158 : tens of thousands are got in three ways, 8 times 4, 3 times 7, and 7 times 2 ; in all 82 : hundreds of thousands from 3 times 4 and 7 times 7 ; in all, 69 : lastly, millions result from the single product 7 times 4, making, with the 6 already in hand, 34.

Perform the following multiplications at once :—

421 by 360.	1050 by 726.	7983 by 1989.
534 by 423.	2563 by 978.	9075 by 2563.
765 by 537.	3594 by 1321.	12738 by 3750.
928 by 652.	5692 by 1786.	57864 by 17649.

By help of a little contrivance, we can render this proceeding very clear, and tolerably free from the chance of mistake ; yet even this might be regarded as curious rather than useful, were it not that we shall have to use the same principle at a future stage of our proceedings. This contrivance consists in writing the multiplier on the lower edge of a slip of paper, but in inverted order, that is, the units being at the left hand. Thus we should write :—

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Having prepared this, we write the multiplicand as usual, and place the bit of paper so that its units shall be above the units of the multiplier. The product is 54 units. To get the tens, we shift the slip of paper, so that its units may be above the tens, and, of course, its tens above the units of the multiplicand, when we find the products 12 and 72. To get the hundreds, we slide the paper along another

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step, so as to show the products 42, 16, and 27, and continue this course until the last figures be brought over each other. One or two trials will render this operation clear enough. The student may apply it to any of the preceding examples.

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F. But of all the modes of performing multiplication, the most advantageous is that in which we begin with the numbers of the highest denomination, and descend to the lower ones. When a sum of money or a quantity of goods is mentioned, we give our attention first to the highest denomination: thus, if there were *two thousand five hundred and forty-three gallons of oil at two shillings the gallon*, we would not try to form an idea of the price by first counting the three odd gallons, but would at once think of the two thousand, and then of the five hundred gallons. Our very system of naming numbers exhibits this; for we do not say three-and-forty and five hundred and two thousand gallons, as the Arabs do. We begin mentally with two thousand gallons at two shillings the gallon; and if we could devise a method of multiplying from the left hand, our work would follow the natural order of our ideas.

On attempting to perform multiplication in this way, we at once encounter the difficulty of knowing how much is to be brought up from the lower figures; as we have not yet multiplied them, it would seem to be impossible to know this. Like many other difficulties, this one vanishes before a little inquiry.

The easiest of all multiplications is that by two. Let us try to double a number, say 7349782, beginning at the highest denomination. The double of the first figure is 14, but there may or may not be unit brought up from the lower figures. Now we readily observe, that if the next figure had been 5 or upwards, its double would have given 1 to the place of millions; but it is less than 5, so there is nothing to be brought up, and without any hesitation we write down the 14. The double of 3 is 6, and the next figure is below 5; so we write the 6 also. The double of 4 is

8, but the next figure is more than 5, so the 8 becomes a 9. The double of 9 is 18, but the first figure of it has already been used, so that we have now only to attend to the 8, which, since the next figure is above 5, becomes a 9. In this way we can proceed until the multiplication be finished. There is not, then, the slightest difficulty in the way of doubling from the left hand.

Before proceeding to multiply by any other number from the left, the learner ought to practise doubling. For this he may write a number at the top of a blank page, double it, double the result, and thus go on doubling until he have rendered himself quite master of the operation. An hour's work at the most will be sufficient for this.

Having thus easily mastered multiplication by 2, let us try to multiply the same number by 3 from the left.

Here we may have 0, 1 or 2, to bring up, and the 7349782 difficulty is to tell which. Three times 7 make 21; 220

I may safely enough write down the 2, but am not so sure about the 1. The next figure is three, and three times three make 9, which does not give anything to the place of millions; however, if anything come up to the 9 it will become a 10, and effectively the next figure 4 does give 1, so that the 9 becomes 10, and the 1 a 2.

Leaving our example for a moment, let us look closely into this matter. If the succeeding figure, instead of being a 4, had been a 3, the triple would again 7333329 have been 9, and the 99 may become 100 by carrying from the place still lower. 21999987 Suppose that that place also had been occupied by a 3, we should have had another 9, and should still have been in doubt; nor can this doubt be removed until we come to a figure not 3; when that figure is less than 3, as in the adjoining example, the 9's are not augmented, and nothing is carried to the 7 333 334 21. But when, as here, the different figure is 22 000 002 more than 3, the 9's are all converted into 0's, and 1 is carried to the 21, making it 22. Thus the turning-

point is easily recognised ; whenever the succeeding figures exceed 3333, carried on indefinitely, there is 1 carried ; when these figures are less than 3333, &c. there is nothing carried. But sometimes we may have to carry 2 ; the figures 9, 8, 7, give 2, but 6 only gives 1, since three times 6 is 18. Yet if the following figure should give 2, this 18 would become 20, or if that figure happen to be 6, the triple would be 198, a number so near 200 that a 2 carried to it would turn the scale. Hence, when the succeeding figures are less than 6666, &c., as in the margin, there is only 1 brought up ; but when they exceed 6666, as in the second example, the 19998 becomes 2000, &c., and 2 is brought up.

7 666 659
22 999 977
7 666 667
23 000 001

Now, although this seem to be a long matter, it is in truth a very easy one, and attention to it enables us to multiply any number by 3 from the left hand without difficulty, since we can recognise at once what the following figures send up.

The student would do well to practise multiplication by 3 in the same manner as I have recommended for 2, until he find it easy. It is not advisable that he try multiplication by 4, until he have become quite familiar with the preceding.

Although logically out of order, I may here point out to the student who is revising his course of study, and such only should be perusing the articles F, that the above 333, &c., 666, &c., are the values of one-third and two-thirds in decimal fractions.

We come now to multiplication by 4. Here we may have to bring up 0, 1, 2, 3, and our business is to find out which of these. Now, four times 25 make 100, four times 50 make 200, and four times 75 make 300, so that, if the succeeding figures be less than 25, we have nothing to bring up ; if they be 25 and under 50, we have to bring up 1 ; if 50 and under 75, we have 2 ; and if the succeeding figures be 75 or upwards, we have to bring up 3. Multiplication by 4 from the left hand is thus not more difficult than multiplication by 3. The learner should proceed to practise in the same way as before. Thus, he may

multiply 314159265359 thirty times successively by 4, or he may set any other example for himself and continue until he find himself quite master of the process.

Multiplication by 5 is quite easy, since 5 times 2 make 10, 5 times 4 make 20, 5 times 6 make 30, and 5 times 8 make 40, so that the figure to be brought up is always the half of the subsequent figure if that figure be even, and just less than the half of the subsequent figure, if it be odd. However, we seldom think of multiplying by 5; we prefer to take the half of the number imagined to be 7830257 transposed one step farther to the left. It is then 39151285 as easy to multiply by 5 as to double. The 195756425 rationale of this operation is, that by transposing one step to the left, we multiply by 10; and that, therefore, the half of the number so transposed, must be the product of the original number by 5.

Our next multiplier is 6; and as we are getting among what may be difficulties to a learner, I have again particularly to entreat the student not to proceed onwards until he have acquired great expertness in the preceding multiplications. Since 6 is just the double of 3, the half of that which gives 1 when we are multiplying by 3 must give 1 when we are multiplying by 6. Now, the half of 3333, &c., is 16666, &c.; and therefore we conclude, that whenever the succeeding figures are more than 166666, &c., we must have to bring up 1. This we see at once from the subjoined examples. Similarly, 33333, &c., is the limit at which we begin to carry 2.

$$\begin{array}{r}
 1666666 \text{ by } 6 \\
 9999996 \\
 \hline
 1666667 \text{ by } 6 \\
 10000002 \\
 \hline
 \end{array}$$

Since 6 times 5 make 30, we have to carry 3 when the subsequent figure is 5; again, 66666 must be the limit at which we begin to carry 4, and 83333 (the half of 16666 removed one step) that at which we begin to carry 5.

From this it is clear that it is a little troublesome at the outset to multiply by 6 from the left hand; after a little practice it becomes easy.

Example, multiply 434294481903 thirty times successively by 6.

We now come to 7, the most troublesome of all the multipliers. If the student have found much difficulty in coming thus far, I would advise him to skip the 7 and proceed at once to 9, then to 8, reserving multiplication by 7 to the last. For the sake of regularity, I shall take them in the order of their magnitudes.

If we can discover what just gives 1, the double, triple, &c. of this will give 2, 3, &c. Now, if the succeeding figure be 2, we have the product 14, which is rather above our limit; the turning-point is then below 2; but if the subsequent figure be only 1, there is 0 to be brought up. Should the next figure give a 3, the 7 would be changed into 10; and in fact 7 times 14 make 98, which is very nearly 100. If the 14 were followed by 3 there would be 1 to come up, since 7 times 143 make 1001. Proceeding in this way we find that 7 times 142857 make 999999, which only needs unit in the last place to send up a 1; and thus it seems that if the subsequent figures be above 142857 142857 142857, &c., carried ever so far, 1 must be brought up. This the advanced student will at once recognise as the development of the fraction one-seventh in decimals.

The order of these figures is remarkable. Taking them in pairs, the first pair 14, is the double of 7; 14
the next, 28, is the double of 14; and the 28
third pair, 57, is the double of 28, with 1 56
brought up from its own double. This is 112
seen in the adjoining calculation, in which 224
we begin with 14, place its double two steps 448
to the right, the double of that again two &c.
steps to the right, and so on. 14285714285 &c.

Having now got that value of the succeeding figures which just give 1, we may take its double, triple, &c., for the purpose of finding what will give 2, 3, &c. It is remarkable that these

products exhibit the same circulation of figures, merely beginning at different places. Thus—

142857142857, &c., give 1
 285714285714, &c., give 2
 428571428571, &c., give 3
 571428571428, &c., give 4
 714285714285, &c., give 5
 857142857142, &c., give 6

so that *to multiply by 7 from the left hand is not quite so difficult as it appeared at first sight.*

When multiplying by 8 from the left, we must keep our eye upon the three subsequent figures. If these be less than 125, we have nothing to bring up, since eight times 125 is just 1000; if the three subsequent figures be from 125 to 249, we have to carry 1, from 250 to 374, to carry 2, and so on, the successive turning-points being 125, 250, 375, 500, 625, 750, 875, which are just the successive multiples of 125.

Lastly, when multiplying by 9, if the succeeding figures be above 11111, &c., we have 1 to bring up; if above 222, &c., we have 2 to bring up; and so on, multiplication by 9 being very easy.

However, it is still more easy to find nine times any number in another way. Let it be required, for example, to take 47827796 nine times. We imagine a 0 to precede the first, and another 0 to follow the last figure; then we subtract each figure from the following figure, and 47827796 write the remainder below it. Thus, 0 from 4 430450164 gives 4; 4 from 7 leaves 3; but we do not take 7 from 8, because the 8 is followed by a figure less than itself; we say 7 from 7 (one less than the 8) leaves 0; 8 from (not 2, but) 12 leaves 4; 2 from 7 (for the 7 followed by another 7 is again followed by a greater figure) leaves 5; 7 from 7 leaves 0; 7 from 8 (because the 6 is less than 9) 1; 9 from 15 (because the imaginary 0 is less than 6) 6; and lastly, 6 from 10 leaves 4.

It may be remarked that, when multiplying from the left hand, our attention is directed mainly to the last figure of each product. Thus, if we have 7 times 6, we attend more to the final figure 2 than to the 42, because the 4 has already been disposed of.

The whole of the results which we have obtained may be exhibited in the following table—

When multiplied by—								
2	3	4	5	6	7	8	9	
5	3333, &c. 6666, &c.	25 50 75	2 4 6 8	1666, &c. 3333, &c. 5000 6666, &c. 8333, &c.	142857, &c. 285714, &c. 428571, &c. 571428, &c. 714285, &c. 857142, &c.	125 250 375 500 625 750 875	111, &c. 222, &c. 333, &c. 444, &c. 555, &c. 666, &c. 777, &c. 888, &c.	Carries 1 .. 2 .. 3 .. 4 .. 5 .. 6 .. 7 .. 8

Having learned how to multiply from the left by every single figure, it is easy to multiply by a large number. A single example is enough to render the matter quite clear. Thus, if the product of 387917 by 29073 be required, we begin to multiply by the highest figure 2, and we see that the first figure of the product is 7; and since the multiplier 2 is four removes from the units' place, this product 7 must be written four places to the left of the 3. When we come to the second multiplier 9, and observe that 9 times 3 make 34, we write this product 34 three places to the left—that is, the units' place of the 34, so that the 3 of the 34 is actually four places to the left.

$$\begin{array}{r}
 387917 \\
 29073 \\
 \hline
 775834 \\
 3491253 \\
 2715419 \\
 1163751
 \end{array}$$

We have seen that multiplication by 5 is equivalent to division by 2, the figures being placed one step to the left. This principle may be carried farther; for since 4 times 25 is 100, instead of multiplying by 25 we may divide by 4, placing the quotient two steps higher on the scale, as in the annexed example, where it is necessary to imagine two zeroes to be appended to the multiplicand when performing the divi-

$$\begin{array}{r}
 3871943 \times 25 \\
 96
 \end{array}$$

sion. Similarly, since 125 is the eighth part of a thousand, we may, instead of multiplying by 125, shift the multiplicand three steps up, and divide by 8, as in the annexed example.

$$\begin{array}{r} 289517 \times 125 \\ 36189625 \end{array}$$

We may even carry this further, and for multiplication by 625 substitute division by 16, removing the quotient four steps ; but this exchange would be attended with little or no advantage.

E. When we have to compute the continued product of several numbers, we first multiply one by another, the product by a third, that product by a fourth, until all the numbers have been multiplied together : but it is quite possible to obtain the continued product at once. Although the direct operation be not, in general, advantageous in practice, the ability to perform it is of great utility in enabling us to understand some of the higher processes connected with the extraction of roots. The student who wishes to become a thorough arithmetician would, therefore, do well to study closely the following remarks.

If the product of two numbers, such as 47 and 83, be required, we at once see that there must be 21 units, 3 times 4, and 7 times 8 tens, and lastly 8 times 4 hundreds, as shown in the adjoining work, in which the parts of the entire product are placed separately. But if we seek the continued product of three numbers, such as 47, 83, and 92, it is not quite so easy to see how all the partial products must go to make up the entire product ; yet, with a little thought, we may make it out. In the first place, the only product which can give units is that of the three figures in the units' place ; we have therefore 42 units. Secondly, tens can only be given by the product of one figure in the place of tens by the units of the other two numbers. Hence the tens must be 4. 3. 2 ; 8. 7. 2, and 9. 7. 3. Thirdly, hundreds can only result from the product of two figures in the place of tens, by the units of the remaining number. Hence the hundreds must be 4. 8. 2 ; 4. 9. 3, and 8. 9. 7. Lastly,

$$\begin{array}{r} 47 \\ 83 \\ \hline 21 \\ 12 \\ 56 \\ 32 \\ \hline 3901 \\ \\ 47 \\ 83 \\ 92 \\ \hline 42 \\ 24 \\ 112 \\ 189 \\ 64 \\ 108 \\ 504 \\ 288 \\ \hline 358892 \end{array}$$

the only product which can give thousands is that of the three figures in the place of tens, viz. 4. 8. 9 ; wherefore the continued product of the three numbers is 358892.

EXAMPLES.

Multiply at once—

43 × 76 × 85.	73 × 69 × 82.	76 × 84 × 98.
72 × 64 × 91.	99 × 47 × 89.	81 × 96 × 75.
89 × 91 × 73.	95 × 86 × 92.	90 × 97 × 88.
58 × 74 × 95.		

When the continued product of four such numbers is wanted, the work, though necessarily longer, is carried on in the same way. Let it be proposed to multiply at once the four numbers, 94, 57, 61, 23.

For the units, we have the continued product,
4. 7. 1. 3 = 84.

For the tens, we have the continued product of each figure in the tens' place into the units of the three other numbers, viz. 9. 7. 1. 3 ; 5. 4. 1. 3 ; 6. 4. 7. 3, and 2. 4. 7. 1.

For the hundreds, we have the product of each pair of figures in the tens' by the supplementary pair of figures in the units' place ; viz. 9. 5. 1. 3 ; 9. 6. 7. 3 ; 9. 2. 7. 1 ; 5. 6. 4. 3 ; 5. 2. 4. 1, and 6. 2. 4. 7.

For the thousands, we have the continued product of each three of the tens into one of the units, viz. 9. 5. 6. 3 ; 9. 5. 2. 1 ; 9. 6. 2. 7 ; and 5. 6. 2. 4.

Lastly, for the tens of thousands, we have the continued product of the four figures in the tens' place, viz. 9. 5. 6. 2.

This work seems long, and the arrangement of it appears embarrassing ; yet, with a little attention, the student may soon render himself familiar with it. On no account can this process be recommended when our object is merely to get the

numerical result : its use is to show how the last result is compounded from the data.

EXAMPLES.

Multiply at once—

$$\begin{array}{ll} 27 \times 52 \times 38 \times 46. & 68 \times 47 \times 59 \times 72. \\ 35 \times 49 \times 28 \times 50. & 87 \times 63 \times 38 \times 77. \\ 48 \times 29 \times 33 \times 46. & 50 \times 75 \times 48 \times 90. \end{array}$$

If there be a greater number of factors, the parts become much more numerous, being doubled in number by the advent of another factor. The continued product of five two-place factors would thus consist of 32 parts. Although numerous, their arrangement is not very difficult, and the student would do well to exercise himself in the work.

EXAMPLES.

Multiply at once—

$$\begin{array}{ll} 17 \times 23 \times 32 \times 41 \times 25. & 47 \times 50 \times 93 \times 84 \times 91. \\ 39 \times 26 \times 54 \times 38 \times 45. & 84 \times 73 \times 69 \times 42 \times 95. \\ 76 \times 58 \times 60 \times 49 \times 52. & \end{array}$$

If the factors contain more than two figures, the number of parts becomes excessive, and the prosecution of the subject becomes both laborious and, at this stage of our attainment, unprofitable. For our subsequent purposes, it is enough to regard such numbers as composed of tens and units ; thus 375 is regarded as the sum of 37 tens and 5 units.

C. Hitherto we have considered the multiplication of numbers, but very often we have to multiply quantities. Those quantities, however, are expressed by numbers, and the actual operation comes to be the multiplication of those numbers.

If I have to repeat seventeen shillings five times, I obtain the result on multiplying the number seventeen by five, and then taking a shilling for each unit of the product ; thus, I find that five times seventeen shillings are eighty-five shillings. Here seventeen shillings is the *multiplicand*, five the *multiplier*, and eighty-five shillings the *product*. It is obvious that the product

must be of the same nature as the multiplicand ; five times seventeen shillings cannot be eighty-five ounces. It is also obvious that the multiplier must be a pure number ; it cannot possibly be a quantity. For example, five yards of cloth, at seventeen shillings the yard, come to eighty-five shillings ; nevertheless, this eighty-five shillings is not the product of seventeen shillings by five yards, it is not five yard times seventeen shillings, but simply five times seventeen shillings.

CHAPTER V.

ON DIVISION.

C. The word *divide* is from the Latin verb *dividuo*, to separate or put asunder, and is used by arithmeticians to denote indiscriminately two operations widely different in their natures and in their results. The figurate portions of these two processes closely resemble each other, and their resemblance has caused, with many, a confusion of ideas.

A. If a merchant have a quantity of grain which he wishes to put into sacks of a certain size, he naturally desires to know how many sacks must be prepared. In order to find this out, the most direct method would be to cause the grain to be measured or *meted* out with a measure containing a sackful. But if he knew how many bushels of grain he had, and how many bushels go to a sackful, he could avoid all this measuring over again. A dealer has seventy gallons of oil which he wishes to put into casks, each holding thirteen gallons; how many casks are needed? To find this out let us lay down seventy counters of any kind to stand for the seventy gallons, and then let us *tale* or count these out in thirteens; there are five thirteens and five over, so that the dealer must prepare five thirteen-gallon casks, and another cask to hold five gallons. Again, a grocer has to send away eighty-three loaves of sugar, and he has waste boxes that hold five loaves each; how many boxes must he order from the cellar? On laying out eighty-three counters and *taling* these in fives, we find that there are sixteen fives with three over; therefore we get sixteen boxes, and a smaller box to hold three loaves.

All of such problems, when put in an arithmetical form, require the repeated subtraction of one number from another until there be no remainder, or a remainder less than the subtrahend: they are all analogous to the question, how many times does seventy-one contain eleven. On trial we find that seventy-one contains eleven six times, with five over.

This process is called *division*; seventy-one is the *dividend* (to be divided); eleven is the *divisor*, or *quota*; six is the *quotient*; and five the remainder.

In this kind of division there may or may not be a remainder; but there is another kind of division which does not admit of any remainder.

Thus, if seventy-one gallons of oil had to be divided among eleven persons, there must be nothing left over, the whole must be divided. Now, if we give six gallons to each person, there still remain five gallons undivided, so that the share is six gallons and the eleventh part of five gallons.

In the case of oil we can contrive some way of dividing the five gallons; but if the question had been to part seventy-one canaries among eleven boys, we should have been quite unable to make it out. Thus, while the first kind of division, which may properly be called *meting* or *taling*, can always be effected, the second, which we may name *sharing* or *parting*, is sometimes impossible.

C. If the question were, having sixty-five oranges, how many boys can I supply with five oranges each, the answer is, thirteen boys. Here sixty-five oranges is the dividend, five oranges is the quota, and thirteen the quotient.

But if the problem had been, divide sixty-five oranges among thirteen boys, the sixty-five oranges would still have been the dividend, thirteen still the quotient, and five oranges still the quota.

In the first question the dividend and quota are known, and the quotient sought for; in the second question the dividend and quotient are known, the quota sought for.

A. When the numbers are small any counters may answer.

EXERCISES.

Tale out sixty-nine in eights.

How often does forty-seven contain nine?

How often does seventy-two contain seven?

How many elevens are contained in ninety-three?

Tale out eighty-four in tens.

How often does fifty-six contain thirteen?

How often does ninety-nine contain eleven?

Tale out eighty-one in nines.

How many seventeens are contained in eighty-seven?

In fifty-five how many sixes are there?

What is the seventh part of fifty-six?

What is the third part of thirty-nine?

What is the fifth part of ninety?

Divide eighty-three into four equal parts.

Divide seventy-two into eight equal parts.

Divide ninety-nine into eleven equal parts.

Divide forty-two into seven equal parts.

Divide forty-three into six equal parts.

Divide one hundred and twenty-one into eleven equal parts.

Divide ninety-seven into five equal parts.

Divide fifty-seven into nineteen parts.

Divide one hundred and sixteen into four equal parts.

But when the numbers are large, we endeavour to shorten the process by employing graduated counters. Thus, if we have to tale out the number *four hundred and seventy-five* in eights, we may represent it by *four peas, seven tares, and five grains*. Instead of taling out *eight* grains at a time, we may count out *eight* tares—that is, *ten times eight grains* at once; or even we may take eight peas, which would be one hundred times eight grains. However, in the present example, there are not eight peas; so we shall change what peas there are into tares, and count out eight tares at a time, laying aside one tare of each group as a counter for it. We find five times eight tares (that

is, fifty times eight grains) with seven tares over. These seven tares are now to be changed into seventy grains, and out of the seventy-five grains we count nine eights with three over. In this way we find that *four hundred and seventy-five* contains eight fifty-nine times, and three over.

EXAMPLES.

How many sixes are contained in one hundred and thirty-two ?
 How many tens are contained in two hundred and thirty-nine ?
 How many eights are contained in one hundred and sixty-eight ?
 Divide one hundred and thirty-two into eleven equal parts.
 Divide three hundred and twenty-four into nine equal parts.
 How often does four hundred and five contain thirteen ?
 How often does three hundred and twenty contain eight ?
 Divide five hundred and seventy-six into twelve equal parts.
 Divide six hundred and eighty-seven into fourteen equal parts.
 How many twelves are contained in five hundred and ninety-one ?

B. The same operation may be performed on the abacus; and after what has been already gone over the learner need experience no difficulty in arranging the work himself.

As neither the graduated counters nor the abacus are now needed, except for the purpose of explaining the principles of our work, I shall discontinue the systematic use of them, recommending the beginner to return to the first chapter to peruse all the articles marked **A** and **B**, and after that to read also those marked **C**.

C. When the numbers are represented by figures, the process of division is readily carried on. Thus, if we have to ascertain how many times *seven* is contained in *nine hundred and fifty-eight*, we may write the subtrahend (or quota), and the minuend (or dividend), as in the margin, and proceed thus: 7 is contained in 9, once $7 \overline{) 958}$
 with two over (that is, in 900, one hun- 136, with 6 over.
 dred times, with two hundred over):
 the two over in this rank are equivalent to twenty in the lower,

and 7 goes in 25 three times with four over (properly in 25 tens, thirty times with four tens over); lastly, 7 goes in 48 six times, with six over; and thus we find that 958 contains 7, 136 times, with 6 over.

Again, if the question had been, what is the seventh part of nine hundred and fifty-eight gallons? we should have proceeded in this way. The $7 \overline{) 958}$ gallons. seventh part of 9 (that is, of 9 hundred) is $136\frac{2}{7}$ gallons. 1 with 2 over (that is, one hundred with 2 hundred over); the seventh part of 25 (that is, of 25 tens) is 3 (that is, three tens), with 4 over; the seventh part of 48 is 6, with 6 over, so that the share wanted is 136 gallons, and the seventh part of six gallons, which we write $\frac{6}{7}$, to show that the remainder, 6 gallons, has still to be divided.

From these examples we see that the actual calculation for the one question is almost identic with that for the other, and that therefore the two fall very naturally to be treated together; yet we must be careful not to confound the true characters of the two operations.

EXAMPLES.

How often is 4 contained in 123?

How often is 6 contained in 192?

What is the 8th part of 640?

Divide 477 into 9 equal parts.

What is the 10th part of 591?

How often is 8 contained in 736?

Divide 854 into 6 equal parts.

Divide 873 into 9 equal parts.

What is the 5th part of 899?

How often is 7 contained in 642?

Divide 872 into 8 equal parts.

Divide 938 into 6 equal parts.

Divide 1053 into 9 equal parts.

Divide 1011 into 10 equal parts.

Divide 1272 into 8 equal parts.

If the divisor be large we can hardly trust ourselves to keep the remainders in mind, and it becomes expedient to write them, and even the multiples, down. Let it be required to divide *three hundred and twenty-four thousand eight hundred and forty-nine*, by *four hundred and thirty-seven*. We write the divisor and dividend near each other, so as to be clearly seen, and usually prepare a place for the quotient, to the right of the dividend; but this is a matter of mere taste or convenience.

$$\begin{array}{r|l|l}
 437 & 324849 & 743 \\
 & 3059 & \\
 \hline
 & 1894 & \\
 & 1748 & \\
 \hline
 & 1269 & \\
 & 1311 & \\
 \hline
 & 158 &
 \end{array}$$

437 does not go once in 324 (that is to say, not one thousand times in 324 thousand; but it goes some 7 times in 3248 (that is, 700 times in 3248 hundreds). On multiplying 437 by 7 we obtain the product 3059 (properly 437 by 700, 305900), which we place below the dividend; on subtracting, the remainder is 189—that is, 189 hundreds, or 1890 tens, which, with the 4 tens, make 1894 tens; we therefore bring down the next figure 4, and proceed in the same way until we have exhausted the dividend.

Sometimes, as in the adjoining example, the remainder after the next figure is annexed to it, may be less than the divisor; in that case we write a zero in the quotient, and bring down the succeeding figure; and we may even have to repeat this until the new dividend contain the divisor.

$$\begin{array}{r|l|l}
 863 & 4369807 & 5063 \\
 & 4315 & \\
 \hline
 & 5480 & \\
 & 5178 & \\
 \hline
 & 3027 & \\
 & 2589 & \\
 \hline
 & 438 &
 \end{array}$$

Although, while actually computing, it appear to be needless to keep note of the rank of the figures on which we are operating, the learner must not imagine that it can ever be neglected. The beautiful regularity of the decimal notation enables us to dispense with the ideas of millions, or thousands, as we are working, but only on the condition that we carefully observe the succession of the ranks.

EXAMPLES.

Divide—

3564 by 132.	57321 by 372.	197256 by 672.
4867 by 157.	69280 by 433.	225951 by 751.
5681 by 247.	97346 by 467.	317017 by 777.
13950 by 279.	173060 by 509.	509292 by 846.
24187 by 361.	156273 by 617.	652152 by 937.
1278468 by 964.	176498253 by 17605.	
1569672 by 1521.	200137945 by 19361.	
2657142 by 1764.	427846320 by 23760.	
3495642 by 2391.	671531427 by 65724.	
21148055 by 691.	976103948 by 73105.	
39652317 by 3672.	7844815168 by 67481.	
22228488 by 4278.	9631729318 by 95731.	
49621875 by 6358.	68294682957 by 136481.	
70621597 by 7236.	173096253161 by 239064.	
81536194 by 8270.	852761304268 by 318257.	
120064137 by 12637.	328962064200 by 539275.	
	3516429751734 by 760913.	
	6388057291165 by 917562.	
	9056732157056 by 3165173.	
	987654321987654321 by 123456789.	

D. Having now shown how to perform division in what is usually called *the long way*, I proceed to point out, to those who wish to become expert computers, the principal abbreviations which may be used.

If the divisor consist of only two figures, the work can be arranged concisely by leaving two lines empty below the dividend, the upper line to receive the units, the under one to receive the tens of the remainders. Thus, in the annexed example, we proceed by saying 7 times 37 are 259 (mentally), leaving 30 over; this remainder we write below, so that the next dividend, 307, may be read

$$\begin{array}{r|l}
 37 & 28977512 \\
 & 01682 \\
 & \hline
 & 3122 \\
 & \hline
 & 783176
 \end{array}$$

obliquely upwards; then 8 times 37 are 296, leaving 11, and so on to the end. But this arrangement can only be used by those who are tolerably expert mental computers.

Perform the following divisions by the above method :—

406813 by 13.	96502843 by 47.	93166783 by 75.
627152 by 23.	15360373 by 49.	136071592 by 79.
915376 by 27.	17462538 by 52.	561094368 by 78.
6534121 by 29.	27039945 by 56.	710581623 by 83.
7215369 by 36.	39157261 by 63.	2911658730 by 87.
8017429 by 43.	71618300 by 71.	684197365 by 89.
918653219586042 by 97.		

E. Even with a divisor consisting of three places the same method may be followed by experienced computers.

EXAMPLES.

146766 by 122.	536921 by 247.	7961325 by 617.
215501 by 143.	960573 by 279.	13865193 by 645.
391125 by 175.	1365984 by 381.	27950336 by 738.
416325 by 213.	2683754 by 426.	650291834 by 897.

D. Division is indicated by the sign $\div$, which is read *divided by*; thus, $91 \div 7$ means the quotient obtained on dividing 91 by 7.

When the divisor is the product of two numbers, we may divide first by one of these numbers, and then the quotient so obtained by the other; thus, to divide by 51 gives the same result as to divide by 3, and again to divide the quotient by 17.

EXAMPLES.

3542 by 14.	16263 by 39.	18018 by 77.
5122 by 26.	9828 by 46.	26226 by 62.
8074 by 22.	32674 by 34.	52026 by 58.
2640 by 33.	29013 by 57.	35805 by 93.
3570 by 35.	11937 by 69.	

The same principle may be applied when the divisor is the product of several numbers.

EXAMPLES.

1260 by 12.	7280 by 52.	43764 by 84.
2556 by 18.	8442 by 63.	91488 by 96.
1540 by 28.	48444 by 66.	90405 by 147.
5922 by 42.	41566 by 78.	320866 by 182.

CHAPTER VI.

ON PRIME AND COMPOSITE NUMBERS.

C. WE have seen in the preceding chapter many cases of numbers which do not contain other numbers exactly. Thus, on trying to divide 35 by 3, we find a remainder 2, and discover that the number 35 cannot be divided into 3 equal parts. If the question were to divide 35 yards, or 35 ounces, into 3 equal parts, we readily see that each part must consist of 11 whole yards, and the third part of the remaining two yards, or of 11 ounces, and the third part of the remaining two ounces; and in such cases the division is quite possible. But for all that, the division of the *number* 35 into three equal parts is quite impossible.

Yet although the number 35 be indivisible by 3, it is divisible by 5. But there are many numbers which cannot be divided at all; thus it is impossible to divide the number 37 into equal parts (other than into 37 units), as we may easily prove by trial. Such indivisible numbers are called *prime numbers*, or, more shortly, *primes*. Numbers, again, which are divisible, must be the product of other numbers; on this account they are called *composite numbers*; thus 35 is the product of 7 times 5; it is said to be a composite number.

Now it is of great use, in many calculations, to be able to tell whether a number be prime or composite, and if composite, by what numbers it may be divided; and many of the most profound algebraists have endeavoured to discover some means of distinguishing prime from other numbers, but hitherto their labours have been unsuccessful, and we have no other course

left than that by trial, which, when the number is large, becomes exceedingly tedious.

The method proposed by Eratosthenes for discovering prime numbers is still, with some slight improvements, the only one in use for that purpose; it is so simple, that it could hardly escape the notice of any one whose attention was called to the subject.

Let it be proposed to make a list of all the prime and composite numbers up to one hundred.

For this purpose, we write all the numbers in succession as below; and having thus prepared our list, we begin with the first number, which is two, and observe that every alternate number must be divisible by 2; so, opposite each, we write $\div 2$ ($\div$ being the sign for *divisible by*, introduced by the accomplished arithmetician Mr Barlow), and we are now sure that not one of the numbers so marked can be prime. The next number not marked as divisible is 3; and after it every third number is divisible by 3—viz. 6, 9, 12, 15, &c.; but the 6 and 12, &c., are already marked as divisible, so we need not mark them again; it is enough to write $\div 3$ at each third number which is not already marked. Having now got rid of all numbers which are divisible by 2 or by 3, we come to the number 4; but it is clear that every number which can be divided into 4 equal parts is divisible by 2, and therefore has been already marked: it is of no use to count each fourth number, and therefore we proceed to seek out those numbers which are divisible by 5. This we do by counting 1, 2, 3, 4, 5; 1, 2, 3, 4, 5; and making the mark $\div 5$ at each fifth number, if it have not been previously marked.

2	9 $\div 3$	16 $\div 2$	23
3	10 $\div 2$	17	24 $\div 2$
4 $\div 2$	11	18 $\div 2$	25 $\div 5$
5	12 $\div 2$	19	26 $\div 2$
6 $\div 2$	13	20 $\div 2$	27 $\div 3$
7	14 $\div 2$	21 $\div 3$	28 $\div 2$
8 $\div 2$	15 $\div 3$	22 $\div 2$	29

30 + 2	49 + 7	68 + 2	87 + 3
31	50 + 2	69 + 3	88 + 2
32 + 2	51 + 3	70 + 2	89
33 + 3	52 + 2	71	90 + 2
34 + 2	53	72 + 2	91 + 7
35 + 5	54 + 2	73	92 + 2
36 + 2	55 + 5	74 + 2	93 + 3
37	56 + 2	75 + 3	94 + 2
38 + 2	57 + 3	76 + 2	95 + 5
39 + 3	58 + 2	77 + 7	96 + 2
40 + 2	59	78 + 2	97
41	60 + 2	79	98 + 2
42 + 2	61	80 + 2	99 + 3
43	62 + 2	81 + 3	100 + 2
44 + 2	63 + 3	82 + 2	101
45 + 3	64 + 2	83	102 + 2
46 + 2	65 + 5	84 + 2	103
47	66 + 2	85 + 5	104 + 2
48 + 2	67	86 + 2	105 + 3

After 5 comes 6, but 6 is itself a divisible number, so that all numbers divisible by 6 are divisible by 2, and have been marked out. We then go to 7, and count each seventh number; and it is worth noticing that 49, 7 times 7, is the first number which gets the new mark + 7. We need not try with 8, or 9, or 10, but make our next trials with 11; but until we reach 11 times 11, that is 121, there can be no new mark made, and so our work is finished—that is, our table shows all numbers that are prime, and the smallest divisor of every composite number up to 100.

By help of this table we can readily find the factors of any number contained in it. Thus, if we wish the factors of the number 84, we find it marked as divisible by 2, and dividing we get the quotient 42. On looking for this number we find again the divisor 2, which gives the quotient 21; 21 has the divisor 3, which gives the quotient 7, a prime number, and therefore 84 is the continued product of 2, 2, 3, 7, or as we write it, $84 = 2 \times 2 \times 3 \times 7$, where the $\times$ is the sign of multiplication.

$$\begin{array}{r}
 2 \overline{) 84} \\
 2 \overline{) 42} \\
 3 \overline{) 21} \\
 7
 \end{array}$$

The student ought to form this table for himself, continuing it say to 200.

The very first step in the construction of such a table suggests a method of lessening the labour.

Every second number is divisible by 2 (or is *even*), and all of these numbers end in 0, 2, 4, 6, or 8, so that even numbers are at once recognisable, and need not be entered at all in our table; and thus we may save one-half of the labour of writing out the numbers and half of that of counting out the composites. It is enough, then, to write out the odd numbers, and to count out each third, fifth, seventh, &c. from among them.

Tables of this kind are of great use for reference. There is one published by M. Burkhardt extending beyond three millions. For a beginner in arithmetic a table as far as one thousand is amply sufficient; and I strongly recommend that each student construct such a table for himself; the labour is trifling, and will render him familiar with numbers; whereas reference to a ready printed table can never impress the characters or individualities of the smaller numbers upon his mind.

EXERCISES.

Find the factors of the following numbers:—

16,	108,	293,	681,	828,	899,	984,
27,	119,	354,	684,	831,	907,	987,
34,	132,	373,	689,	833,	916,	991,
39,	156,	392,	717,	847,	927,	994,
42,	167,	438,	728,	859,	934,	996,
51,	168,	471,	732,	862,	946,	999,
58,	198,	523,	746,	875,	962,	1001,
65,	217,	585,	753,	876,	967,	1017,
72,	238,	605,	768,	882,	973,	1021,
81,	246,	660,	773,	883,	975,	1024,
84,	247,	672,	786,	891,	979,	
97,	261,	677,	795,	894,	981,	

D. When the number of which we want the factors is beyond the limits of our table, we have no help but to try in succes-

sion one prime number after another, until we either find a divisor, or make sure that there is no divisor. For example, if we wish to find the divisors of the number 3589, we try to divide it by 2, but the number is odd since it ends in an odd figure, so we try 3, and find the quotient 1196 with 1 over; then we try 5, next 7, 11, 13, and so on, and it is only when we come to try 37 that we find an exact quotient 97. In this way we ascertain that 3589 is the product 37×97 .

In this example we have succeeded in finding a divisor, but if the number had been prime we should have had to go on with our trials until we had made sure that the number has no divisor whatever. Thus, if 3593 be proposed, we find no divisor up to 37, and have to go on trying 41, 43, &c.; but it appears to be a formidable undertaking if we must try in succession every prime number up to 3593, and we begin to inquire, must we really do this? Let us examine: on trying 41 we find the quotient 87, with the remainder 26; on trying 43 we find the quotient 83, with the remainder 24; and it is quite clear that as the divisor is increased the quotient must be lessened. So with 47 we have the quotient 76; with 53 we have 67; and with 59 we have the quotient 60, in all cases with remainders. I say that we need try no more, for if we were to make the trial with the next prime number 61, we must have a quotient less than 59; and if it were possible to divide 3593 by any number above 60, it would also be possible to divide it by some number less than 59. We have, however, tried all numbers up to 59 without success, and therefore we conclude that 3593 has no divisor whatever.

From this we readily see that in seeking for the divisors of any number we need only go on until we come to a prime the product of which by itself is greater than the proposed number. If thus far we have found no divisor, we are sure that the number is prime.

EXERCISES.

Find the factors of the following numbers :—

1096,	5184,	7681,	10757,	14472,	36953,
1258,	5209,	7763,	11659,	14561,	47103,
1313,	5436,	7811,	11735,	14603,	59167,
1473,	5647,	8064,	11927,	14641,	72773,
1575,	5723,	8246,	12531,	14681,	86109,
1692,	5805,	8651,	12658,	14707,	89231,
1721,	5917,	8729,	12749,	14784,	89764,
1927,	5998,	8844,	12860,	14832,	90011,
1998,	6292,	8977,	13103,	15627,	90432,
2602,	6354,	9161,	13672,	15701,	92751,
2809,	6497,	9401,	13857,	16352,	93165,
3179,	6531,	9523,	13926,	17943,	94434,
3577,	6667,	9684,	13989,	18629,	96852,
3751,	6812,	9753,	14062,	22177,	97723,
4024,	6919,	9836,	14273,	25647,	98653,
4620,	6984,	9897,	14291,	27639,	99993,
4736,	7007,	9899,	14362,	27954,	100631,
4889,	7534,	10236,	14413,	29382,	102611.

C. Numbers have many properties in regard to their divisors, but a discussion of these properties belongs to a higher department of our subject ; for the present I shall content myself with pointing out some which every learner ought to know.

Those numbers—as 4, 6, 8, 10, 12, &c.—which can be halved, are called even ; the others—3, 5, 7, 9, 11, &c.—are called odd ; on these classes of numbers we have the following remarks to make :—

The sum of two or more even numbers is even : thus 18 and 26 make 44, which is even.

The difference of two even numbers is even : thus 22 all but 8 is 14, which is even.

The sum of two odd numbers is even : thus 13 and 29 make 42 ; and the reason of this is, that the odd unit of the one and the odd unit of the other make 2, which is even.

The difference of two odd numbers is even : thus 23 taken from 47 leaves 24.

The sum or the difference of two numbers, one of which is even and the other odd, is always odd.

The product of two even numbers is always divisible by 4.

The product of an odd by an even number is always even.

Lastly, the product of two odd numbers is always odd.

These are the leading properties of odd and even numbers ; that is, of numbers considered in reference to the divisor 2.

Every number which is divisible by 10 ends in 0. For to divide a number by 10, we have only to write each figure of it one step lower ; and if there were any figure in the last place (0 being properly no figure, but merely the sign of a blank), it could not be moved any lower. Indeed we have already learned, that in order to multiply a number by 10, we have only to raise the rank of each figure, by writing 0 at the end of the number.

Now 5 is a divisor of 10, and every number which is divisible by 10 is also divisible by 5 ; so that all numbers ending in 0 may be divided by 5 ; and if we add 5 to one of these numbers, the sum must also contain 5 exactly ; wherefore all numbers ending in 0 or 5 are divisible by 5.

It costs us no trouble then to ascertain whether a proposed number be or be not divisible by 2 or by 5.

D. The number 9 has this peculiar property, that if a number be divisible by 9, the sum of its digits is also divisible by 9. Thus, nine times 7 are 63, and 6 and 3 make 9 : again, nine times 418 make 3762, and 3, 7, 6, and 2, make 18, the double of 9. It is very easy to convince ourselves of the truth of this statement by trial ; but we may obtain a much more satisfactory explanation of it by an appeal to the abacus.

If we put nine balls in the units' column of the abacus, we indicate the number *nine* ; but to add nine to that, we do not require any more balls ; for if I take one ball from the units' column, and place it in the tens' column, I have added ten and taken away one ; that is, I have increased the number by *nine*. To add nine again I take one ball from the units' place and

lay it in the place for tens, thus obtaining 27 ; and in this way I can proceed until I have taken all the balls out of the first column, and placed them all in the second column ; this brings me to 90, the tenth multiple of nine. To add nine now, I have no alternative but to put nine new balls in the column for units, and thus to mark the number ninety-nine we need eighteen counters.

I add nine to this number by taking one from the units and placing it among the tens ; but as there are now ten counters in one column, we may remove these and place one of them in the higher column ; thus we take away nine of the eighteen counters and leave the original nine counters to represent the number 108.

It seems then that as we go on adding nine to the number we may not change the entire number of counters ; we may need nine more, or else we may remove nine of what we have, and so we must always have some multiple of nine counters employed. Or, what is the same thing, the sum of all the digits which represent a multiple of nine, is itself a multiple of nine.

If we had had some counters, say 5, in the units' column at the first, and had gone on adding nine, we should have found always 5, 14, 23, &c. counters ; that is, always 5 counters, or 5 counters increased by some multiple of 9. Hence we see that if on dividing, or rather trying to divide, a number by 9, we have a remainder, we shall obtain the same remainder on trying to divide the sum of the digits by 9.

The number 3 is a divisor of 9 ; and if the remainder, after the 9's are taken out, be 3 or 6, the number must be divisible by 3 ; so that we readily recognise numbers which can be divided by 3 or by nine ; we have only to sum up all the digits, and try whether the sum be divisible by 3 or by 9.

This property of 9 was considered by the earlier writers on arithmetic to be very wonderful, and was imagined to furnish a means of checking the accuracy of additions, subtractions, multiplications. It is, however, much more wonderful that this

egregious mistake is perpetuated in the great majority of modern books on arithmetic. That the striking out of the 9's can afford no test whatever becomes clear when we recollect that a complete inversion of the order of the figures, or an interchange of them in any way, does not affect their sum, and that the test of striking out the nines cannot distinguish among numbers so exchanged. This property does not belong to 9 as nine, but as the number immediately under 10, the root of our scale of numeration.

E. If, instead of counting by tens we had counted by twelves—that is, if twelve had been the root of our numeration-scale—the number eleven would, in that scale, have had properties analogous to those which nine has in the decimal scale; and so for any other scheme of arrangement.

Thus if we had counted by *hundreds*—that is, if the second column in the abacus had been for hundreds, the third for ten-thousands, and so on, the number ninety-nine (one less than a hundred) would have had this property, that the number of counters representing any multiple of ninety-nine must also be a multiple of ninety-nine. Instead of placing counters in the columns of such an abacus we may write the appropriate figures in them. In this way the first line in the adjoining figure would represent 7 millions, 7

7	7	28	53
7	7	29	52

ten-thousands, 28 hundreds, and 53 units; or if a zero had been written before the second 7, the number 7072853. To add ninety-nine to this number, we take one counter out of the units' column and place it in the hundreds' column, thus obtaining 29 52, and so on. Hence all the reasoning which we applied to the multiples of nine may be applied to the multiples of ninety-nine; and thus we arrive at this conclusion, that if the digits of any multiple of 99 be grouped in pairs and the groups summed, their sum is a multiple of 99: and also that if the digits of any number be grouped in pairs and summed, the sum and the original number have the same

remainder when 99 is taken out of each of them. Now 99 is divisible by 3, by 9, by 11, and by 33, and thus we see that if the sum of the digits of any number grouped in pairs be divisible by 3, 9, 11, 33, or 99, the number itself is so. Thus if the number 8350741 be proposed, we group its digits in pairs beginning from the units, and find the amount to be 91. This is divisible neither by 3 nor by 11 (the prime divisors of 99), and therefore we conclude that the number 8350741 is indivisible by either of these numbers.

As 9, one less than the root of the scale, possesses peculiar properties, so 11, one more than the same root, has its peculiarities. The chief of these is this, that the sums of the alternate digits of any multiple of 11 are either equal to each other or differ by a multiple of 11.

We may examine the truth of this readily by help of the abacus. Place counters to represent any number whatever in the columns of the abacus, and mark the alternate columns with two distinguishing letters, say A and B. Then if at the same time I take one counter out of each of two adjoining columns, I diminish the entire number by a number

A	B	A	B	A
o o o	o o o	o o	o o o	o
o o	o		o o o	o
			o	o
				o

of 11's, and therefore do not change its divisibility by 11, nor alter the difference between the total number of counters in the columns A, and the total number in the columns B. But again, if I take a counter from one of the columns B and place it in another column B, or from a column A and place it in another column A, I shall not alter the divisibility by 11, for this removal is equivalent to adding or to subtracting 99, according as the removal is to the second column above or to the second column below. We may thus collect all the counters B in one column (say that for tens), and all the counters A in one column (say that for units), without altering the divisibility by 11: and then we may proceed to throw out two counters at a

time—viz. one from each column, until one of the columns be exhausted. If the original number be divisible by 11, the residual counters must also be so.

When we are seeking for the divisors of a very large number, we are glad to lessen the labour by any contrivance in our power. The following process enables us very readily to test the divisibility by 3, 7, 11, 13, and 37.

If we remove a figure *six* steps to the right we diminish the number by a multiple of 999999, and therefore if we group the digits of a large number in periods of six each, and sum all these groups,

23041 890732 593047	
carrying, if need be, from the million	890732
place down to the units, the sum so	<u>23041</u>
obtained must be the remainder on	506821

taking 999999 as often as possible

out of the proposed number. If this remainder be divisible by any of the divisors of 999999, the original number must also be so. Now 999999 is the product of 999 by 1001; hence if we divide the figures of the remainder into two

groups of three each, and take the sum of these	506 821
carrying from the thousands to the units, we	<u>506</u>
shall obtain the remainder after taking 999 as	328
often as possible from the original number; and if	315 ÷ 7

this remainder be divisible by 3, 9, or 27 or 37,

the original number must be so: or if we take the difference of these two groups, and if this difference be divisible by 7, 11, 13, the factors of 1001, the original number is so.

There are many other important properties of divisors, but these are more conveniently discussed with the help of arbitrary signs: the reader is therefore referred to the proper chapter in the Theory of Numbers for further information on this subject. There is, however, one general property which ought to be familiar to every one engaged in figuring, and which, without attempting to investigate, I shall, in conclusion, briefly state.

C. The product of two prime numbers is not divisible by any

other number but those two primes. Thus 437, the product of 19 and 23, is not divisible by any other number.

The product of three primes is only divisible by those primes and by their products in pairs ; and in general the continued product of any prime numbers is indivisible by any other prime.

CHAPTER VII.

ON FRACTIONS.

C. WE have seen that, in attempting to divide one number by another, we very often come upon a remainder. Now if the dividend represent a number of individuals, it is impossible to perform the division ; but if the dividend stand for a quantity or for a number of things dividual, we may manage the division. Thus although we cannot divide 23 canary birds equally among 5 boys, since a canary is individual, we may divide 23 hanks of kite-string quite easily among them. We would give each of them 4 hanks, and then set ourselves to divide the remaining 3 hanks among them. The share that each should get of the remainder being a broken hank, we call it a *fraction* (from the Latin verb *frango*, I break).

As such fractions cannot be indicated by numbers, a particular mode of writing them is used. We place the dividend above and the divisor below a horizontal stroke thus $\frac{3}{5}$, which means that the three units over is still to be divided into five equal parts : so that we would read this *the fifth part of three units*. But in dividing the remaining 3 hanks of string among the 5 boys, we might tie the hanks end to end so as to make one piece of string, and then fold that in five ; or we might fold each hank separately in five, and give each boy three of those fifths : so that we may read this mark $\frac{3}{5}$ either as the fifth part of three hanks, or as three-fifth parts of one hank : the latter reading is the most usual ; and on this account the dividend is called the *numerator*, and the divisor the *denominator* (namer) of

the fraction. It seems then that on dividing 23 hanks among 5 boys the share of each would be *four* hanks and three-fifths of a hank or $4\frac{3}{5}$ as it is usual to write it.

By this means we can readily indicate the quotient, even although the quantity to be divided be represented by an indivisible number. Thus 239 divided by 8 gives the quotient $29\frac{7}{8}$.

EXAMPLES.

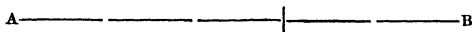
Divide—

13 by 4.	63 by 10.	591 by 20.	5794 by 67.
16 by 5.	99 by 8.	653 by 27.	6101 by 66.
9 by 8.	115 by 8.	791 by 32.	6534 by 69.
17 by 6.	137 by 12.	899 by 39.	6792 by 72.
22 by 7.	159 by 4.	1256 by 43.	6953 by 76.
25 by 4.	172 by 12.	1698 by 48.	7872 by 81.
29 by 6.	296 by 13.	1979 by 51.	8652 by 83.
30 by 7.	299 by 15.	2603 by 54.	9561 by 87.
34 by 11.	324 by 17.	2951 by 57.	14317 by 96.
37 by 9.	379 by 19.	3478 by 62.	18937 by 102.
52 by 6.	384 by 21.	4913 by 65.	20561 by 115.
57 by 7.	473 by 22.	5371 by 64.	

D. It is clear, however, that this is not performing the division, but merely explaining what division is to be performed: so that fractional quotients can only have a meaning when they have a reference to objects susceptible of division. Such an expression as $3\frac{2}{5}$ can never be a number, it can only have a meaning when it stands for a quantity. I mention this because some writers erroneously speak of and treat such expressions as if they were numbers, and regard $3\frac{1}{2}$ as a *number* greater than the number 3 and less than the number 4. The slightest consideration is enough to convince us that there is no number whatever intermediate between 3 and 4; although there be many weights intermediate between 3 lb and 4 lb, or many lines longer than three feet and shorter than four feet: to play upon the word; a table cannot stand on $3\frac{1}{2}$ feet, it may have three feet or four feet, but certainly not 3 and any fraction of a foot.

Rightly considered $3\frac{1}{2}$ is the number 7 in disguise, and $4\frac{3}{5}$ is the number 23; the one is seven halves of the unit and the other is 23 fifths.

C. The same fractional quantity may be written in an endless variety of ways: thus the fifth part of three hanks is the same as the tenth part of six hanks, so

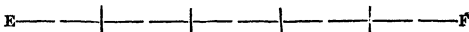


that we may

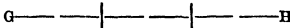


write indiffer-

ently $\frac{3}{5}$ or $\frac{6}{10}$;



we may also

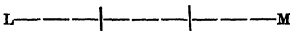


write $\frac{9}{15}$, or $\frac{12}{20}$,

and so on. Thus



if the line AB



be the unit, we

shall obtain the quantity $\frac{3}{5}$ by dividing AB into 5 equal parts and measuring off 3 of them on CD, so that CD would be three fifth-parts of AB.

Again, having measured EF equal to AB, and divided it into ten equal parts, if we measure off six of these tenths on GH, this GH must be exactly equal to CD.

Or yet if we make IK equal to the unit, divide it into fifteen equal parts, and make LM nine of those fifteenths, LM is also equal to CD: so that the three fractions, $\frac{3}{5}$, $\frac{6}{10}$, and $\frac{9}{15}$, are of the same value.

From this we perceive that if the numerator and denominator of a fraction be both multiplied by the same number, the value of the fraction is not altered.

EXERCISES.

Write out a number of fractions equal to each of the following fractions:—

$$\frac{1}{2} = \frac{2}{4} = \&c.$$

$$\frac{3}{4} = \&c.$$

$$\frac{2}{3} = \&c.$$

$$\frac{1}{7} = \&c.$$

Of course, if we be at liberty to multiply both members of a fraction by any number without changing its value, we may

also divide both members by any number that will divide them, and yet not alter the value of the fraction. Of the series of equal fractions, $\frac{2}{5}$, $\frac{4}{10}$, $\frac{6}{15}$, $\frac{8}{20}$, &c., the first is expressed by help of the smallest numbers, and is therefore, for many purposes, a more convenient form than any of the others; and it may happen that a fraction given in high numbers may be reducible to simpler terms; thus we see that $4\frac{85}{85}$ may be simplified to $\frac{97}{147}$, by dividing each of its members by 5. Thus arises the problem—*To reduce a fraction to its lowest terms.*

In order to convert a fraction as $2\frac{84}{62}$ into another which shall have the same value, but have its terms smaller, we must seek for some number which can divide both the numerator and the denominator.

One very convenient method of doing this is to resolve each into its factors, thus :—

$$\frac{294}{462} = \frac{2.3.7.7}{2.3.7.11} = \frac{7}{11},$$

and to observe if any factor be common to both; if so, we have only to leave that common factor out: in this way we find that $2\frac{84}{62}$ is just equal to $\frac{7}{11}$.

Reduce the following fractions to their lowest terms :—

$\frac{2}{4}$	$\frac{8}{32}$	$\frac{14}{98}$	$\frac{54}{404}$	$\frac{709}{1073}$	$\frac{8592}{41349}$
$\frac{3}{6}$	$\frac{11}{44}$	$\frac{19}{95}$	$\frac{61}{256}$	$\frac{832}{1088}$	$\frac{9721}{31744}$
$\frac{3}{12}$	$\frac{7}{28}$	$\frac{17}{53}$	$\frac{72}{574}$	$\frac{1062}{1208}$	$\frac{939}{21878}$
$\frac{4}{16}$	$\frac{9}{54}$	$\frac{20}{28}$	$\frac{12}{1428}$	$\frac{2571}{7062}$	$\frac{11583}{11319}$
$\frac{5}{15}$	$\frac{12}{36}$	$\frac{23}{69}$	$\frac{391}{437}$	$\frac{3908}{5662}$	$\frac{12348}{56628}$
$\frac{7}{21}$	$\frac{13}{65}$	$\frac{16}{84}$	$\frac{921}{2633}$	$\frac{5044}{5064}$	$\frac{13565}{41721}$
$\frac{9}{27}$	$\frac{12}{66}$	$\frac{34}{87}$	$\frac{529}{1403}$	$\frac{6673}{10697}$	

When the numbers are not very large, or when we have a table of divisors, this method is rapid and convenient; but with large numbers it becomes exceedingly troublesome. It is clear that if one of the members of a fraction have a divisor which does not divide the other, this fact is of little use to us. What we want is a number, if there be any such, which is a divisor of both.

Now it is quite clear that any number which is contained exactly in the numerator and also exactly in the denominator, must be contained without a remainder in the difference between them. Thus if the proposed fraction had been

$$\frac{79469}{265603},$$

any number which divides both 79469 and 265603 must divide the difference 186134, so that we may now look for a divisor common to both 79469 and 186134 ; but this must also be a divisor of the difference 106665, or still of the difference 27196. That is, if we take the lesser of our two numbers as often as possible out of the greater, the common divisor must be the remainder or a divisor of the remainder. In the present example it must be a divisor of 27196.

We have now to seek for some number which may divide both 27196 and 79469 ; and we are thus much advanced towards our aim that we have to deal with smaller numbers, while we are certain that any common divisor of these two must be a divisor of both our first numbers.

Taking 27196 out of 79469, we obtain the remainder 25077 ; taking this out of 27196, we obtain 2119 ; taking this as often as possible out of the last subtrahend, we get 1768 ; so that if our first two numbers have a common divisor, that divisor must be common to 1768 and 2119.

The most convenient form for carrying on this work is shown in the margin. Having drawn a column vertically on our page, we write the one number on the right hand, the other on the left hand of this column, placing the larger number one line higher than the other : 79469 goes 3 times in 265603, and leaves the remainder 27196 ; this goes twice in the former subtrahend, and so on, the middle column serving for a place to receive the quotients. In this way we proceed until we find no remainder ; and then the last subtrahend is the common divisor ;

79469	3	265603
54392	2	238407
25077	1	27196
23309	11	25077
1768	1	2119
1755	5	1768
13	27	351

or till we find the remainder *unit*, in which case unit is the only common measure of the two numbers, so that they can have no common divisor. In our present example we obtain the common divisor 13, and our fraction may be put under the form $\frac{6113}{20431}$, which is the simplest of all fractions having the same value.

Reduce the following fractions to their lowest terms :—

$\frac{6127}{6721}$	$\frac{17441}{19387}$	$\frac{57229}{61759}$	$\frac{99764}{249410}$	$\frac{150211}{152320}$
$\frac{8203}{8671}$	$\frac{25632}{37032}$	$\frac{61601}{62879}$	$\frac{106327}{107861}$	$\frac{241703}{271733}$
$\frac{9353}{9331}$	$\frac{21751}{23467}$	$\frac{90041}{97426}$	$\frac{103303}{110828}$	$\frac{240823}{242993}$
$\frac{11849}{12709}$	$\frac{31302}{46983}$	$\frac{91756}{106372}$	$\frac{117856}{120808}$	$\frac{314631}{339230}$
$\frac{12678}{18283}$	$\frac{39719}{43021}$	$\frac{95437}{98743}$	$\frac{132503}{134323}$	$\frac{463298}{561368}$

D. The method just explained is the natural, and for certain collateral purposes the *best* method for finding the greatest common divisor of two numbers: yet, when our only object is to discover that divisor, we may often shorten the process by help of the following considerations.

If, of two numbers, one have a *prime* divisor which does not divide the other, its quotient by that divisor will have the same common divisor with the second number that the first number had; or, in other words, if one of the numbers have a prime divisor which the other has not, we are at liberty to divide it by that prime divisor.

In our last example the first remainder is 27196, which has the divisor 4 or 2×2 , while the corresponding number on the other side, viz. 79469 is odd. Instead then of seeking the common divisor of 79469 and 27196, we may seek that of 79469 and 6799, so that in this way the labour may be considerably reduced. The next remainder, 4680, is obviously divisible by 40, and the quotient 117 by 9, wherefore if the two original numbers have any common divisor that divisor must be 13.

79469	3	265603
		238407
74789	11	27196 ÷ 4
		6799
40 ÷ 4680		
9 ÷ 117		
13		

Reduce the following fractions by the above method :—

$\frac{94717}{148841}$	$\frac{234248}{256109}$	$\frac{421723}{440647}$
$\frac{106003}{135863}$	$\frac{276127}{278939}$	$\frac{414259}{560081}$
$\frac{131072}{363216}$	$\frac{312827}{350011}$	$\frac{486513}{688327}$
$\frac{153109}{170819}$	$\frac{295159}{384583}$	$\frac{789047}{800303}$
$\frac{162761}{188047}$	$\frac{342293}{438529}$	$\frac{865341}{991757}$
$\frac{198419}{199361}$	$\frac{397121}{403633}$	

E. When either of the proposed numbers is within the range of a table of divisors we can readily resolve it into a continued product, and then we have only to try in succession each one of its prime factors with the other number : and also, whenever any of the remainders is brought within the limits of the table we may have recourse to the same expedient. Thus we see that a knowledge of prime and composite numbers is of great use in the management of fractions.

CHAPTER VIII.

ON THE ADDITION AND SUBTRACTION OF FRACTIONS.

C. IF we have to add together two fractions of the same name or denomination, we add together the numerators, thus $\frac{2}{7}$ and $\frac{3}{7}$ make $\frac{5}{7}$; but if the fractions be of different names, the addition is not quite so easy.

Thus if it were proposed to add $\frac{3}{5}$ of a yard to $\frac{2}{7}$ of a yard, we could not add the numerators, since *three-fifths* and *two-sevenths* will not make *five* any-things.

However, the fraction $\frac{3}{5}$ may be written under a great variety of forms, as $\frac{6}{10}$, $\frac{9}{15}$, $\frac{12}{20}$, $\frac{15}{25}$, $\frac{18}{30}$, $\frac{21}{35}$, $\frac{24}{40}$, and so on; while the fraction $\frac{2}{7}$ is equivalent to any one of the series $\frac{4}{14}$, $\frac{6}{21}$, $\frac{8}{28}$, $\frac{10}{35}$, $\frac{12}{42}$, &c., and among these we may find two having a common denominator: indeed we have not long to seek, for in the first series we find $\frac{24}{40}$ and in the second series $\frac{10}{35}$ which have the common denominator 35, and which we can at once add, their sum being $\frac{34}{35}$.

For the sake of clearness, we may represent the above operation thus:—

$$\frac{3}{5} + \frac{2}{7} = \frac{24}{35} + \frac{10}{35} = \frac{34}{35}.$$

Thus it seems that, in order to add together two fractions of different denominations, we must first convert them into others of equal values and of a common denomination, and then add these together as we add any two numbers of like things. Now this conversion is always possible; for whatever may be the two denominators, if we multiply each member of the first fraction by the denominator of the second, and each member of the

second fraction by the denominator of the first, each of the new fractions so obtained must have, for its denominator, the product of the two original denominators. It is quite clear that the same transformation enables us to obtain the difference between two fractions.

Add together—

$\frac{1}{2}$ and $\frac{2}{3}$	$\frac{2}{3}$ and $\frac{1}{6}$	$\frac{9}{24}$ and $\frac{35}{24}$
$\frac{2}{3}$ and $\frac{2}{5}$	$\frac{18}{25}$ and $\frac{13}{25}$	$\frac{27}{25}$ and $\frac{91}{25}$
$\frac{1}{3}$ and $\frac{4}{4}$	$\frac{46}{41}$ and $\frac{2}{35}$	$\frac{137}{2}$ and $\frac{68}{99}$
$\frac{4}{4}$ and $\frac{2}{6}$	$\frac{32}{43}$ and $\frac{15}{54}$	$\frac{396}{357}$ and $\frac{372}{487}$
$\frac{6}{11}$ and $\frac{5}{7}$	$\frac{43}{84}$ and $\frac{28}{59}$	$\frac{627}{336}$ and $\frac{425}{462}$
$\frac{7}{9}$ and $\frac{10}{13}$	$\frac{57}{71}$ and $\frac{63}{78}$	$\frac{757}{965}$ and $\frac{669}{1037}$
$\frac{12}{13}$ and $\frac{9}{16}$	$\frac{53}{82}$ and $\frac{34}{91}$	$\frac{1375}{2162}$ and $\frac{987}{3611}$
$\frac{14}{17}$ and $\frac{19}{25}$	$\frac{73}{95}$ and $\frac{61}{84}$	$\frac{1566}{3765}$ and $\frac{279}{6571}$
$\frac{16}{21}$ and $\frac{13}{29}$	$\frac{75}{98}$ and $\frac{23}{105}$	

Although this method of conversion be always successful, it is not always the most convenient that can be found. For example, if we be required to add $\frac{1}{4}$ to $\frac{3}{8}$ we do not need to convert these into $\frac{8}{32}$ and $\frac{12}{32}$ respectively, for we see at a glance that the first denominator is contained in the second, so that for $\frac{1}{4}$ we put $\frac{2}{8}$ and obtain the sum $\frac{5}{8}$ thus :

$$\frac{1}{4} + \frac{3}{8} = \frac{2}{8} + \frac{3}{8} = \frac{5}{8},$$

and this is a much more concise proceeding than the former. *Indeed we never have recourse to the general method when we can avoid it.*

Add together—

$\frac{1}{3}$ and $\frac{7}{9}$	$\frac{19}{18}$ and $\frac{11}{18}$	$\frac{72}{164}$ and $\frac{33}{41}$	$\frac{552}{358}$ and $\frac{43}{43}$
$\frac{7}{9}$ and $\frac{11}{18}$	$\frac{15}{84}$ and $\frac{9}{28}$	$\frac{78}{4}$ and $\frac{35}{35}$	$\frac{98}{72}$ and $\frac{354}{432}$
$\frac{4}{6}$ and $\frac{5}{9}$	$\frac{17}{32}$ and $\frac{13}{28}$	$\frac{192}{108}$ and $\frac{24}{33}$	$\frac{396}{428}$ and $\frac{93}{107}$
$\frac{12}{16}$ and $\frac{5}{4}$	$\frac{53}{64}$ and $\frac{19}{16}$	$\frac{327}{366}$ and $\frac{37}{45}$	$\frac{695}{3676}$ and $\frac{44}{44}$

This operation also enables us to add a fraction to an integer or whole number, and to express the result in one fraction.

Thus if we wish to make $3\frac{1}{5}$ into one fraction, we observe that each unit is 5-fifths, and that therefore three units make 15-fifths; or

$$3 + \frac{1}{5} = 1\frac{5}{5} + \frac{1}{5} = 1\frac{6}{5},$$

so that the 3 units might have been written $\frac{3}{1}$ to give it the appearance of a fraction, and then the operation would have been exactly as before.

Add the following fraction and integer numbers together, expressing the result in one fraction:—

$7\frac{3}{8}$	$65\frac{3}{81}$	$300\frac{7}{9}$	$5690\frac{1273}{2884}$
$12\frac{7}{3}$	$157\frac{32}{5}$	$374\frac{123}{282}$	$13907\frac{167}{3805}$
$9\frac{7}{15}$	$198\frac{6}{54}$	$439\frac{371}{455}$	$23713\frac{67}{2341}$
$17\frac{7}{4}$	$236\frac{6}{83}$	$1365\frac{141}{882}$	$694\ 307\frac{156}{30841}$
$23\frac{1}{11}$	$257\frac{1}{92}$	$2798\frac{1}{981}$	$965\ 158\frac{3967}{706573}$
$27\frac{7}{31}$	$279\frac{3}{5}$	$3759\frac{7}{99}$	$2691\ 532\frac{9987}{85832}$

It may happen that the denominators of the two proposed fractions have a common divisor; in such case we can simplify our proceedings. Thus let it be proposed to add $\frac{7}{15}$ to $\frac{11}{25}$. Here we see that each denominator is divisible by 5, the one being 3.5, the other 5.5; and that it is enough to multiply each member of the first fraction by 5, each member of the second fraction by 3; thus the work stands

$$\frac{7}{15} + \frac{11}{25} = \frac{35}{75} + \frac{33}{75} = \frac{68}{75}.$$

3.5 5.5.

Add together—

$\frac{5}{12}$ and $\frac{11}{12}$	$\frac{72}{81}$ and $\frac{65}{108}$	$\frac{1011}{6032}$ and $\frac{909}{4624}$
$\frac{13}{18}$ and $\frac{25}{36}$	$\frac{146}{155}$ and $\frac{78}{186}$	$\frac{596}{4710}$ and $\frac{1079}{3297}$
$\frac{7}{15}$ and $\frac{15}{36}$	$\frac{173}{74}$ and $\frac{265}{222}$	$\frac{6591}{7251}$ and $\frac{3765}{3453}$
$\frac{12}{21}$ and $\frac{43}{53}$	$\frac{69}{164}$ and $\frac{155}{198}$	$\frac{5863}{2662}$ and $\frac{8761}{8746}$
$\frac{38}{36}$ and $\frac{19}{27}$	$\frac{211}{230}$ and $\frac{96}{345}$	$\frac{1076}{30715}$ and $\frac{10991}{23828}$
$\frac{34}{43}$ and $\frac{57}{70}$	$\frac{199}{248}$ and $\frac{218}{276}$	$\frac{752}{36983}$ and $\frac{1532}{78063}$
$\frac{37}{39}$ and $\frac{59}{121}$	$\frac{67}{209}$ and $\frac{106}{399}$	
$\frac{18}{16}$ and $\frac{21}{84}$	$\frac{126}{281}$ and $\frac{157}{167}$	

When we have to add or subtract quantities composed of whole units (or *integers*) and fractions, we now find no difficulty, for we can operate upon the fractions and upon the integers separately.

Thus if I have to add $372\frac{1}{4}$ bushels to $491\frac{3}{8}$ bushels, I proceed almost as in common addition; only as quarters and eighths are of dissimilar denomination I make a jotting of their equivalents at the side, and obtain the sum $863\frac{3}{8}$ bushels. But if it had been proposed to add $562\frac{1}{4}$ yards to $398\frac{3}{5}$ yards, the amount of the fractional parts would have come out $\frac{4}{5}$ of a yard: now $\frac{3}{5}$ make unit, and therefore instead of $\frac{4}{5}$ we may put $1\frac{0}{5}$, so that the whole result comes out $961\frac{0}{5}$.

$$\begin{array}{r} 372\frac{1}{4} \\ 491\frac{3}{8} \\ \hline 863\frac{3}{8} \end{array}$$

Again in subtraction we proceed much in the same way, thus to take $2761\frac{4}{5}$ from $3843\frac{1}{5}$ we proceed as in ordinary cases of subtraction after having prepared the fractions; taking care to write the remaining fraction in its lowest terms. But if the fractional part of the subtrahend had happened to be the greater we must have borrowed one from the units as in this example, where we cannot take the $\frac{2}{3}$ from $\frac{1}{3}$; and must borrow unit or $\frac{3}{3}$ from the 8 of the minuend to make the fractional part $\frac{5}{3}$; just as we have been accustomed to borrow from the higher rank to support a defective figure.

$$\begin{array}{r} 3843\frac{1}{5} \\ 2761\frac{4}{5} \\ \hline 1082\frac{2}{5} \end{array}$$

$$\begin{array}{r} 5248\frac{4}{8} \\ 3106\frac{7}{8} \\ \hline 2141\frac{3}{8} \end{array}$$

Perform the following additions and subtractions:—

Add together—

$$157\frac{6}{11} \text{ and } 98\frac{2}{4}$$

$$259\frac{1}{3} \text{ and } 165\frac{5}{8}$$

$$279\frac{7}{8} \text{ and } 216\frac{3}{4}$$

$$569\frac{1}{5} \text{ and } 492\frac{2}{3}$$

$$956\frac{1}{3} \text{ and } 351\frac{2}{3}$$

$$1039\frac{1}{3} \text{ and } 1349\frac{1}{6}$$

$$1693\frac{1}{3} \text{ and } 1258\frac{1}{3}$$

$$3964\frac{1}{3} \text{ and } 561\frac{5}{6}$$

$$19707\frac{1}{3} \text{ and } 38641\frac{1}{3}$$

Subtract—

$$\begin{array}{ll}
 317 \frac{2}{5} \text{ from } 495 \frac{4}{5} & 1352 \frac{1}{364} \text{ from } 7230 \frac{2}{485} \\
 194 \frac{6}{9} \text{ from } 275 \frac{8}{9} & 17364 \frac{2}{376} \text{ from } 39047 \\
 573 \frac{7}{121} \text{ from } 619 \frac{8}{143} & 5321 \frac{1}{396} \text{ from } 8695 \frac{1}{88} \\
 912 \frac{1}{119} \text{ from } 1294 \frac{1}{187} & 9072 \frac{5}{3872} \text{ from } 56931 \frac{7}{1243} \\
 694 \frac{2}{66} \text{ from } 1797 \frac{2}{120} & 13625 \frac{5}{324} \text{ from } 781514 \frac{7}{1831}
 \end{array}$$

So much for the sum or difference of *two* fractions. When we have more than two to deal with we may first add two, then the sum of these to another; but such is a very clumsy procedure, since the entire amount may be obtained by one operation.

Sometimes the denominator of one fraction may be a multiple of each of the others: as in this case, $\frac{5}{8} + \frac{7}{12} - \frac{3}{4} + \frac{2}{3} - \frac{1}{2}$. Here we can turn all the fractions into 24ths thus:—

$$\frac{1}{4} + \frac{1}{4} - \frac{1}{2} + \frac{1}{2} - \frac{1}{2} = \frac{1}{4} = \frac{1}{2}.$$

Sometimes again there may be a divisor common to two or three of the denominators: as in this example, $\frac{1}{7} + \frac{2}{9} + \frac{2}{14}$. In such cases, and indeed in general, it is convenient to note below each denominator the factors of which it is the product;

$$\begin{array}{ccccccc}
 \frac{1}{7} & + & \frac{2}{9} & + & \frac{2}{14} & = & \frac{247}{1001} + \frac{253}{1001} + \frac{203}{1001} = \frac{703}{1001} \\
 7 \cdot 11 & & 7 \cdot 13 & & 11 \cdot 13 & & \\
 & & 7 \cdot 11 \cdot 13 & & & & \text{common denominator.} \\
 13 & & 11 & & 7 & &
 \end{array}$$

we then compose a common denominator by writing down the factors of the first, then each new factor which is found in the other denominators. In this way we obtain a continued product which contains each of the denominators. We then observe what factor or factors may be wanting in each denominator to bring it up to the common denominator. Thus, to find the common denominator in our example, I write down 7. 11 the factors of the first denominator: then proceeding to those of the second I find 7 which I have already and do not need again, and 13 which I have not; I write the 13, making now 7. 11. 13, and go on to the next denominator where I find no new factor: thus I determine that the continued product 7. 11. 13 or 1001 is to be the common denominator. Re-

turning to the factors of the first denominator I find 13 wanting, and therefore multiply both members of the first fraction by 13. Similarly those of the second fraction by 11, and those of the third by 7.

If we now wish to know whether the result be reducible to simpler terms, we have only to try whether its numerator be divisible by any of the prime factors of the denominator.

Collect the following fractions :—

$$\begin{aligned}
 & \frac{1}{3} + \frac{4}{9} + \frac{5}{12}; \quad \frac{9}{14} + \frac{5}{28} + \frac{20}{21}; \quad \frac{4}{5} + \frac{12}{20} - \frac{17}{35}; \quad \frac{6}{4} + \frac{17}{37} + \frac{11}{111}; \\
 & \frac{6}{7} + \frac{4}{8} + \frac{11}{35}; \quad \frac{4}{9} + \frac{13}{18} + \frac{10}{24}; \quad \frac{7}{13} - \frac{32}{39} + \frac{72}{78}; \quad \frac{17}{32} + \frac{23}{24} + \frac{5}{36}; \\
 & \frac{5}{18} + \frac{20}{36} + \frac{68}{108} - \frac{84}{54}; \quad \frac{9}{14} - \frac{17}{18} + \frac{25}{32} + \frac{11}{16}; \quad \frac{28}{39} + \frac{123}{195} - \frac{64}{117} + \frac{59}{78}; \\
 & \frac{101}{133} + \frac{45}{57} - \frac{31}{209} - \frac{127}{247}; \quad \frac{77}{122} - \frac{392}{427} + \frac{492}{671} + \frac{94}{183}; \\
 & \frac{69}{113} + \frac{97}{113} + \frac{173}{205} - \frac{84}{11}; \quad \frac{369}{551} + \frac{574}{6003} + \frac{455}{437} - \frac{511}{337}; \\
 & \frac{295}{384} + \frac{173}{572} + \frac{949}{4004} + \frac{69}{77}; \quad \frac{54}{51} + \frac{67}{48} + \frac{39}{42} - \frac{25}{36}; \\
 & \frac{219}{312} + \frac{133}{336} - \frac{623}{1840} + \frac{281}{360}; \quad \frac{25}{39} - \frac{64}{42} - \frac{35}{45} + \frac{41}{51}; \\
 & \frac{174}{214} - \frac{98}{203} + \frac{105}{126} - \frac{37}{182}; \quad \frac{631}{704} - \frac{73}{192} + \frac{237}{248} - \frac{55}{320}; \\
 & \frac{784}{1469} + \frac{391}{1261} - \frac{1205}{1937} + \frac{1996}{1939} + \frac{532}{1703}; \quad \frac{48}{91} + \frac{105}{187} - \frac{39}{169} + \frac{74}{183}; \\
 & \frac{357}{496} + \frac{695}{1104} + \frac{749}{1422} - \frac{631}{1026} - \frac{949}{1580}; \\
 & \frac{654}{833} + \frac{513}{637} - \frac{641}{1127} + \frac{729}{1519} - \frac{432}{931}; \\
 & \frac{903}{1367} - \frac{651}{772} - \frac{103}{6544} - \frac{72}{193} - \frac{65}{852}; \\
 & \frac{1562}{3151} + \frac{530}{2599} - \frac{1692}{2389} + \frac{965}{2231} + \frac{782}{3473}; \\
 & \frac{3654}{9971} - \frac{10417}{12337} - \frac{324}{6253} - \frac{1652}{7287} - \frac{7055}{14027}.
 \end{aligned}$$

When it happens that the denominators have no common divisors, the denominator of the equivalent fractions must clearly be the continued product of all the denominators, and therefore each numerator has to be multiplied into all the denominators except its own. The labour of these operations is often very great, and therefore we always anxiously inquire if there be no divisor whereby it may be lessened.

CHAPTER IX.

ON THE MULTIPLICATION AND DIVISION OF FRACTIONS.

C. THERE is little difficulty in multiplying a fraction, for just as five times three apples make fifteen apples, so five times three-sevenths make fifteen-sevenths ; that is,

$$\frac{3}{7} \times 5 = \frac{15}{7} = 2\frac{1}{7}.$$

But this very easy operation can sometimes be better performed ; thus if three times five-twelfths be wanted we can at once obtain fifteen-twelfths for the product ; but of this fraction both members are divisible by three, so that we can simplify it to five-fourths, as below :

$$\frac{5}{12} \times 3 = \frac{15}{12} = \frac{5}{4}.$$

However it is clear that we might have avoided this last operation by observing that three-twelfths make exactly one quarter, and that therefore three times five-twelfths must just make five quarters ; in fact, instead of multiplying the numerator, we might have divided the denominator by three, and obtained at once

$$\frac{5}{12} \times 3 = \frac{5}{4},$$

and it is always preferable to divide the denominator if it can be done.

Again it may happen that although the denominator be not divisible by the multiplier it is divisible by some factor of the

multiplier, and in that case also the work may be shortened, as in the following example :

$$\frac{13}{4} \times 21 = 9\frac{1}{2} = 11\frac{3}{2},$$

3.7

where the multiplier 21 is the product of 3 by 7, and where one of the factors, viz. 3, divides the denominator although the other factor 7 do not : we then divide the denominator by 3 and multiply the numerator by 7 to get the result $9\frac{1}{2}$.

Thus it seems that when we have to multiply a fraction by a number we shorten the work by considering whether the denominator of the fraction and the multiplier have any common factor ; and that a knowledge of the factors of numbers is of great use to us in the management of fractions.

Perform the following multiplications :—

$\frac{3}{5}$ by 7.	$\frac{59}{2}$ by 60.	$\frac{267}{29}$ by 281.
$\frac{5}{12}$ by 6.	$\frac{63}{2}$ by 71.	$\frac{354}{374}$ by 319.
$\frac{4}{11}$ by 9.	$\frac{73}{8}$ by 72.	$\frac{65}{29}$ by 517.
$\frac{19}{8}$ by 11.	$\frac{78}{5}$ by 91.	$\frac{72}{21}$ by 515.
$\frac{8}{15}$ by 12.	$\frac{99}{119}$ by 85.	$\frac{81}{32}$ by 474.
$\frac{5}{19}$ by 9.	$\frac{99}{80}$ by 128.	$\frac{57}{69}$ by 641.
$\frac{28}{5}$ by 15.	$\frac{136}{175}$ by 98.	$\frac{53}{64}$ by 783.
$\frac{27}{35}$ by 30.	$\frac{147}{195}$ by 127.	$\frac{49}{33}$ by 622.
$\frac{28}{45}$ by 42.	$\frac{112}{209}$ by 143.	$\frac{358}{1052}$ by 263.
$\frac{27}{47}$ by 53.	$\frac{85}{236}$ by 177.	$\frac{107}{2599}$ by 972.
$\frac{39}{48}$ by 52.	$\frac{239}{206}$ by 247.	$\frac{964}{2953}$ by 6531.
$\frac{42}{65}$ by 63.	$\frac{263}{289}$ by 187.	$\frac{5728}{9753}$ by 6502.
$\frac{47}{9}$ by 67.	$\frac{241}{306}$ by 263.	

If the multiplicand consist of whole units and fractions, we have to multiply each part separately and then collect the results together. Thus if 7 times $23\frac{5}{9}$ yards be wanted, we observe that 7 times $\frac{5}{9}$ make $3\frac{5}{9}$ or $3\frac{5}{9}$, and carry the three units to the units of the 7th multiple $164\frac{8}{9}$ of 23.

When the multiplier is a small number, the operation can be carried straight on ; but with a large multiplier we must multiply the fraction *at once*. Thus to multiply $279\frac{1}{3}$ by 287, although we multiply the 279 first by 7, then by 80, and next by 200, it would be inconvenient to treat the fractional part so : it is better to multiply the fractional part at once by 287 as shown in the margin.

$$\begin{array}{r}
 279\frac{1}{3} \\
 \times 287 \\
 \hline
 242\frac{1}{3} \\
 1953 \\
 2232 \\
 558 \\
 \hline
 80315\frac{1}{3}
 \end{array}$$

D. When multiplying the fraction $\frac{1}{3}$ by 287, we may observe that 287 is 22 times 13 and one over ; so if we multiply $\frac{1}{3}$ by 286, we obtain at once 242, or 22 times 11, units ; and thus the product is $242\frac{1}{3}$. This kind of abbreviation can always be used when the result contains several integers. Thus if we have to multiply $\frac{1}{7}$ by 284, we may observe that 284 is 16 times 17 and 12 over : or is $272 + 12$, wherefore 284 times $\frac{1}{7}$ being regarded as 272 times $\frac{1}{7}$ and 12 times $\frac{1}{7}$, gives 208 together with $\frac{5}{7}$ or $8\frac{1}{7}$, and thus the entire result is $216\frac{1}{7}$.

EXAMPLES.

C. Multiply—

$3\frac{4}{5}$ by 3.	$91\frac{9}{11}$ by 59.	$1079\frac{65}{133}$ by 604.
$7\frac{3}{13}$ by 6.	$97\frac{1}{13}$ by 75.	$2158\frac{11}{107}$ by 635.
$8\frac{1}{6}$ by 13.	$131\frac{7}{8}$ by 79.	$2761\frac{19}{65}$ by 594.
$15\frac{7}{9}$ by 11.	$165\frac{4}{5}$ by 76.	$3472\frac{53}{14}$ by 755.
$19\frac{2}{5}$ by 17.	$192\frac{3}{4}$ by 98.	$4169\frac{27}{81}$ by 834.
$25\frac{8}{9}$ by 14.	$267\frac{9}{2}$ by 117.	$5630\frac{86}{33}$ by 956.
$27\frac{6}{23}$ by 16.	$273\frac{9}{4}$ by 108.	$8715\frac{38}{5}$ by 1470.
$31\frac{5}{22}$ by 33.	$283\frac{3}{3}$ by 126.	$9356\frac{58}{8}$ by 329.
$39\frac{7}{7}$ by 27.	$329\frac{7}{4}$ by 139.	$10673\frac{52}{6}$ by 429.
by 36.	$378\frac{2}{1}$ by 157.	$12994\frac{7}{1872}$ by 1631.
by 24.	$527\frac{4}{23}$ by 246.	$17356\frac{73}{1564}$ by 1955.
$65\frac{3}{2}$ by 63.	$694\frac{8}{13}$ by 355.	$29657\frac{57}{74}$ by 2691.
$85\frac{1}{6}$ by 57.	$973\frac{96}{88}$ by 572.	$47538\frac{76}{68}$ by 2955.
$89\frac{7}{11}$ by 69.		

Having learned how to multiply a fraction, we proceed to consider how we may divide one.

If it be proposed to divide $\frac{1}{4}$ by 4, we perceive at once that the quotient must be $\frac{1}{16}$, just as the fourth part of 12 oz. is 3 oz. : it is, thus, enough to divide the numerator of the fraction, and when this division can be performed we have no farther trouble.

But if the dividend had been $\frac{1}{17}$, of which the numerator is not divisible by 4, we must have proceeded differently. As here we have one-seventeenth over, we must pass to the quarter of a seventeenth : now if unit be divided into seventeen parts, and if we take the quarter of one of these, that quarter of a seventeenth must be contained sixty-eight times in the unit, and thus we might write our quotient $\frac{1}{17}$ and $\frac{1}{68}$. However, we may regard $\frac{1}{17}$ as meaning the seventeenth part of 13, and therefore the quarter of this as the sixty-eighth part of 13, and so we may obtain the result in the more convenient form $\frac{1}{68}$.

Thus we see that there are two processes by either of which we can effect the division of a fraction, the one being to divide the numerator, the other to multiply the denominator of the fraction ; and it is clear, from the very nature of fractions, that each of these processes leads to a true answer.

Perform the following divisions :—

$\frac{9}{13}$ by 3.	$\frac{7}{18}$ by 21.	$\frac{371}{551}$ by 189.
$\frac{7}{16}$ by 7.	$\frac{13}{53}$ by 23.	$\frac{542}{723}$ by 271.
$\frac{13}{18}$ by 6.	$\frac{49}{54}$ by 28.	$\frac{831}{956}$ by 554.
$\frac{3}{17}$ by 9.	$\frac{52}{63}$ by 31.	$\frac{473}{1372}$ by 769.
$\frac{13}{18}$ by 12.	$\frac{33}{72}$ by 36.	$\frac{972}{1735}$ by 1188.
$\frac{13}{18}$ by 11.	$\frac{59}{78}$ by 41.	$\frac{1273}{2657}$ by 1273.
$\frac{15}{22}$ by 10.	$\frac{98}{133}$ by 63.	$\frac{627}{3178}$ by 2090.
$\frac{18}{22}$ by 15.	$\frac{99}{167}$ by 69.	$\frac{7592}{9737}$ by 5329.
$\frac{31}{27}$ by 18.	$\frac{23}{231}$ by 92.	$\frac{657}{16959}$ by 4599.
$\frac{37}{27}$ by 16.	$\frac{81}{236}$ by 108.	$\frac{985}{16321}$ by 3946.
$\frac{39}{27}$ by 19.	$\frac{27}{306}$ by 157.	$\frac{9765}{39774}$ by 7973.

We can often combine both of these processes with advantage. When the divisor is a composite number of which one factor is a factor of the numerator of the fraction, we can use that factor as a divisor of the numerator, and the remaining factor as a multiplier of the denominator: thus if the 15th part of $\frac{2}{3}$ be wanted, we observe that in order to divide by 15 we may first divide by 3 and then the quotient by 5. Now the 3rd part of $\frac{2}{3}$ is $\frac{2}{9}$, and the 5th part of $\frac{2}{9}$ is $\frac{2}{45}$. This artifice is the counterpart of that which we used (see page 121) for the multiplication of a fraction.

EXAMPLES.

Divide—

$\frac{2}{3} \frac{7}{2}$ by 24.	$\frac{9}{15} \frac{7}{4}$ by 269.	$\frac{53}{25} \frac{7}{6}$ by 4833.
$\frac{1}{4} \frac{7}{2}$ by 51.	$\frac{12}{18} \frac{3}{5}$ by 287.	$\frac{2}{5} \frac{7}{3} \frac{1}{9}$ by 3749.
$\frac{2}{5} \frac{9}{4}$ by 48.	$\frac{2}{5} \frac{6}{9} \frac{3}{1}$ by 317.	$\frac{1}{5} \frac{9}{9} \frac{6}{5} \frac{8}{1}$ by 5709.
$\frac{3}{5} \frac{4}{9}$ by 66.	$\frac{3}{4} \frac{5}{6} \frac{2}{9}$ by 396.	$\frac{5}{8} \frac{9}{6} \frac{2}{1} \frac{2}{5}$ by 7593.
$\frac{5}{9} \frac{7}{1}$ by 105.	$\frac{5}{7} \frac{9}{9} \frac{4}{7}$ by 333.	$\frac{9}{10} \frac{2}{6} \frac{7}{2} \frac{6}{5}$ by 12368.
$\frac{6}{10} \frac{4}{5}$ by 168.	$\frac{7}{9} \frac{9}{6} \frac{6}{7}$ by 873.	$\frac{8}{10} \frac{7}{3} \frac{2}{2} \frac{1}{5} \frac{1}{1}$ by 43757.
$\frac{5}{8} \frac{6}{9}$ by 161.	$\frac{9}{13} \frac{6}{6} \frac{3}{7}$ by 856.	$\frac{5}{9} \frac{9}{7} \frac{2}{3} \frac{2}{4} \frac{8}{2}$ by 42696.
$\frac{7}{11} \frac{2}{9}$ by 189.	$\frac{3}{10} \frac{7}{6} \frac{1}{5} \frac{1}{2}$ by 4809.	

I now proceed to explain a process which is so closely allied to multiplication as to be mistaken for it and to be called by the same name, even although the effects of the two processes be often exactly opposite. It would be convenient if we had a distinct name for this process, but it is no easy matter to find one free from objections; so I shall follow the usual course and call it multiplication, premising that in using this name all idea of meaning attached to the word *multiply* must be abandoned.

If I purchase seven yards of calico at fivepence per yard, I ascertain the price of the whole by repeating the price of one yard as often as there are yards; in other words, I multiply the five pence by seven, and obtain thirty-five pence as the price of my purchase. This is an example of pure multiplication, but I must observe that we do not, as a careless student may think,

multiply the five pence by the seven yards; we multiply it simply by the number seven. Every one must perceive that a *multiplier* never can be anything else than a pure or absolute number.

If the price of the calico had been fivepence halfpenny, I should still have multiplied the $5\frac{1}{2}$ pence by 7, the number of the yards, and would have obtained the result $38\frac{1}{2}$ pence. This also is an example of proper multiplication.

But if the quantity had been $7\frac{3}{4}$ yards at the price 5d. per yard, I should have had to multiply the 5d. by 7 to obtain the price of the seven yards, and to take three quarters of 5d., that is $3\frac{3}{4}$ d., to get the price of the three quarters of a yard; and thus the whole price would have been $38\frac{3}{4}$ pence. Now this process is not pure multiplication; it indeed combines multiplication and division; multiplication to get the price of the entire yards and division to obtain that of the fractional parts. To speak then of *multiplying* by $7\frac{3}{4}$ is improper, yet it would also be exceedingly inconvenient to have two distinct names, particularly because in the higher branches of arithmetic we use indeterminate characters to represent quantities, and we seldom know whether these be fractional or not. Our language accommodates itself in some degree to the difficulty of the case, for we speak of taking five pence, seven times and three-quarters of a time.

To place the matter in a stronger light, suppose that I had purchased only half a yard. Here in our usual language I have to *multiply* 5d. by $\frac{1}{2}$, although in truth I have to *divide* 5d. by 2.

Lastly let us consider the case in which both the price and the quantity are expressed by fractions. Suppose the quantity $7\frac{3}{4}$ yards, and the price $5\frac{1}{2}$ d. per yard. In the first place seven yards at $5\frac{1}{2}$ d. would cost 7 times $5\frac{1}{2}$ d. or $38\frac{1}{2}$ pennies, and in the second place three quarters of a yard at $5\frac{1}{2}$ d. must cost just three quarters of fivepence halfpenny, or *one quarter* of three times $5\frac{1}{2}$, that is of $16\frac{1}{2}$, which is $4\frac{1}{8}$ d. Hence the whole cost is $42\frac{5}{8}$ d.

From these examples it is easy to see that, in order to multiply by a fraction, we must multiply by the numerator and divide by the denominator of the fraction, and that the product is greater or less than the multiplicand according as the multiplier is greater or less than unit.

EXAMPLES.

Multiply—

$\frac{1}{3}$ by $\frac{1}{2}$	$6\frac{9}{10}$ by $3\frac{7}{15}$	$173\frac{67}{8}$ by $56\frac{32}{4}$
$\frac{2}{5}$ by $\frac{1}{4}$	$7\frac{13}{16}$ by $5\frac{17}{18}$	$258\frac{76}{5}$ by $67\frac{9}{50}$
$\frac{7}{6}$ by $\frac{2}{7}$	$13\frac{13}{26}$ by $9\frac{2}{7}$	$357\frac{7}{9}$ by $89\frac{7}{12}$
$1\frac{3}{4}$ by $\frac{1}{5}$	$26\frac{14}{17}$ by $16\frac{11}{23}$	$493\frac{97}{13}$ by $146\frac{17}{93}$
$1\frac{7}{8}$ by $\frac{6}{7}$	$29\frac{14}{27}$ by $23\frac{9}{31}$	$937\frac{51}{178}$ by $154\frac{3}{91}$
$2\frac{5}{12}$ by $1\frac{3}{5}$	$37\frac{41}{57}$ by $12\frac{17}{32}$	$1372\frac{156}{275}$ by $162\frac{70}{153}$
$3\frac{2}{11}$ by $1\frac{6}{17}$	$58\frac{33}{40}$ by $37\frac{6}{11}$	$1960\frac{137}{380}$ by $273\frac{14}{65}$
$5\frac{7}{17}$ by $2\frac{3}{16}$	$94\frac{25}{48}$ by $47\frac{5}{24}$	

The product of fractions can be simplified whenever there is a factor common to the numerators and denominators; thus if we have to multiply $7\frac{1}{3}$ by $4\frac{2}{7}$ and $13\frac{1}{4}$ successively, the work is capable of being much shortened; for writing these mixed quantities in the fractional form we have $\frac{22}{3} \times \frac{30}{7} \times \frac{49}{4}$ and the product in the usual way would be $\frac{22 \times 30 \times 49}{3 \times 7 \times 4}$; but we observe that 30 is divisible by 3, 49 by 7, and that the factors 2 of the 22, and the 2 of the 30, are just the factors of 4, so that the result is $11 \times 5 \times 7 = 385$.

EXAMPLES.

Multiply—

$6\frac{11}{12}$ by $3\frac{14}{15}$	$8\frac{16}{17}$ by $5\frac{23}{25}$	$122\frac{0}{3}$ by $18\frac{17}{17}$
$5\frac{8}{21}$ by $7\frac{7}{12}$	$11\frac{23}{32}$ by $10\frac{13}{15}$	
$21\frac{7}{13}$ by $16\frac{21}{32}$	$4\frac{9}{16}$	$19\frac{5}{4}$ by $28\frac{16}{39}$ by $22\frac{4}{17}$
$24\frac{3}{17}$ by $18\frac{15}{26}$	$13\frac{14}{16}$	$32\frac{14}{17}$ by $25\frac{7}{8}$ by $15\frac{7}{15}$

$$\begin{array}{ll}
 35\frac{1}{2}\frac{2}{3} \text{ by } 29\frac{1}{4}\frac{2}{7} \text{ by } 17\frac{1}{8}\frac{2}{3} & 93\frac{5}{8}\frac{2}{3} \text{ by } 78\frac{3}{4}\frac{7}{10} \text{ by } 23\frac{9}{14} \\
 57\frac{3}{4}\frac{2}{3} \text{ by } 34\frac{8}{9}\frac{2}{3} \text{ by } 39\frac{2}{4}\frac{7}{8} & 120\frac{1}{8}\frac{2}{3} \text{ by } 71\frac{8}{9}\frac{2}{3} \text{ by } 27\frac{5}{10}\frac{1}{1} \\
 73\frac{5}{8}\frac{5}{4} \text{ by } 51\frac{1}{2}\frac{2}{3} \text{ by } 47\frac{3}{8}\frac{2}{3} & 156\frac{8}{9}\frac{2}{3} \text{ by } 107\frac{7}{9}\frac{2}{3} \text{ by } 56\frac{3}{4}\frac{2}{3} \\
 84\frac{7}{8}\frac{2}{3} \text{ by } 64\frac{2}{5}\frac{2}{3} \text{ by } 32\frac{3}{8}\frac{2}{3} & \\
 234\frac{2}{4}\frac{2}{3} \text{ by } 147\frac{4}{10}\frac{7}{9} \text{ by } 53\frac{3}{8}\frac{2}{3} \text{ by } 41\frac{1}{4}\frac{1}{8} \\
 317\frac{9}{13}\frac{2}{7} \text{ by } 257\frac{7}{8}\frac{7}{11} \text{ by } 191\frac{5}{8}\frac{4}{3} \text{ by } 18\frac{2}{8}\frac{7}{3} \\
 544\frac{5}{8}\frac{2}{3} \text{ by } 356\frac{1}{5}\frac{4}{11} \text{ by } 165\frac{4}{7}\frac{5}{9} \text{ by } 154\frac{7}{8}\frac{7}{5} \\
 637\frac{4}{8}\frac{2}{3} \text{ by } 558\frac{2}{7}\frac{2}{10} \text{ by } 489\frac{3}{4}\frac{2}{3} \text{ by } 271\frac{2}{3}\frac{2}{9} \text{ by } 108\frac{2}{5}\frac{2}{9} \\
 625\frac{1}{2}\frac{9}{8}\frac{2}{3} \text{ by } 597\frac{2}{8}\frac{2}{3} \text{ by } 327\frac{4}{5}\frac{5}{8} \text{ by } 273\frac{9}{12}\frac{2}{7} \text{ by } 256\frac{3}{7}\frac{7}{1} \\
 1832\frac{2}{5}\frac{3}{9}\frac{5}{4} \text{ by } 650\frac{1}{6}\frac{9}{5}\frac{5}{3} \text{ by } 538\frac{4}{7}\frac{2}{5} \text{ by } 490\frac{3}{5}\frac{1}{4}\frac{2}{7} \text{ by } 257\frac{5}{8}\frac{2}{4} \\
 2569\frac{1}{3}\frac{5}{18}\frac{4}{5} \text{ by } 1840\frac{2}{5}\frac{7}{3}\frac{2}{3} \text{ by } 736\frac{2}{4}\frac{2}{3} \text{ by } 281\frac{3}{9}\frac{1}{9} \text{ by } 176\frac{2}{10}\frac{2}{6} \\
 7433\frac{4}{1}\frac{5}{10}\frac{2}{9} \text{ by } 3229\frac{7}{8}\frac{2}{9} \text{ by } 2391\frac{5}{7}\frac{2}{7} \text{ by } 1398\frac{1}{10}\frac{1}{10} \text{ by } 477\frac{3}{6}\frac{2}{1} \\
 13968\frac{1}{2}\frac{7}{9}\frac{2}{3} \text{ by } 9605\frac{2}{10}\frac{5}{3}\frac{7}{3} \text{ by } 8954\frac{8}{9}\frac{7}{4}\frac{7}{11} \text{ by } 5690\frac{5}{12}\frac{2}{7} \text{ by } 2671\frac{1}{3}\frac{2}{4}\frac{2}{9}
 \end{array}$$

If we know the cost of a quantity of goods and desire to learn what has been the price per unit, that is, per yard, per pound, &c., we have to *divide* the entire cost into as many equal parts as there are units in the quantity. Thus if I paid 35 shillings for 7 pounds of tea and desire to know what has been the price of each pound, I divide the 35 shillings by 7 and obtain 5 shillings as the price of one pound. In this case we have a *pure* division. But if the quantity had been fractional the operation would not have been quite so simple.

For instance if $5\frac{2}{3}$ yards cost me 68 pence, and if I wish to know how much that has come to per yard, we can hardly say *divide* 68 pence into $5\frac{2}{3}$ equal parts, unless indeed we depart considerably from the ordinary meaning of the word *divide*. However that may be, let us try to accomplish our task. Now $5\frac{2}{3}$ yards make just 17 third parts of a yard; and since the cost of these 17 parts was 68 pence we obtain the price of *one* part by dividing 68 pence by 17; thus we learn that each third part of a yard cost 4 pence, and this by a pure division. Then since the third part of a yard cost 4 pence, the whole yard must

have cost 12 pence. The operation then usually called *division by* $5\frac{2}{3}$ consists of a division by 17 and a multiplication by 3. The order in which these two operations are performed is of no moment, for we might have proceeded thus: Since $5\frac{2}{3}$ yards cost 68 pence, 3 times $5\frac{2}{3}$ yards, that is 17 yards, must, at the same rate, cost 204 pence, and therefore 1 yard must cost the 17th part of 204 pence, that is 12 pence.

Just as we saw that what is called multiplication by a fraction is not properly multiplication, so we now find that what is usually called division by a fraction is not properly division; in general both of these processes require multiplication and division, and in many cases they are exactly the reverse of what their names imply.

Thus if a person paid 17 shillings for two-thirds of an ounce of ultra-marine, and wish to know its price per ounce, he must, in the general language, divide 17 shillings by $\frac{2}{3}$; in reality he must multiply by 3 to obtain 51 shillings, the price of two ounces, and divide by 2 to get $25\frac{1}{2}$ shillings, the price of one ounce.

Or again if $\frac{1}{5}$ of a pound of spice cost 4 shillings, we would obtain the price of one pound by *dividing* 4 shillings by $\frac{1}{5}$, in reality by *multiplying* the 4 shillings by 5.

When we have to deal with fractions, then, the words *divide* and *multiply* lose entirely their primitive significations.

To multiply any quantity by a fraction is to multiply by the numerator and to divide by the denominator of the fraction.

To divide any quantity by a fraction is to divide by the numerator and to multiply by the denominator.

Hence to divide by $\frac{2}{3}$ gives the same result as to multiply by $\frac{3}{2}$, so that division by a fraction can always be converted into multiplication by the inverse fraction.

EXAMPLES.

$3\frac{1}{2}$ yards of ribbon cost $10\frac{1}{2}$ pence, what is the price of 1 yard?

1 pound of biscuit cost $11\frac{1}{2}$ pence, what will $3\frac{1}{2}$ pounds cost?

1 herring cost $\frac{3}{4}$ of a penny, what will 31 cost ?

$7\frac{1}{2}$ yards of cloth cost 60 pence, what is the price of 1 yard ?

$13\frac{3}{4}$ yards of silk cost 60 pence, what is the price of 1 yard ?

$15\frac{5}{16}$ pounds of sugar cost $61\frac{1}{4}$ pence, what is the price of 1 pound ?

23 bobbins of linen thread cost $126\frac{1}{2}$ farthings, what did each cost ?

$45\frac{3}{7}$ tons of coal cost $545\frac{1}{4}$ shillings, what is the price of 1 ton ?

1 yard of rope cost $2\frac{7}{11}$ pence, what will $27\frac{1}{2}$ yards cost ?

$57\frac{3}{8}$ pounds of tea cost $205\frac{9}{32}$ shillings, what is the price of 1 pound ?

If 67 pounds of coffee cost $93\frac{5}{8}$ shillings, what will 1 pound cost ?

168 oranges cost $471\frac{9}{8}$ pence, what will the price of 1 be ?

$289\frac{3}{7}$ pounds of sugar cost $1281\frac{9}{32}\frac{2}{9}\frac{9}{1}$ pence, what will 1 pound cost ?

If 1 pound of lead cost $3\frac{1}{8}$ pence, what will $765\frac{7}{8}$ pounds cost ?

If 1 ton of iron cost $11\frac{3}{10}$ pounds, what will $832\frac{1}{2}$ tons cost ?

$273\frac{8}{11}$ gallons of wine cost $5112\frac{2}{7}\frac{8}{8}$ pence, what will 1 gallon cost ?

If 1 yard of silk cost $14\frac{3}{4}$ shillings, what will $312\frac{3}{8}\frac{1}{5}$ yards cost ?

$1965\frac{7}{8}\frac{3}{5}$ pounds of tea cost $13366\frac{1}{17}\frac{1}{5}\frac{1}{6}$ shillings, what will 1 pound cost ?

$24\frac{3}{9}$ yards of velvet cost $627\frac{1}{11}\frac{7}{8}$ shillings, what is the price of 1 yard ?

$194\frac{1}{3}\frac{5}{9}$ yards of lead pipe cost $19924\frac{1}{10}\frac{9}{4}$ shillings, what is the price of 1 yard ?

CHAPTER X.

ON THE NOTATION AND CONVERSION OF DECIMAL FRACTIONS.

C. WE have seen that, for business purposes, it is often necessary to divide the unit into parts. With only a gallon measure we are unable to measure out small quantities : we must have measures of fractions of a gallon. Now it would be both expensive and troublesome to provide ourselves with all manner of fractional measures, while it would be inconvenient to have one dealer measuring in fifths, another in sevenths. It would, clearly, be for the general advantage to have a definite system of division agreed upon, and it is important to consider what that division ought to be.

Our scale of numeration goes on by tens, and we have experienced the prodigious facilities which the use of a uniform system gives us : we may then anticipate that the continuation of the same system into the notation of fractions should extend those facilities. If we divide the unit into ten parts, each of these into ten parts, and so on, we obtain a series of fractions easily expressed in the decimal system. The principle of that system is that the value of a figure is augmented tenfold for every remove to the left ; or that its value is reduced ten times for each remove to the right. We have only to carry out this system in order to obtain the notation of decimal fractions. The place immediately to the right of units becomes the place of tenths, the next the place of hundredths, and so on. But if we put a figure to the right of units, we raise the rank of all the other figures : we must therefore, to prevent this, put some mark

for the purpose of distinguishing the units' place. The distinction is usually made by placing a dot or comma after the units, so as to separate the integer from the fractional part of the expression: thus 437.362 means four hundred and thirty-seven units, three tenths, six hundredths and two thousandth parts of a unit; it is clear that, since a tenth part is just ten hundredth parts, these fractions may be read *thirty-six hundredths and two thousandths*; or that, since one hundredth part is equal to ten thousandth parts, they may be read *three hundred and sixty-two thousandths*; and that the above quantity may be written in the common fractional form thus: $437\frac{3}{10} + \frac{6}{100} + \frac{2}{1000}$ or $437\frac{362}{1000} + \frac{2}{1000}$, or yet $437\frac{362}{1000}$.

It is to be observed that a zero written to the right of a decimal fraction has neither any value nor any effect, because the decimal point indicates the ranks of the figures: thus 437.3620 or 437.3620000 have the same meaning with 437.362: just in the same way a zero written to the left of an integer has no effect, for 0437 or 000437 mean just 437, since the last figure is in the units' place.

EXAMPLES.

Read in words and write in the common fractional form the following decimals:—

23.7052; 38.027; 273.1804; 1.76283; 5.027619; 0.31827;
 .98263; .00715; 15.602100; 3.000005; .0000009;
 28.10065701; 51.075149271; 378.6500072504;
 470.3197046357; 29.701035621573;
 6915.735; 162.7859473; 165.0000003; .0147153.

By this extension the decimal notation is rendered capable of indicating quantities of extreme minuteness: thus if our unit be an inch, .001 represents the thousandth part of an inch, that is less than half the thickness of a hair, while .0000001 stands for the ten-millionth part of an inch, a distance more minute than can be measured by help of the most powerful microscope.

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From the very nature of the notation, it follows that if the decimal point be removed one step to the right the expression becomes ten times greater; if to the left, ten times less: thus 4373.62 is ten times 437.362, and 43.7362 is its tenth part. To multiply or to divide by 100 we must shift the decimal point two steps to the right or to the left; and so on for 1000, 10000, &c.

EXAMPLES.

Ten times 38.7 is :
 The tenth part of 38.7 is :
 The tenth part of 410. is :
 Ten times 6730. is :
 One hundred times 3600.04 is :
 The hundredth part of 8573.605 is :
 One thousand times 356.51004 is :
 One thousand times 600.00005 is :
 The one thousandth part of 7615.0061 is .

Sometimes it happens that the decimal point has to be placed beyond the range of the actual figures; in such cases we must indicate the blanks by zeroes. Thus 10000 times 5.76 is 57600. and the ten-thousandth part of it .000576 .

EXAMPLES.

The hundredth part of 5.096 is :
 The thousandth part of 0.605 is :
 One hundred times 470.4 is :
 One thousand times 1571.3 is :
 The ten-thousandth part of 2.957 is :
 The ten-thousandth part of .00653 is :
 Ten thousand times 271.5 is :
 One hundred thousand times 8394.61 is
 The one hundred-thousandth part of 65.9472 is
 The one hundred-thousandth part of 185.732 is

One million times 159.6543 is :

The millionth part of 451.65 is .

Having thus arranged the notation of fractions of which the denominators are 10, 100, 1000, &c., we have to consider how other fractions are to be converted into these, or how these are to be reconverted into common fractions. In some cases we experience no difficulty: thus $\frac{1}{2}$ is just $\frac{5}{10}$ or .5, so that $3\frac{1}{2}$ would be written 3.5; $\frac{1}{4}$ is $\frac{25}{100}$, $\frac{3}{4}$ is $\frac{75}{100}$, so that $6\frac{1}{4}$ is 6.25; $7\frac{3}{4}$ is 7.75; $\frac{1}{5}$ is $\frac{2}{10}$ or .2; and in general all fractions of which the denominators are products of the prime numbers 2 and 5 can be put in the decimal form. But it is otherwise with such a fraction as $\frac{1}{3}$, the denominator of which is not contained exactly in 10, in 100, or in any term of the series of decimal denominators. Neither the fraction $\frac{1}{3}$ nor any other fraction into the denominator of which (when in its simplest form) a prime number other than 2 or 5 enters, can be expressed decimally. This is as much as to say that hardly any fractions can be so expressed, and would seem to offer a serious objection to the use of decimal fractions. However a few steps in the descending scale bring us to quantities so minute as to be imperceptible, so that although we cannot give the value of a fraction with absolute precision, we can easily attain to any prescribed degree of exactitude.

Let it be required to express in decimals the value of the fraction $\frac{7}{16}$; that is of the sixteenth part of seven units. As each unit consists of
 ten tenths, seven units consist of seventy
 tenths, the sixteenth part of which is 4
 tenths, with 6 tenths over. Now each
 tenth is ten hundredths, so that the 6
 tenths make 60 hundredths, of which the
 16th part is 3 hundredths with 12 hun-
 dredths over. These 12 hundredths are
 equivalent to 120 thousandths, of which
 the 16th part is 7 thousandths with 8 thousandths over. These

16		7.0		.4375
		6.4		
		<u>60</u>		
		48		
		<u>120</u>		
		112		
		<u>80</u>		
		80		
		<u>80</u>		

are equivalent to 80 ten-thousandths, of which the 16th part is 5 ten-thousandths exactly, so that the fraction $\frac{7}{16}$ is of the same value with .4375. The work here closely resembles a common division.

To convert the fraction $\frac{7}{16}$ into decimal fractions we proceed exactly in the same way; but the division never terminates, so that however far we may proceed we cannot accurately express the fraction $\frac{7}{16}$ on the decimal scale. Yet if we were counting in inches, the last digit of .368421 would stand for the millionth part of an inch, and .368421 cannot differ from the true value by so much as the millionth part of unit. This degree of accuracy is sufficient for almost every purpose; and if it be not considered so, we may proceed still farther and take .36842105, so that no practical inconvenience need arise.

$$\begin{array}{r}
 .36842105, \&c. \\
 19 \overline{) 7.0} \\
 \underline{5.7} \\
 1.30 \\
 \underline{1.14} \\
 160 \\
 \underline{152} \\
 80 \\
 \underline{76} \\
 40 \\
 \underline{38} \\
 20 \\
 \underline{19} \\
 100 \\
 \underline{95}, \&c.
 \end{array}$$

In performing a division the remainder can never exceed the divisor, and therefore however far we carry on the above operation, we can only have the remainders 1, 2, 3,.....16, 17, 18, so that we must soon come upon an old remainder again. Whenever this happens, the subsequent figures of the quotient must be copies of the preceding ones. Thus on computing 18 terms of the fraction $\frac{7}{16}$ we find them to be .368421052631578947 with a remainder 7, and the subsequent terms of the decimal must just be copies of the preceding eighteen figures following each other in endless succession.

Decimals of this kind are called *circulating* decimals, and are said to circulate in *periods*.

As soon as we have established the circulation it is unnecessary to continue the division.

EXAMPLES.

Express in decimals the following fractions :—

$\frac{1}{3}$	$\frac{29}{49}$	$\frac{42}{83}$	$\frac{36}{131}$	$\frac{9801}{18096}$
$\frac{2}{7}$	$\frac{35}{57}$	$\frac{16}{81}$	$\frac{42}{167}$	$\frac{549}{23631}$
$\frac{25}{25}$	$\frac{9}{64}$	$\frac{59}{89}$	$\frac{63}{271}$	$\frac{35671}{39642}$
$\frac{5}{14}$	$\frac{47}{87}$	$\frac{27}{98}$	$\frac{85}{346}$	$\frac{64934}{82917}$
$\frac{11}{24}$	$\frac{12}{67}$	$\frac{7}{99}$	$\frac{43}{576}$	$\frac{5721}{63780}$
$\frac{17}{32}$	$\frac{31}{72}$	$\frac{13}{100}$	$\frac{71}{6947}$	$\frac{86544}{93975}$
$\frac{16}{45}$	$\frac{25}{72}$	$\frac{64}{115}$	$\frac{4758}{5767}$	$\frac{573419}{698239}$
$\frac{21}{47}$	$\frac{59}{79}$	$\frac{75}{125}$	$\frac{743}{2436}$	$\frac{7346912}{9387336}$

D. The number of figures in the period of a circulating decimal can never exceed the number immediately less than the denominator of the *vulgar* or common fraction to which it is equivalent, and often it is an aliquot part of that number. Thus the period for the fraction $\frac{1}{7}$ consists of 6 figures: that for $\frac{17}{9}$ consists of 18 figures, but the period for $\frac{1}{3}$, instead of having 36, has only 3 figures.

The complete inquiry into these circumstances is too difficult for the present state of our attainments in arithmetic: it is enough for me, here, to indicate in a simple way the leading facts connected with them.

When the divisor is 9 the period consists of only one figure, thus $\frac{1}{9} = .1111$, &c., $\frac{2}{9} = .2222$, &c., and so on. We call such decimals repeaters, and usually mark them by a dot over the repetend thus $.22\dot{2}$. It is easy to see that this is because 9 is one less than the root of the scale of progression: if that root had been 100, 99 would have had the same property, and indeed we find $\frac{1}{99} = .010101$, &c., whence $\frac{43}{99} = .434343$: again if the root of the scale had been 1000, the division by 999 would have given a repeater, and on the decimal scale it does give a circulating period of three figures, thus $\frac{1}{999} = .001001001$, &c., $\frac{173}{999} = .173173173$, and so on of all similar

divisors. From this we easily conclude that a period of so many (say 7) figures may result from a division by 9999999, the number represented by a period of as many nines.

All fractions having the denominator 9 are single repeaters ; now among these, those with the denominator 3 must be found, since $\frac{2}{3} = \frac{1}{3}$, $\frac{4}{9} = \frac{2}{3}$; hence we conclude that thirds and ninths give single repeaters.

Again all fractions having 99 for their denominator repeat in periods of two figures : but among these are those having the denominators 3, 9, 11, 33 ; hence we infer that a circulating decimal of which the period has two figures must have resulted from a division by 11, by 33, or else by 99.

Fractions having the denominator 999 include those of which 27, 37, 111, 333, are denominators : and these divisors are the only ones which give a period of three figures.

In the same way 9999 is a multiple of 101, 303, 909, 1111, 3333, and these are the only divisors which can give a period of four figures.

Further 99999 is the continued product of 3.3.41.271, and thus the only divisors which can give a period of five figures are 41, 123, 271, 369, 813, 2439, 11111, 33333, and 99999.

To find those divisors which give a period of six places we resolve 999999 into its factors : these are 3. 3. 3. 7. 11. 13. 37, and those products of these which are not already given as producing repeaters or periods of two or of three figures are the only divisors which give a period of six figures. It may be a good exercise for the student to compute them all.

Periods of seven places are produced by the divisor 9999999, of which the factors are 3. 3. 239. 4649.

Periods of eight figures by the divisor 99999999 = 3. 3. 11. 73. 101. 137.

Periods of nine places by 999999999 = 3. 3. 3. 37. 333667.

Periods of ten places by 9999999999 = 3. 3. 11. 41. 271. 9091.

Periods of eleven places by 99999999999 = 9. 1111111111

(the ambitious student may try whether this last number 11111111111 be prime or composite).

Periods of twelve places are produced by the divisor 999999999999 = 3. 3. 3. 7. 11. 13. 37. 101. 9901 ; so that 9901 is the smallest divisor which can give a period of twelve figures.

For thirteen places we have 9. 53. 20964360587.

For fourteen places 3. 3. 239.4649. 11.909091, and so on.

From this it seems that very few fractions likely to occur in business give decimals which circulate in periods of moderate length.

EXAMPLES.

Express in decimals the following fractions :—

$\frac{2}{3}$	$\frac{32}{41}$	$\frac{136}{137}$	$\frac{478}{999}$	$\frac{542}{11111}$
$\frac{5}{9}$	$\frac{69}{73}$	$\frac{142}{239}$	$\frac{43}{2439}$	$\frac{654}{9999}$
$\frac{7}{9}$	$\frac{64}{111}$	$\frac{34}{369}$	$\frac{54}{1111}$	$\frac{7238}{37037}$
$\frac{6}{11}$	$\frac{151}{303}$	$\frac{543}{813}$	$\frac{784}{3333}$	$\frac{596}{33333}$
$\frac{10}{11}$	$\frac{23}{333}$	$\frac{632}{909}$	$\frac{652}{2439}$	$\frac{6580}{99999}$
$\frac{5}{27}$	$\frac{84}{123}$	$\frac{57}{813}$	$\frac{745}{999}$	$\frac{956}{11111}$
$\frac{36}{37}$	$\frac{95}{271}$	$\frac{723}{1001}$	$\frac{274}{3333}$	$\frac{109370}{111111}$
$\frac{25}{33}$	$\frac{15}{369}$	$\frac{65}{909}$	$\frac{6530}{9091}$	$\frac{9806}{333333}$
$\frac{71}{99}$	$\frac{372}{369}$	$\frac{354}{1111}$	$\frac{2171}{4049}$	$\frac{7065}{33333}$
$\frac{10}{101}$	$\frac{147}{333}$	$\frac{99}{1001}$	$\frac{891}{9999}$	$\frac{999997}{999999}$

C. When the denominator of a fraction ends in one or more zeroes, the circulation of the equivalent decimal will, in general, begin after as many decimal places as there are zeroes. Thus $\frac{537}{370} = 1.4513513$, where the period 513 is separated from the decimal point by the non-circulating figure 4 : and $\frac{867}{1300} = .66692307692307$, the first period being preceded by the non-circulating figures 66.

EXAMPLES.

$\frac{343}{640}$	$\frac{237}{9500}$	347 $\frac{9473}{67000}$
$\frac{478}{510}$	$\frac{2791}{3600}$	1681 $\frac{42632}{128000}$
$\frac{171}{730}$	$\frac{9461}{12700}$	2800 $\frac{1591}{189000}$
$\frac{1942}{2700}$	$\frac{13763}{25600}$	7412 $\frac{254322}{370000}$
$\frac{2651}{5480}$	$\frac{6541}{35000}$	13478 $\frac{421717}{3030000}$

Having now seen how we may convert common into decimal fractions, it remains for us to consider how decimal may be changed into the equivalent common fractions.

If the decimal be finite there is no difficulty in putting it in the form of a vulgar fraction : for if .648 were proposed we see at once, from the very nature of the notation, that it is $\frac{648}{1000}$ written in another form : this may be simplified if the numerator be divisible by 2 or by 5 : in the present example the fraction becomes $\frac{81}{125}$.

EXAMPLES.

Convert the following decimal fractions into common fractions :—

.3	45.1003	300000.75000
.12	681.0002	0.492800
.24	954.2030	38721.000075364
0.374	000.693	97653.728960004
.653	5732.7296	70.12007640
5.942	1007.3154	1957.04568
0.776	395.00412	2395.1003964637
357.75	4257.3336	7496.35491768
10.2544	10000.002576	

D. But when the decimal includes a circulating period the conversion of it into a common fraction is a little more difficult. If the period begin immediately after the decimal point as in .43564356, we see at once that it is equivalent to the fraction of which the numerator is the period and the denominator

period of as many nines, thus $\frac{4356}{9999}$, which may be simplified to $\frac{44}{101}$.

When the first period is preceded by non-circulating figures we may divide the decimal into two parts and treat these separately. For example the fraction $.73574\dot{5}74$ may be regarded as the sum of $.73$ and $.00574\dot{5}74$, now $.574\dot{5}74 = \frac{574}{999}$, wherefore $.00574\dot{5}74 = \frac{574}{99900}$, while $.73 = \frac{73}{100}$, hence $.73574\dot{5}74 = \frac{73}{100} + \frac{574}{99900} = \frac{73591}{99900}$.

EXAMPLES.

Convert the following decimal fractions into common fractions :—

$.666\dot{6}$	$.3782$	$2765.045\dot{0}45$
$.7777$	$.72277227722\dot{7}$	$27.0715\dot{0}715$
$2.8518\dot{5}18$	$.17658536585\dot{3}$	$4379.921771\dot{2}1771$
$0.38759259\dot{2}$	578.87878	$38.49650349650\dot{3}$
$25.3781\dot{8}1$	$1697.85285285\dot{2}$	$236.95204795204\dot{7}$
$0.5827\dot{2}7$	$357.9306930\dot{6}$	$623.7990965799096\dot{5}$
$371.64864\dot{8}$	$.772357723\dot{5}$	$3651.23985\dot{2}3985$
0.081818	$54086.2409240\dot{9}$	

E. The value of a circulating decimal is readily found by help of a simple equation. One example is enough to explain the process. Let it be required to find the value of the decimal $.28739063906$. Denote this value by f , then

$$.28739063906\dot{3}906 = f,$$

and multiplying by 10000,

$$2873.90639063906\dot{3}906 = 10000 f,$$

then subtracting the first equation from the second,

$$2873.619 = 9999 f, \text{ whence}$$

$$\frac{2873.619}{9999} \text{ or } \frac{2873619}{9999000} = f.$$

In ordinary work the dots over the repetends or circulators are omitted.

EXAMPLES.

Convert the following decimal fractions into common fractions :—

.986666	0.021988519885
0.091111	2560.56063063063
.23333	57110.0071129707112970
10.8272727	0.54796747967
0.0148148	235.8891780821917808219
142.5837837	875.99793719793719
64.0252252252	628.0156545654
30.0091089108	1652.063099630896
5374.0813813813	14007.016435435435
13.0008241094710947	6031.0734579945799
1651.09361638361638	95482.0020752107521

CHAPTER XI.

ON THE ADDITION, SUBTRACTION, MULTIPLICATION, AND DIVISION OF DECIMAL FRACTIONS.

C. THERE is nothing to distinguish the addition and subtraction of decimal fractions from the addition and subtraction of integers. All that we must take care of is to avoid mixing quantities of different ranks; and this we accomplish by keeping the one decimal point under the other.

Thus to add together 37.1065, 432.07, 4.20733, and 11.706, we write them so as to have all their units' places in one column thus—

$$\begin{array}{r} 37.1065 \\ 432.07 \\ 4.20733 \\ \underline{11.706} \\ 485.08983 \end{array}$$

and proceed exactly as in common addition. Many would prefer, in such an example, to fill up all the blank spaces with zeroes so as to bring the decimal parts to appear to have the same number of places thus—

$$\begin{array}{r} 37.10650 \\ 432.07000 \\ 4.20733 \\ \underline{11.70600} \\ 485.08983 \end{array}$$

and this has the advantage of leading the eye along the columns; but after all it is a matter of taste.

If the subtrahend should have more decimal places than the minuend it is also usual to supply zeroes; thus the subtraction

$$\begin{array}{r} 421.763 \\ - 73.280974 \\ \hline 348.482026 \end{array}$$

would be done by many

$$\begin{array}{r} 421.763000 \\ 73.280974 \\ \hline 348.482026 \end{array}$$

and all that can be said about the matter is, that while the additional zeroes do no harm they do as little good.

EXAMPLES IN ADDITION.

Add together—

63.507 and 7.412; 47.397 and 23.201; 98.7439 and 42.60751; 793.65003 and 158.34996; 1380.006571 and 329.6073528; 256.35007 and 23.9463769; 7593.613270056 and 2406.370064; 754.69135472, 17653.000000731968 and 6392.15031541; 35749.15364107850026 and 1398.6791547318; 91632.5470813168542, 2937.13861542956154173, and 6219.300075610347968, 5714.93627153001, 6954.00000000095173623, 61473.95172651547, 1573.615920739852047 and 379562.540000397162.

EXAMPLES IN SUBTRACTION.

Subtract—

27.61 from 3985; 35.276 from 81.694; 57.3801 from 137.615; 253.57026 from 391.2654; 358.48924 from 763.5100472; 162.000053 from 271.3; 792.65935472 from 1093.25713468154; 1698.400973416507 from 3942.073165427973546; 57143.651240000739 from 93780.000000000923651; 1473.2069978147658412 from 137926.572041603; 71591.65415398457291105 from 957314.279568419536715; 7465.93000051269173051009265 from 739165.401562; 15920.0729156748519 from 961002.161804253610785913.

In multiplying a decimal fraction by any number we proceed exactly as in the multiplication of integers. Thus to multiply 48.7023 by 7 we begin 7 times 3 make 21 (in reality 7 times 3 ten-thousandths make 21 ten-thousandths), and write the 1 in the same column with the multiplicand. But if the multiplier had been 70 we should have raised every figure one rank, because 70 being ten times 7, 70 times any quantity must be ten times seven times that quantity. Again if the multiplier had been .7, we should have depressed the rank of each figure, because seven-tenths of any quantity is the tenth part of seven times that quantity; and the same remarks may be applied to multiplications by 700 or by .07.

$$\begin{array}{r}
 48.7023 \\
 7. \\
 \hline
 340.9161 \\
 \\
 48.7023 \\
 70. \\
 \hline
 3409.161 \\
 \\
 48.7023 \\
 .7 \\
 \hline
 34.09161
 \end{array}$$

If, then, the multiplier be in the place of units, the product has to be written directly under the multiplicand; but if the multiplier be to the right or to the left of the units' place, the product must be shifted as many steps to the right or to the left of the multiplicand. Thus, in the annexed example, when multiplying by the 2, which is two removes to the left of units, we place the product two steps to the left of where it would have been if the multiplier had been units; or when multiplying by the 9, which is three removes to the right of units, we shift the product three steps to the right.

$$\begin{array}{r}
 514.736 \\
 283.459 \\
 \hline
 1028.195 \\
 41178.88 \\
 1544.208 \\
 205.8944 \\
 25.73680 \\
 \hline
 4.632624 \\
 145906.551824 \\
 \\
 514.736 \\
 283.459 \\
 \hline
 4.632624 \\
 25.73680 \\
 205.8944 \\
 1544.208 \\
 41178.88 \\
 102947.2 \\
 \hline
 145906.551824
 \end{array}$$

When we keep this principle in view we may take the figures of the multiplier in any order: yet, in general, it is convenient to take them in succession, beginning either with the lowest or with the highest as we

Thus we see that the multiplication of decimals differs in no respect from that of integers.

EXAMPLES.

Multiply—

17.62 by	9	4658.01573 by	135.0416
26.53 by	0.8	1347.654 by	256.1
125.68 by	1.47	65327.09007 by	1530.95728
196.043 by	3.61	75643.1475 by	93.007
365.473 by	13.825	87390.80735 by	369.500103
571.9701 by	0.0274	197368.571983 by	671.0000053
290073.6918572 by		3491.796154	
5732916.573 by		176908.5073915	
27865700.19250478 by		0.000965483	
15726185.1263594 by		16953267.14	
57562917.56008152 by		6150000	
7265003.15970462 by		39172.0573641	
447161910.57290071 by		7491682.5755679	
350069175.01 by		735.65174732	

D. When each factor contains a decimal fraction the produce must contain one having as many places of decimals as both of the factors : thus in the above instance the lowest figures of the factors are $\frac{6}{1000}$ and $\frac{8}{1000}$, the product of which is $\frac{54}{1000000}$, so that the 4 must be written in the place of millionths, that is in the sixth decimal place ; and similarly of any others. The degree of minuteness, therefore, of the product far exceeds that of its factors, and may be beyond any attainable precision. Thus although in some very precise operations, we may be able to count the thousandth part of a unit beside five hundred units, that is, although we may be sure of the last figure in the multiplicand 514.736, no one can imagine the possibility of estimating the millionth part of a unit alongside of one hundred thousand units. The last figure then of the product 145906.351824 is altogether useless : even the four or five last figures may be rejected without practically vitiating the result ; and it is desirable to avoid the labour of obtaining these unnecessary figures. For this purpose we *contract* the multiplication : the process may be best seen by an example.

Let the product of 3.8640372 and 1.2479603 be required true to the seventh decimal place. Beginning with the highest figure of the multiplier we obtain the first line of the product ; taking then the second figure we put the second line one step out, thus obtaining one decimal place more than we wish to preserve : this we separate by a line. The product by the $\frac{4}{100}$ would go out two steps ; and the last figure may be rejected, so we may cut off one place from the multiplicand, and begin 4 times 7 make 28, &c. Rather than mark off the figures of the multiplicand it is better to write a small 4 above the last figure multiplied by 4. Taking now the $\frac{7}{1000}$ we begin 7 times 3 make 21 (or with 5 brought up from below 26) ; and so on. In this way the work is curtailed as we proceed : and when we sum up we reject the eighth decimal as unnecessary. The use of carrying the multiplication one step farther than the desired result is to obtain what is carried up from that step.

$$\begin{array}{r}
 3069742 \\
 3.8640372 \\
 1.2479603 \\
 \hline
 3.8640372 \\
 77280744 \\
 15456148 \\
 2704826 \\
 347763 \\
 23184 \\
 115 \\
 \hline
 4.8221650
 \end{array}$$

EXAMPLES.

Multiply 27.653 by 9.157 contracting to 3 decimal places.

47.0431 by 32.61796 contracting to 4 decimal places.

78.6129 by 42.5346 contracting to 4 decimal places.

157.60524 by 90.00785 contracting to 5 decimal places.

1.957 by 637.951706 contracting to 6 decimal places.

7955.600054034 by 6538.90705932 contracting to 8 decimal places.

15830.1576572101 by 351.7965149 contracting to 7 decimal places.

1764.3951072651 by 857.61935146 contracting to 8 decimal places.

95738.005349654987 by 1647.39150072 contracting to 8 decimal places.

697052.39756410392 by 8795.60325471516 contracting to 9 decimal places.

E. Contracted multiplication can only give us an approximate result, and we are in doubt whether the last figure be true, on account of the rejection of the subsequent parts, which, though small individually, may in the aggregate amount to several units in the last place kept. The likelihood of this error is diminished by taking not the last figure of any product, but that figure which is nearest to the truth, whether it be above or below. Thus if the true value be 3.849073, and we desire to keep only four places of decimals, we write 3.8491, because this is only 27 in excess, whereas 3.8490 is 73 in defect. In this way there is, to some degree, a balance of errors, since the amounts taken in excess form a set-off against those in defect. Even with this caution, however, the last figure cannot be depended upon; and hence it is usual, in very precise work, to carry the computations two or three places farther than we intend to keep the results.

All doubt as to the accuracy of the last figure may be removed by the following modification of the process for multiplying described in chapter 4, page 74.

Let the multiplication be done in the usual contracted way, but without bringing up anything from those figures of the multiplicand which have been rejected; as in the adjoining example, in which the product is sought to be true to eight places; jotting the figures of the multiplier in inverted order on the lower edge of a slip of paper. The last figures of this result are, of course, too small.

In order to compute the correction we shift the slip of paper one step along, and take the sum of the product thus indicated; in the present example there are

281828172	
3.14159265358979	
2.71828182845904	
6.28318530	
2.19911482	
3141592	
2513272	
62830	
25128	
314	
248	
6	
8.53973402	180
	201
	224
	1170 ?
8.53973422	234 too little.
	351 too much.

$24 + 2 + 32 + 1 + 40 + 18 + 16 + 6 + 35 + 6 = 180$, and this sum we place one step out. Shifting the paper another step we have the products $12 + 8 + 8 + 8 + 5 + 72 + 4 + 48 + 5 + 21 + 10 = 201$; this we write again one step out. Continuing in the same way we have $15 + 4 + 32 + 2 + 40 + 9 + 16 + 12 + 40 + 3 + 35 + 16 = 224$; and now it is evident that we need proceed no farther, for although all the figures had been 9's we should only have 13 times 81 or 1053, which with what, in that case, would have been still to follow, would not have reached 13 times 99 or 1170; wherefore $8.53973422 \mid 234$ being too small, $8.53973422 \mid 351$ must be too much; so that the product thus obtained is true to the eighth place.

If we had gone a step farther, the extreme possible error in that place would have been 9 times 14 or 126, and so on; and as soon as the figures of one factor are exhausted this number remains stationary until the figures of the other factor be exhausted, when it decreases. A very small amount of labour is sufficient in this way to certify the last figure: however there is one case in which the supplementary calculation may have to be carried a great way: it is when the next figure beyond the place to be kept is a 4, for if the subsequent figure be a 9, we observe that a very small addition may convert the 49 into 50; now while for $3 \mid 499$ we should write 3, for $3 \mid 501$, we must write 4, and therefore we must continue the work until we be sure that some subsequent figure is an 8 or less.

EXAMPLES.

Perform the following multiplications:—

15.6372 by 7.4327 contracting to 4 decimal places.

3.65703 by 9.68125 contracting to 5 decimal places.

27.00738 by 17.69831 contracting to 5 decimal places.

35.6150739487 by 26.32795071815 contracting to 7 decimal places.

78.870000436925721 by 69.3792650387157 contracting to 12 decimal places.

317.46543985711639532 by 47.603955726478587279 contracting to 17 decimal places.

4915.627000000005499 by 6271.395617934 contracting to 15 decimal places.

7285.65073914006543392571 by 1760.99976582730051471-726 contracting to 20 decimal places.

947269.35065718932564779800047 by 3695.4137201 contracting to 21 decimal places.

976273.000573921698705419872958 by 654.729600655473-669281725614 contracting to 18 decimal places.

C. The division of a decimal by an integer number is readily accomplished ; thus if we have to divide the quantity represented by 5.870632 into 17 equal parts, we proceed exactly as in common division. The 17th part of 5 units is not so much as one unit, therefore we turn the 5 units into 50 tenths, which with the 8 tenths make 58 tenths, the 17th part of which is 3 tenths with 7 tenths over ; and so on, as in the margin, or in the more concise form—

17	5.870632000	.34533129, &c.
	795525672	17 5.870632
	1	5.1
		77
		68
		90
		85
		56
		51
		53
		51
		22
		17
		50
		34
		160

When the divisor has a fractional part, the question is to ascertain how many times or how many parts of a time it is contained in the dividend.

Thus if it be asked how many times is 23.767 contained in 481.02368 we see that 23 is contained twice in 48, or 20 times in 480, and therefore we write the quotient 2 in the tens' place, and then proceed in the usual way—

$$\begin{array}{r}
 20.24 \\
 23.756 \overline{) 481.02368} \\
 \underline{475.12} \\
 5.9036 \\
 \underline{4.7512} \\
 1.15248 \\
 \underline{95024}
 \end{array}$$

EXAMPLES.

Divide—

25.132 by 6.283	216.03078 by 71.652
196.5348 by 6.1417	7153.00653 by 793.156
1256.210506 by 13.709	2694.495368308 by 71.6354
1423.85235523 by 31.6523	3917.05621 by 174.31764
5002.85881454939 by	84.70063
5880.322151408 by	270.57361
7461.397428 by	187.619421
10835.321499873 by	170.6003
31296.43732939 by	231.7501
29570.150073628 by	397.61429
1189829.4973528 by	1562.0054
3599838.289332 by	5713.6952
6532947.5007965 by	7942.647391

Even when the divisor has no integer part the same method has to be followed; thus if the question were, how often is .27803 contained in 17396.17? we readily see that 27 hundredths go about six times in 173 hundredths; now our 173 is not hundredths, but hundreds or ten thousand times as great, wherefore 27 hundredths go sixty thousand times in 17300, and we write the quotient 6 in the place for tens of thousands. It is to be remarked that when we come to the units' place of the quotient the

$$\begin{array}{r}
 62569. \\
 .27803 \overline{) 17396.17} \\
 \underline{16681.8} \\
 714.37 \\
 \underline{556.06} \\
 158.310 \\
 \underline{139.015} \\
 19.2950 \\
 \underline{16.6818} \\
 2.61320
 \end{array}$$

decimal point of the divisor comes just under that of the dividend.

Perform the following *divisions*.—

34.3643 by .503	824.404548 by .5907
44.2854 by .278	139.063133 by .9053
39.15 by .0725	28.40055499047 by .107103
41.69442 by .571	3.5205212592 by .025317
124.90658886 by .7193	421.605703 by .1092573
176.23 by .98705	11.81633976 by .03176
602.82296 by .0030571	
23281.3522217515 by .0000097385	

Or again if it be asked “how often does .43179 go in .0264183?” we observe that 43 goes 6 times in 264; now 43 hundredths go 6 times in 264 hundredths, but our 264 instead of being hundredths is ten-thousandths, that is one hundred times less, wherefore 43 hundredths go in 264 ten-thousandths only six-hundredths of a time, and therefore we write the quotient 6 in the place for hundredths.

$$\begin{array}{r}
 .061 \quad \&c. \\
 .43179 \overline{) .0264183} \\
 \underline{259074} \\
 51090 \\
 \underline{43179} \\
 79110
 \end{array}$$

Thus we see that while the figurate part of the work is an exact counterpart of that for common division, an ordinary attention to the nature of the question enables us to determine the rank of each figure.

EXAMPLES.

Divide—

.7426 by .47	.00406855356 by .14739
.137088 by .128	.000542016 by .0941
.053952 by .27414	.0104368053971 by .0479803
.0000047719480532 by .00654301	
.00108027361988394 by .15003694	

.000266033808 by	.003728
.00028513195836 by	.0069532
.000003029974 by	.000398
.00396005273851 by	.0003917
.00010960350363 by	.0157203
.000000084961526 by	.0000278
.000071629047 by	.0071492
.000009348380908635 by	.127000515
.00000000000888188098042 by	.00000051629
.000000539065728003725 by	.0000076900073
.00000000000800322676473772 by	.0000000057236

D. As in the multiplication so in the division of decimals there is a degree of precision beyond which it is unnecessary to push our calculations ; and it becomes a question how to obtain the desired degree of accuracy with the least labour. If we continue to use all the figures of the divisor we give ourselves a great deal of needless work, since the last figures of the remainder can have no perceptible effect upon the last figure of the quotient : we can therefore contract the work by cutting off the successive figures of the divisor as in the adjoining example.

The dividend and the divisor having been determined by some operation true to the last decimal, cannot be supposed correct beyond that : thus the dividend if carried a step farther might have been 15.62897414 or 15.62897406 : we are only sure that the last figure 1 is the nearest to the truth. Being then uncertain of the succeeding digits it would be pretending to an unattained degree of precision to bring

3947414	4.1474930
3.7682943	15.6289741
	15.0731772
	.5557969
	1789675
	282357
	263780
	18577
	15073
	3504
	3391
	113
	113
	0

down zeroes ; and therefore we rather cut off places from the divisor. However, for the purpose of preventing the accumulation of errors, we may carry the work one or two steps farther ; without venturing to exhaust these.

EXAMPLE.

When the diameter of a circle is unit its circumference is known to be 3.1415926535897932384626434 true to the last decimal place. Required the diameter of a circle of which the circumference is unit. For this purpose we must divide unit by the above number.

E. Although in contracting these divisions we take care to use the nearest last figures, so that the excesses may counteract the defects, it is not impossible that in some cases the errors lie all on one side, and that thereby the last figure of the quotient may be vitiated. The method of division invented by M. Fourier, and called by him *Division Ordonnée*, enables us to make sure of the last figure of the quotient without unnecessary labour. In the following explanation of this process I have modified, with a view to simplification, the description given by Herr J. A. Grunert in his Supplement to Klügel's Wörterbuch (Leipzig 1836). The subjoined manipulation as well as the reasoning differs considerably from Herr Grunert's.

Let it be proposed to divide unit by the number

2.302585092994

(that is, to compute the modulus of the common or Briggs' system of logarithms from the Neperian logarithm of 10). For conciseness I place the divisor above the dividend ; a slip of paper is then prepared to receive the figures of the quotient on its lower edge. Things being thus prepared, we separate two or three of the first figures of the divisor, drawing a line over them as a mark. In this case I have separated four figures, viz. 2302. If I were to use this alone as the divisor the quotient would come out too great ; yet the first one or two figures might be right ; how-

ever it is necessary at every step to correct the error which arises from assuming 2.302 to be the entire divisor. The first figure of the quotient is clearly .4 : this figure I write over the last digit of the 2302, upon the movable slip of paper : and take the product 9208 from the dividend, bringing down the next figure 0 ; the remainder 7920 so obtained has again to be divided by 2302 ; but this remainder has first to be corrected for the neglect of the next figure 5 of the divisor : to make this correction I shift the slip of paper one step along, so as to bring the 4 over the 5 : this indicates the product 20, which falls to be subtracted from the 7920, and leaves 7900, in which 2302 goes 3 times, we write the 3 on the slip, to the left of the 4, and to avoid much writing we combine mentally the two subtrahends 20 and 6906, subtract and bring down the next figure, sliding the slip of paper along another step. To correct the remainder 9940 for the neglect of the two figures 5 and 8 we must subtract the two products 15 and 32 : 2302 goes in the residue 4 times ; this 4 we write above the last digit of the separated divisor, and obtain the new subtrahend 9208, which with the 15 and 32 makes 9255, and leaves the new remainder 685 ; bringing down the next figure and moving the slip another step we find that 6850 has to be corrected by subtracting $20 + 24 + 20$: the residue 6786 contains the separated divisor 2 times, and $4604 + 64$ gives 4668 for the subtrahend, which leaves the remainder 2282 : bringing down the next figure and moving the slip another step, we have the correction $10 + 32 + 15 + 0 = 57$, and

	9184492434.
2.302585092994	
1.000000000000	
9208	
7920	
6926	
9940	
9255	
6850	
4668	
21820	
20775	
10450	
9325	
11250	
9345	
19050	
18591	
4590	
2483	
21070	
20971	
990	

in the residue 2302 is contained 9 times, whence the entire subtrahend is $20718 + 57 = 20775$, and in this way we continue until the division be carried as far as we require.

Nothing remarkable occurs in this division till we arrive at the last step which is printed; the last quotient ⁹ is represented in its place on the edge of the movable slip of paper, and the remainder is only 99. Now this small remainder may be less than what has to be brought up from the omitted products, and if this be the case our last quotient must have been too great: however we have only nine multipliers, and the product of each of these can in no possible case amount to more than 9, and therefore 81 is the utmost value of the correction that ever can be brought at this stage of the work, wherefore we conclude that the last figure is safe. Our actual multipliers are 1, 8, 4, 4, 9, 2, 4, 3, 4, and whatever may have been the figures neglected in the divisor the numbers brought up by these multipliers can never be equal to themselves, whence their sum 39 is greater than any error, and only if the remainder had been less than 39 need we fear to have taken too high a quotient.

If, instead of taking the first four, we had only taken the first two figures of the divisor, such occurrences would have been more fréquent; thus our first remainder is 8 which is greater than the quotient 4, therefore although the divisor had been 239999 instead of 23, there would have been the same quotient. The next remainder 11 is greater than the sum of 3 and 4, so that we are safe with the quotient 3. But the remainder 10 is less than $4 + 3 + 4$, so that the quotient 4 is uncertain; how-

	<u>3434.</u>		
	2.302585092994		
	1.000000000000		
	<u>92</u>		
80		$8 > 4$	4 certain.
<u>69</u>			
110		$11 > 3 + 4$	3 certain.
<u>100</u>			
100		$10 < 4 + 7$	4 uncertain.
<u>26</u>			
74			
<u>69</u>			
50		$5 < 3 + 11$	3 uncertain.
55			3 wrong.

ever bringing down another figure we find the correction to be only 26, thus leaving 74 to be divided by 23. Here the remainder 5 is much less than 14 the sum of all the digits of the quotient, and the figure 3 is very doubtful. In fact bringing down a figure and computing the corrections we find 55, which exceeds 50, and therefore the 3 is wrong: we must make it a 2.

Thus we see that it is most convenient to take several of the first figures of the divisor at once.

In general this method of M. Fourier is convenient: it has the advantage of en-

abling us to stop at any stage and to take up the work if it should need to be carried farther; but at rare times it fails to aid us. Thus in the ad-

joining example, having separated the first three figures of the divisor we find the second remainder to be 35, but as this is greater than $1 + 8$, the last quotient 8 is safe.

Bringing down another figure of the dividend we find the correction 57 and the quotient 0; and afterwards we come to the remainder 9, which is less

$$\begin{array}{r} 5.8769738214 \\ \hline \end{array}$$

$$\begin{array}{r} 10.6079376726 \\ \hline \end{array}$$

$$\begin{array}{r} 5.87 \\ \hline \end{array}$$

$$\begin{array}{r} 4\ 737 \\ \hline \end{array}$$

$$\begin{array}{r} 4\ 702 \\ \hline \end{array}$$

$$\begin{array}{r} 359 \\ \hline \end{array}$$

$$35 > 9.$$

$$\begin{array}{r} 57 \\ \hline \end{array}$$

$$\begin{array}{r} 3023 \\ \hline \end{array}$$

$$\begin{array}{r} 3014 \\ \hline \end{array}$$

$$\begin{array}{r} 97 \\ \hline \end{array}$$

$$9 < 14 \quad 5 \text{ uncertain.}$$

$$\begin{array}{r} 89 \\ \hline \end{array}$$

$$\begin{array}{r} 86 \\ \hline \end{array}$$

$$8 < 14 \quad 0 \text{ uncertain.}$$

$$\begin{array}{r} 77 \\ \hline \end{array}$$

$$\begin{array}{r} 97 \\ \hline \end{array}$$

$$9 > 14 \quad 0 \text{ uncertain.}$$

$$\begin{array}{r} 101 \\ \hline \end{array}$$

$$0 \text{ wrong.}$$

$$\text{for } 5000 \text{ put } 4999.$$

$$\begin{array}{r} 5967 \\ \hline \end{array}$$

$$\begin{array}{r} 101 \\ \hline \end{array}$$

$$\begin{array}{r} 5866 \\ \hline \end{array}$$

than $1 + 8 + 0 + 5 + 0$, and therefore the last figure 5 may be too great. On taking down another figure we find that 97 is greater than the next correction 89, but the remainder 8 is less than $1 + 8 + 0 + 5 + 0 + 0$, so this last zero is uncertain: again we have the 9 less than $1 + 8 + 0 + 5 + 0 + 0 + 0$, so that the third zero is uncertain too: and at the next step we find the correction 101 greater than the remainder, so that the third

zero is wrong, and for 5000 we must write 4999. In order to correct our work it is only necessary to add 5870 to the 97 (properly 587 to the 9) in order to obtain 5967, the remainder which we would have had if we had used the quotient 4999.

On comparing the amount of labour in this process with that required in the ordinary way to obtain the same degree of precision, we find that every product which is used in the one process is used in the other, and that the difference consists in the arrangement; the conciseness being rather in favour of the common method.

Those who have to make long calculations are aware that when an error is committed the same computer may make it again and again: hence when the importance of the results requires that the calculations be revised, either another person must do the revision, or the same person must do the work in another way. This mode of performing long divisions offers an excellent change to the computer.

CHAPTER XII.

ON RATIO AND PROPORTION.

C. It is not easy, or rather it is not possible, to explain in words what we mean by *ratio*; yet the idea of ratio is readily formed. If we compare one magnitude with another magnitude of the same kind, we observe, according to the case, that it is less than, is equal to, or is greater than that other magnitude. When we find it equal to the other our comparison is at an end: but if we find it to be less or greater, we naturally inquire in what degree it is less or greater.

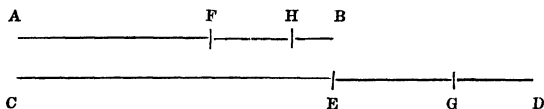
To avoid the obscurity attending the constant use of the pronoun *it* and of the demonstrative articles *this*, *that*, *the one* and *the other*, let us call the magnitudes A and B; then if A be less than B, we observe that it is about the half of B, a little more than the half or a little less than the half as it may happen. Perhaps A may be three quarters of B.

To form in our minds an idea of the relation between the quantities A and B, we imagine one of them, say B, to be divided into equal parts, and estimate how many of these parts make up A. Thus if on dividing B into three parts we find that A contains two of them, we conclude that A is two-thirds of B, and we say that the ratio of A to B is that of 2 to 3; in that case B is three halves of A, and the ratio of B to A is that of 3 to 2.

When we have discovered that part of B which goes exactly in A, and counted how many times it does go, we have obtained two numbers which serve to express the ratio of A to B. But how is this discovery to be made? If we divide B into four

equal parts, one of these may not measure A exactly : if we try the division of B into five parts we may be again unsuccessful ; and may have to make very many trials before we obtain the desired result. We must therefore seek for some regular procedure by means of which the *common measure* of two quantities may be discovered.

Suppose, to take an easily managed case, that it be required to find the ratio of the line AB to the line CD. For this

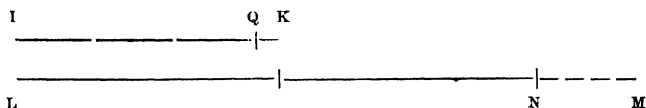


we must discover some small line that measures AB and also CD, that is which goes exactly from A to B and also exactly from C to D. Now it is clear that, if such a line be found, when we step it along CD we shall come at some one step to a distance exactly equal to AB, and that the common measure must be contained exactly in the remainder ; wherefore let us measure off from CD a portion CE equal to AB, and the line for which we are seeking must be a measure of ED : instead then of having to seek a common measure of AB and CD we may seek a common measure of ED and AB.

Reasoning in the same way, whatever line measures both ED and AB exactly, must measure their difference exactly, so having cut off AF equal to ED, the line for which we are seeking must measure FB, and we may now seek the common measure of FB and ED : it is evident that the common measure must either be FB itself or some part of FB. Let us then try FB upon ED, we find that it goes once and leaves GD, wherefore either GD or some part of GD must be the common measure : on trying GD upon FB we find that it goes once and leaves HB ; and on trial we find that HB goes exactly twice in GD, and therefore HB must measure AB and CD each exactly. Having got the common measure HB, we have to count how

often it goes in each of the lines AB and CD. Since GD is 2 HB, FH must be twice and FB or EG must be three times HB, wherefore ED or AF is 5 HB, and AB is 8 HB, while CD is 13 HB; hence it turns out that the thirteenth part of CD is contained 8 times in AB, or that the ratio of AB to CD is that of 8 to 13.

As another example let it be proposed to find the ratio of the line IK to the line LM. Taking IK as often as possible out of LM, we find that it goes twice with NM over: on trial NM goes three times in IK with QK over, and QK goes exactly four times in NM; therefore QK is the desired common measure.



Since $NM = 4 QK$, $IQ = 3.4 QK = 12 QK$ and $IK = 13 QK$; also $LN = 2.13 QK = 26 QK$ and $LM = 30 QK$, wherefore the ratio of IK to LM is that of 13 to 30.

It is easy to see that the same method may be applied to other kinds of magnitude: thus if we have to ascertain the ratio of one weight to another weight, we may prepare a quantity of sand or of material easily parted, equally heavy with the larger of the two weights, and weigh out from it as often as possible the lesser weight: then we may weigh out the remainder as often as possible from the former subtrahend, and so go on until we find no remainder left.

The learner ought to set himself exercises in this operation: the most convenient kind of magnitude is the linear. If he have a balance at hand he may apply the process to the finding of the ratio of the weights of any two stones picked up at hazard.

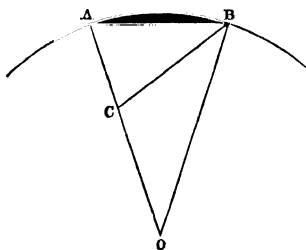
D. One cannot think long on the nature of this process without being led to put the question—"What if it never end?" In seeking the common measure of AB and CD, we found AB to go once in CD, the remainder ED once in AB, the remainder

FB once in ED,—what if it had gone on for ever so? Let us reflect a little. Our problem was “to find the common measure of the two lines AB and CD;” by means of repeated subtractions we changed this problem into another, viz. to find the common measure of HB and GD. In principle then we have made no advance, for we are still at the problem “to find the common measure of two straight lines.” However if there be a common measure we are nearer to it: the process cannot create a common measure, it can only discover it if it exist.

Now it does very often happen that we have to deal with and compare magnitudes which have no common measure; and the question very naturally arises, “How are we to proceed in such a case?”

Our business is to acquire knowledge by all the means which we can command, and therefore I shall not hesitate to break through the rules of strict arrangement in order to place before the student one or two geometrical illustrations of this subject.

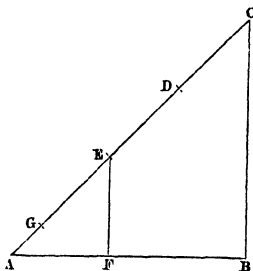
Having divided the circumference of a circle into ten equal parts: let us seek the ratio of the chord of one of those parts to the radius of the circle. Suppose that O is the centre of the circle, and that the arc AB is the tenth part of its circumference: our object is to find the ratio of the chord AB to AO. Now from the well-known property that the three angles of a triangle make together half a revolution, we easily see that each of the angles OAB and OBA is double of AOB. Thus much premised, let us proceed.



We take AB as often as we can out of OA; it goes only once and leaves CA over: we have to take CA out of BA. But a glance at the figure shows us that the line BC bisects the angle ABO, and that AB, BC, CO are all equal to each other. The trigon ABC has the very same characters with AOB: its base

AC must go once in BA and leave a remainder : that remainder must go once in CA with a remainder, and, although we soon cease to be able to measure off the parts on account of their minuteness, this succession must go on for ever ; so that AB and AO cannot have a common measure ; they are *incommensurable*, and no two numbers whatever can serve to express the ratio of the one to the other.

Again let it be proposed to find the ratio of the side to the diagonal of a square. ABC being half of a square, it is proposed to find the ratio of the side AB to the diagonal AC : for this purpose we have to find their common measure. Cut off AD equal to AB ; and we have now to seek the common measure of CD and DA. Measure off DE equal to DC, and draw EF parallel to CB, it can be shown that AF and FE are severally equal to DE, so that AE is greater than DC which may be taken again out of it : hence DA contains twice CD with GA over. Now GA bears to GE the same relation that CD bears to DA, it is the excess of the diagonal above the side of the square of EG : therefore AG goes twice in GE with a remainder, that remainder goes twice in AG with another remainder, and so on for ever. Wherefore AB and AC are incommensurable.



Many other analogous instances may be adduced ; and the student who is acquainted with the elements of geometry may find it advantageous to attempt the solution of the following case.

A trigon has one of its angles equal to the fourth part, and another of its angles equal to five fourth-parts of a right angle, to find the ratio of the one to the other of the opposite sides.

If then the two quantities A and B have no common measure

their ratio cannot be expressed accurately by numbers ; yet we may obtain two numbers of which the ratio shall agree very closely with that of A to B. I shall now proceed to explain how such numbers may be found by help of the operations which we have just performed.

C. The calculation of the number of times that the common measure is contained in each of the two quantities is most readily carried on in this way. Having arranged, as in the adjoining example, a column for each of the two quantities separated by a

	AB		CD
		1	1. AB + 1. ED
	1. ED + 1. FB	1	2. ED + 1. FB
	2. FB + 1. GD	1	3. FB + 2. GD
	3. GD + 2. HB	1	5. GD + 3. HB
	8. HB	2	13. HB

column for the quotient : we write the larger one CD a step higher than the other. On trial we found CD to contain AB once with ED over : or that CD is equal to 1. AB + 1. ED. On the next trial AB was found to contain ED once with FB over, and therefore CD must contain ED twice with FB over. Again, ED contains FB once with GD over, wherefore AB contains FB twice with one GD : but each ED of CD contains one FB and one GD, so that $CD = 3 \text{ FB} + 2 \text{ GD}$. Farther, FB contains one GD and one HB, so that $AB = 3 \text{ GD} + 2 \text{ HB}$, and $CD = 5 \text{ GD} + 3 \text{ HB}$. Lastly, GD is twice HB, therefore AB is twice 3 times 2.HB, and 2 HB's more, in all 8 HB, while CD is twice 5 times 2.HB and three HB's more, or 13 HB.

The operation may be better traced on the next example. LM contains 2. IK with NM over : IK contains 3. NM with QK, wherefore $LM = 2.3 \text{ NM} + 1 \text{ NM} + 2. \text{ QK}$; lastly, $NM = 4. \text{ QK}$, wherefore $IK = 4.3 \text{ QK} + \text{QK} = 13 \text{ QK}$, and $LM = 4.7 \text{ QK} + 2 \text{ QK} = 30 \text{ QK}$. In this way we can carry on the calculation as the work of measurement proceeds.

D. In order that we may better perceive the formation of

these successive expressions let us make up an imaginary case. Suppose that we have two magnitudes a and b ; and that on trial a is found to contain b three times with c over; b to contain c twice with d over; c to contain d four times with e over, and so on, as shown in the subjoined list, where the supposed measurements and their results are put down together.

$a = 3b + c$	$1a$	3	
$b = 2c + d$	$3b + 1c$	2	$1b$
$c = 4d + e$	$7c + 3d$	4	$2c + 1d$
$d = 1e + f$	$31d + 7e$	1	$9d + 2e$
$e = 2f + g$	$38e + 31f$	2	$11e + 9f$
$f = 3g + h$	$107f + 38g$	3	$31f + 11g$
$g = 2h + i$	$359g + 107h$	2	$104g + 31h$
&c.	$825h + 359i$	&c.	$239h + 104i$
	&c.		&c.

The formation of one of these expressions from the preceding is thus: having found that a is, for example, made up of $38e$ and $31f$; and that $e = 2f + g$, it follows that $a = 38(2f + g) + 31f = 76f + 38g + 31f = 107f + 38g$. From this it follows that the numbers in the second column of the values of a are the same with those in the first column deferred one step: and so of the values of b .

If the remainder c had been zero, the ratio of a to b would have been that of 3 to 1; so that actually the ratio 3 to 1 gives a too small in comparison with b . Let us take into account the remainder c but reject d ; in that case it is evident that we take b too small, so that the ratio 7 to 2 gives a too great in comparison with b . Let us take the remainder d into account but neglect e , and we obtain the ratio 31 to 9, which must give a too small. Thus the successive ratios 3 to 1; 7 to 2; 31 to 9; 38 to 11; 107 to 31, &c. give a alternately too small and too great in comparison with b ; or in other words—

$$\begin{array}{l}
 a \text{ is greater than } \frac{3}{1} b, \quad \frac{81}{9} b, \quad \frac{107}{31} b, \quad \frac{825}{239} b, \quad \&c. \\
 a \text{ is less than } \quad \frac{7}{2} b, \quad \frac{38}{11} b, \quad \frac{359}{104} b, \quad \&c.
 \end{array}$$

The series of fractions $\frac{3}{1}, \frac{7}{2}, \frac{31}{9}, \frac{38}{11}$, &c. is very easily formed by the following abbreviation of the above process :

$$\begin{array}{ccccccc} 3 & 2 & 4 & 1 & 2 & 3 & 2 & \&c. \\ \frac{0}{1} \div \frac{1}{0} \div \frac{3}{1} \div \frac{7}{2} \div \frac{31}{9} \div \frac{38}{11} \div \frac{107}{31} \div \frac{359}{107} \div \frac{825}{359}, \&c. \end{array}$$

Having written, for form's sake, the two fractions $\frac{0}{1} \div \frac{1}{0}$, we write above the latter the first quotient 3, multiply each member of it by 3 and add in the corresponding member of the preceding fraction : this gives $\frac{3}{1}$. Above this we write the second quotient 2, multiply each member by 2 adding in the member of the preceding and obtain $\frac{7}{2}$; and we continue this process until the series of quotients be exhausted: the last fraction so found expresses the relation of a to b , exactly if there have been no remainder and approximately if there have been one: the same computation may be arranged in columns as shown in the margin.

0		1
1	3	0
3	2	1
7	4	2
31	1	9
38	2	11
107	3	31
359	2	104
825	&c.	239
&c.		&c.

Now while these fractions are alternately above and below the true value, they differ less and less among themselves as we proceed along the series. Thus the difference between $\frac{3}{1}$ and $\frac{7}{2}$ is $\frac{1}{2}$; between $\frac{7}{2}$ and $\frac{31}{9}$ is $\frac{1}{18}$; between $\frac{31}{9}$ and $\frac{38}{11}$ is $\frac{1}{99}$; and between $\frac{38}{11}$ and $\frac{107}{31}$ is $\frac{1}{858}$. If then $\frac{38}{11}b$ be more than a , and $\frac{107}{31}b$ be less than a , neither of them can differ from the true value of a by so much as the 24856th part of b .

It is a remarkable property of such series of fractions that the cross products of the numerator of one into the denominator of the other, and of the denominator of the one into the numerator of the other of two contiguous fractions always differ by *unit*: in other words, that the numerator of the fraction which expresses their difference is always *unit*.

E. As very extensive application is made of this property to the higher branches of arithmetic, it may be worth while here to satisfy ourselves of its universality. I say, for example, that the product of 359 by 239 differs from the product of 825 by

104, by unit. 239 was obtained by doubling 104 and adding 31, that is $239 = 2.104 + 31$, and similarly $825 = 2.359 + 107$; therefore it follows that 359 times 239 is made up of twice 359 times 104, together with 359 times 31, or $359 + 239 = 2.359.104 + 359.31$, and in the same way $825 + 104 = 2.359.104 + 104.107$, now the product 2.359.104 is common, and therefore the difference between the cross products 359.239 and 825.104 must be the same as the difference between the cross products 359.31 and 104.107. But in the very same way we can show that this difference is the same as that of the preceding pair of cross products, and thus we conclude that this difference remains the same all along. Now at the outset the difference of the cross products 0.0 and 1.1 is unit, so that the difference must be unit throughout; and it is to be observed that the excess alternates from one to the other, the greater products being those which are shown by the oblique lines.

D. This method of finding those numbers which serve to express approximately the ratio of two given quantities is applicable whether these quantities have or have not a common measure. Thus in seeking the ratio of the side of a decagon to the radius of the circumscribing circle we found the quotients to be 1, 1, 1, 1, 1, 1, &c. for ever; hence the successive approximations are—

$$\begin{array}{cccccccccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \&c. \\ \hline 0 & \frac{0}{1} & \frac{1}{1} & \frac{1}{2} & \frac{2}{3} & \frac{3}{5} & \frac{5}{8} & \frac{8}{13} & \frac{13}{21} & \frac{21}{34} & \frac{34}{55} & \frac{55}{89} & \frac{89}{144} & \frac{144}{233} & \&c.; \end{array}$$

and again in seeking the ratio of the diagonal to the side of a square we found the quotients 1, 2, 2, 2, 2, &c.; hence the approximations—

$$\begin{array}{cccccccc} 1 & 2 & 2 & 2 & 2 & 2 & 2 & \&c. \\ \hline \frac{0}{1} & \frac{1}{0} & \frac{1}{1} & \frac{2}{2} & \frac{2}{5} & \frac{2}{7} & \frac{2}{12} & \frac{2}{20} & \frac{2}{30} & \frac{2}{40} & \frac{2}{50} & \&c. \end{array}$$

On account of the great utility of this process the learner ought to accustom himself to the use of it; I proceed to give a few examples of its application to numbers.

C. When two numbers are very large it is not easy to form an idea of the ratio of the one to the other: we naturally desire to reduce them to smaller numbers for comparison; now the process just described enables us to do this with great rapidity. For instance let it be proposed to find the successive approximations to the ratio of 24319259 to 54031573. Proceeding in the same way as for the ratio of two magnitudes we take the lesser out of the greater as often as possible; the remainder out of the lesser, the new remainder out of the former, and so on until the numbers be exhausted, as shown in the adjoining work, which exhibits the successive quotients 2, 4, 1, 1, 26, 5, 4, 1, 14, 1, 12. By help of these we compose the adjoining series of successive approximations—viz. 1 to 2; 4 to 9; 5 to 11; 9 to 20, &c.; the last of which, 1279961 to 2843767, is just what would have been obtained by dividing each of the proposed numbers by their common measure 19.

24319259	2	54031573
21572220	4	48638518
2747039	1	5393055
2646016	1	2747039
101023	26	2646016
97090	5	2626598
3933	4	19418
3686	1	15732
247	14	3686
228	1	3458
19	12	228

1		0
0	2	1
1	4	2
4	1	9
5	1	11
9	26	20
239	5	531
1204	4	2675
5055	1	11231
6259	14	13906
92681	1	205915
98940	12	219821
1279961		2843767

D. Another example may suffice. The results of the most accurate astronomical observations show that the length of the year is 365.242217 mean solar days: let it be required to determine what ratio the excess .242217 above the common year of 365 days bears to one day.

Proceeding in the usual way we obtain the quotients 4, 7, 1,

3, 1, 1, 4, 3, 2, 1, 1, 1, 3, 7, and from these the successive approximations—

$$\begin{array}{cccccccccccc}
 4 & 7 & 1 & 3 & 1 & 1 & 4 & 3 & 2 & 1 & 1 & 1 \\
 \frac{1}{0} & \frac{0}{1} & \frac{1}{4} & \frac{7}{29} & \frac{8}{33} & \frac{31}{128} & \frac{39}{161} & \frac{79}{289} & \frac{319}{1317} & \frac{1027}{4240} & \frac{2373}{9727} & \frac{8400}{14037} & \frac{5773}{23884} \\
 & & 3 & & 7 & & & & & & & & \\
 \frac{9173}{7871} & \frac{33292}{137447} & \frac{242217}{1000000} & & & & & & & & & &
 \end{array}$$

The first of these approximations ($\frac{1}{4}$) shows that in four years the common year of 365 days must fall one day short, and hence the use of the leap year, which is counted to have 366 days. In four years we have 1461 days, so that the average of these gives $365\frac{1}{4}$ days for the length of the year. This, the Julian year, is a little too long; the next approximation gives us $365\frac{7}{29}$; if we were to adopt this value of the year we would have to postpone every seventh leap year till the year following; that is to say, we should make six intervals of four and one interval of five years, so as to have 7 leap in 29 years; the years so determined would be short. The next approximation gives us 8 leaps in 33 years; the following 31 leaps in 128 years, and so on.

EXERCISES.

C. Find the successive approximations to the following ratios: 3843 : 7129; 46973 : 93765; 826397 : 47869971; 80097 : 9721; 173627 : 41283; 970029 : 68733.

D. The diameter of a circle being unit, its circumference is, as already stated (p. 152), 3.14159, &c. Required the successive approximations to it.

The ratio of common to Neperian logarithms is as

$$.43429 \ 44819 \ 03251 \ 82765 \ 11289 : 1.$$

Required the successive approximations.

The basis of Neperian logarithms is

$$2.71828 \ 18284 \ 59045 \ 23536 ;$$

it is required to express it by vulgar fractions.

This process enables us to supply a want in the reversion of decimal into common fractions. When the decimal has been carried out until it begins to circulate we can readily turn it into a common fraction: but though it have not been carried so far we can still obtain the fraction from which it was derived by computing the successive approximations to it. Thus if the fraction .16279 06977 be

proposed, we find, as in the adjoining work, the quotient 6 and then the

.16279 06977	6	1.00000 00000
16279 07666		.97674 41762
		2325 58238

quotient 7 very nearly; so that the fraction $\frac{7}{43}$ is a very close approximation to the given decimal, and indeed on trial we find that $\frac{7}{43}$ gives the very decimal proposed.

EXAMPLES.

Convert the following decimals into vulgar fractions:—

0.3103448275; 4.460829493; 2.618025751;

.3902439; 13.7704918; .001025641; .073712114.

C. When the ratio of one pair of magnitudes is the same with the ratio of another pair, these quantities are said to be proportional: thus if the ratio of A to B were as (say) three to seven; and if the ratio of X to Y were also as three to seven, the quantities X and Y are said to be proportional to A and B; and the proportion is usually written $A : B :: X : Y$, which is read “A is to B as X is to Y.” The mark $A : B$ here means the ratio of A to B; $X : Y$ means the ratio of X to Y, and the equality of these two ratios used to be denoted by the expression $A : B = X : Y$. However, when we come to look into the matter we perceive that ratios are not susceptible of equality: no ratio can be equal to another ratio. The ratio of A to B may be the very same as the ratio of X to Y, but it cannot with propriety be said to be equal to it; hence instead of the sign $=$, we use some other sign $::$ to mean *the same as*; this may be called the sign of identity, while $=$ is the sign of equality. Here A

and B may be of one kind, X and Y of another kind of magnitude ; thus A and B may be two weights, X and Y two distances : but A never can be of a kind different from B, since we cannot compare one magnitude with another of a different kind.

In order to ascertain the ratio of A to B we try how often some part of A goes in B : hence the definition of proportion, *Four quantities are said to be proportional when a part of the first is contained in the second as often as a like part of the third is contained in the fourth.* (Leslie's *Elem. of Geom.* book v.)

D. This definition fails in the case of incommensurable quantities, since no aliquot part of the one is contained exactly in the other : yet, by help of the operation which I have described, we can always find a part of A that shall be contained in B as accurately as we may desire ; so that this definition may be allowed to extend even to incommensurables.

C. When both terms of a ratio are multiplied or divided by the same number, the ratio is not altered. Thus if the two numbers 17 and 23 be each tripled, the product 51 and 69 have the same ratio with 17 and 23 or $51 : 69 :: 17 : 23$. The truth of this is manifest from the consideration that the seventeenth part of 17, which is unit, goes twenty-three times in 23 ; while the seventeenth part of 51, which is 3, goes twenty-three times in 69.

To take another illustration, let A and B be two lines, and let C be the triple of A while D is the triple of B, then I say that

M —

N —

A —

B —

C —

D —

A : B :: C : D.

Suppose that some short line M measures both A and B, and make N three times M : then as often as M goes in A so often must N go in C ; and again as often as M goes in B just so often must N go in D ; so that whatever numbers express the ratio of A to B the same numbers must express the ratio of C

to D: and it is clear that the same argument would hold good for any other multiples, or for any parts of A and B. This property of ratios is analogous to that of fractions explained in Chapter VII., page 109.

Although the quantities C and D had not been complete multiples of A and B; if they had been say three and a half times A and B respectively, they would still have been proportional to them. That is if A be to C as B is to D, then A is to B as C is to D.

The most important property of proportion is this, that *if four numbers be proportional the product of the extremes is the same with the product of the means*; we make constant use of this in computations.

Thus if the numbers 143, 247; 341, 689 be proportional, that is if $143 : 247 :: 341 : 689$, I say that the product of the extremes 143 and 689 must be the same with the product of the means 247 and 341. For if we seek the common measure of 143 and 247 we must find either unit or some number which divides them both: and so also of the members of the second ratio 341, 689; and thus we find that while 143 and 247 are equimultiples of two numbers 11 and 19, 341 and 689 are also equimultiples of the same numbers, for else while the eleventh part of 143 is contained nineteen times in 247, the eleventh part of 341 would not be contained nineteen times in 689; and the four numbers could not have been proportional. Hence $143 = 11.13$; $247 = 19.13$; $341 = 11.31$ and $689 = 19.31$, and the product of 143 by 689 is the continued product of $11.13.19.31$, while the product of the means 247 and 341 is the continued product of the very same numbers.

The same property holds good when the terms of one of the ratios are quantities: *If two quantities be proportional to two numbers the product of the extremes is equal to that of the means.*

Thus if the two quantities A and B be in the ratio of 3 to 7, that is if $A : B :: 3 : 7$, seven times A is equal to 3 times B. For if M be the third part of A, it must be the seventh part of B, that is $A = 3M$, $B = 7M$, and therefore 7A must be 7 times

3M, while 3B is 3 times 7M: now we know that 3 times 7 is the same with 7 times 3, and therefore we conclude that $7A = 3B$.

But we must not generalise and say that, in every proportion, the product of the extremes is equal to that of the means, for if all the four terms be *quantities*, as when A and B are two weights, C and D two measures, we can attach no meaning to the product of A by D. However, *When four quantities are proportional the product of the numbers which represent the extremes is equal to the product of those numbers which represent the means*, and stated in this form the proposition is universally applicable.

A knowledge of this truth enables us to compute any term of a proportion when the other three are given; and on this account it is often called *The Rule of Three*: because of its extensive use it has also been called *The Golden Rule*.

If in the proportion $A : B :: X : Y$ the three first terms be known and the last term required, we have only to suppose a, b, x, y , the four numbers which stand for those quantities, and to observe that, since a times y must be equal to b times x , y must be the a th part of b times x . Or, if the three last were known and the first required, a must be the y th part of b times x ; similarly x must be the b th part of a times y , and lastly b must be the x th part of a times y .

Every one perceives that the prices of two quantities of goods of the same kind and quality ought to be proportional to the quantities, for if 5 yards of calico cost 12 pence, 10 yards of the same cotton should surely cost 24 pence, or $2\frac{1}{2}$ yards should cost 6 pence; and thus if the cost of one quantity of an article of commerce be known that of any other quantity of the same article can be computed.

For example if 17 gallons of oil cost 67 shillings, the price of 29 gallons may be found by forming this proportion—

17 gallons : 29 gallons :: 67 shillings : the price of 29 gallons,

whence the required price is found to be 67 shillings $\times$ 29

$\div 17 = 114\frac{5}{17}$ shillings. In this work we use 17 and 29 as pure numbers, the ratio of 17 gallons to 29 gallons being the same as that of 17 to 29.

The same result may be arrived at in this way. Since 17 gallons cost 67 shillings, one gallon must cost $\frac{67}{17}$ of a shilling, and therefore 29 gallons must cost $\frac{67 \times 29}{17} = 114\frac{5}{17}$ shillings.

It often happens that the calculation may be shortened. Thus if the proportion were $77 : 91 :: 240$, we readily observe that the terms of the first ratio are both divisible by 7, and that it may be put $11 : 13 :: 240$.

Or it may be that both antecedents are divisible by the same number as in the example $18 : 23 :: 45$: where the antecedents 18 and 45 are divisible by 9, so as to allow of the proportion being changed into $2 : 23 :: 9$:

Sometimes simplifications in both ways may be had as in this example, $84 : 75 :: 105$: in which the members of the first ratio may be divided by 3 giving $28 : 25 :: 105$; and again in which the antecedents may be divided by 7 so as to change the proportion into $4 : 25 :: 15$: but although the second and third terms be both divisible by 5, this circumstance is of no avail in simplifying the work.

EXAMPLES.

In working these examples the learner should give the answers which are fractional both in common and in decimal fractions. He should also be on the look-out for shortenings.

14 : 27 :: 21 :	.	67 :	84 :: 173 :	.
23 : 65 :: 46 :	.	132 :	91 :: 429 :	.
34 : 35 :: 204 :	.	535 :	814 :: 1763 :	.
57 : 64 :: 304 :	.	3581 :	4076 :: 6270 :	.

If 9 yards of cloth cost $108\frac{3}{4}$ shillings, what will 28 yards cost?

If 25 yards of ribbon cost $3\frac{1}{2}$ pence, what will 97 yards cost?

If 63 pints of vinegar cost $86\frac{5}{8}$ pence, what will 189 pints cost?

- If 91 tons of coals cost $1107\frac{1}{2}$ shillings, what will 125 tons cost ?
- If 38 pounds of rice cost $85\frac{1}{2}$ pence, what will 251 pounds cost ?
- If 157 bottles of ink cost 1042 pence, what 295 cost ?
- If 16902 feet of tube cost 34074 shillings, what will 7941 feet cost ?
- If silk ribbon 3 inches broad cost $7\frac{1}{2}$ shillings per yard, what should ribbon 1 inch broad be sold at ?
- When silk velvet 1 yard wide cost $15\frac{7}{2}$ shillings per yard, what should velvet $\frac{2}{3}$ of a yard wide cost ?
- When yard wide cloth is sold at 38 pence per yard, what should five-eighth wide cloth be sold at ?
- If yard and a-half wide paper is sold at $1\frac{7}{2}$ shillings per yard, what should three-fourths wide paper be sold at ?
- If 5 men could together lift a stone weighing 734 pounds, how many pounds could 17 men lift ?
- If 2 horses could pull a loaded cart weighing $3\frac{1}{2}$ tons along, how many tons could 13 horses pull ?
- If it required 26 men to lift a weight weighing $3846\frac{1}{7}$ pounds, how many men will be required to lift a lump of iron weighing $19978\frac{2}{4}$ pounds ?
- A furnace consumed 28 tons of coal in 77 days, in how many days will it burn $293\frac{3}{4}$ tons ?
- A man can walk $48\frac{3}{4}$ miles in 13 hours, how many miles will he walk in 29 hours ?
- If a pump draw $36\frac{4}{5}$ gallons of water in 5 minutes, how many minutes will it take to draw 21 gallons ?
- There are 145 casks of equal capacity, 23 of which hold exactly 352 gallons of oil, how many gallons will be required to fill all ?
- Two persons P and Q contribute £58 and £93 to purchase a quantity of goods which they afterwards sell for £197, what share of this belongs to each ?
- A, B and C purchased a ship, A advancing £320 ; B £1750 and C £830 ; now the vessel made a voyage by which £627 was cleared : required each person's share of the profit.

A number of neighbouring proprietors agreed to make a joint feuing-plan for a town. D gave 147 acres, E gave 231 acres, F gave 93 acres, and G gave 77 acres. Now the entire ground produced £3978 of feu-duty (ground rent) : required the share which each should get of it.

Two grocers accommodate each other with sugar of different qualities. H gives K 628 lb of sugar worth $5\frac{1}{4}$ per pound, How much sugar at $7\frac{1}{2}$ d. must K give in return ?

To a joint venture in fruit at Smyrna L contributed 2317 okas of figs at 73 paras, M 4782 okas of nuts at 47 paras, N 1732 okas of pomegranates at 51 paras, and O 4739 okas of raisins at 85 paras ; the gain when the fruit was sold in London was £37 : required each person's share thereof.

The same method may be applied to the solution of many analogous questions. For example, 5 yards of rope weighed 23 ounces : required the weight of 37 yards of the same rope.

The circumference of a circle 113 inches in diameter is 355 inches almost exactly : required the circumference of a wheel 43 inches diameter.

A cable $16\frac{7}{8}$ fathoms long cost $17\frac{1}{2}$ shillings, how much will 1973 fathoms cost ?

A man charged $4121\frac{1}{4}$ shillings for building a brick wall of which the face is 2355 square feet, what will be the cost of the same kind of wall with 739560 square feet of face ?

A horse can run $84\frac{3}{10}$ miles in $5\frac{7}{10}$ hours, how long will it take to run 24 miles ?

A ship took $76\frac{1}{2}$ days to sail 8507 miles, how many miles will it sail in 365 days at the same rate ?

If a lamp-lighter take $13\frac{1}{2}$ minutes to light 25 lamps, how long will he take to light 235 lamps ?

If $36\frac{1}{2}$ cubic feet of water weigh $2265\frac{3}{8}$ pounds, what do 17562 cubic feet weigh ?

If 15 men can dig $31\frac{1}{2}$ tons of coal in 1 day, how many tons will 87 men dig in the same time ?

A man was able to carry on board a ship $29\frac{1}{2}$ tons of pig iron

in $8\frac{1}{2}$ hours, how long will it take him to put on board 987 tons?

If 17 men can plant 3879 trees in $3\frac{1}{2}$ hours, how many will they plant in $14\frac{1}{2}$ hours?

A man found that he could cut down at an average 145 trees in 17 days, how long would it take him to cut down a forest of 250796 trees?

A furnace could melt 115 tons of iron in $6\frac{1}{2}$ days, how many days will it take to melt 7650 tons?

A pipe delivered 1692 gallons of water in $57\frac{1}{2}$ seconds, how long will it take to deliver 109876 gallons?

If 15 men could lay 195 square feet of causeway in $7\frac{1}{2}$ hours, how long will it take them to lay 37065 square feet?

Interest is the price paid by a borrower for the use of money; it is usually counted at so much *per cent* (that is, per hundred) *per annum* (for each year): and is held to be due at the end of the year. Thus if A borrow £100 from B for one year at 3 per cent, he has at the year's end to repay B the original sum of £100, with £3 for the use of it, in all £103. Or if B still allow the principal sum to remain on loan, A pays him the £3 for interest and the bargain goes on for another year.

The computation of interest is a mere variety of proportion, with this facility, that the divisor is 100. One example may suffice. Required the interest of £4397 at $4\frac{3}{4}$ per cent. The proportion here is £100 : £4397 :: $4\frac{3}{4}$: £208.8575, and the actual operation is to multiply 4397 by 4.375 and divide by 100, or, what comes to the same thing, to multiply it by .04375.

If the money be lent for a part of a year the interest is in proportion to the time. Thus if £8395 be lent for 30 weeks at $5\frac{1}{2}$ per cent per annum; the interest for one year is computed; and then the proportion made, as 52 weeks to 30 weeks so must the interest for one year be to the interest for 30 weeks: and both

operations may be combined thus : $8395 \times \frac{5\frac{1}{2}}{100} \times \frac{30}{52} = £266.38.$

EXAMPLES.

Required the interest of £7893 at $3\frac{1}{2}$ for 5 years.

Required the interest of £8374.7 at $4\frac{1}{2}$ for 10 months.

The sum of £4876 was lent on the 10th March 1843 at $4\frac{3}{4}$ per cent; on the 17th September 1849 to what has it amounted?

The sum of £8795 was deposited in a bank on 1st February 1841; now the bank allowed $3\frac{1}{2}$ per cent in 1841, $3\frac{1}{4}$ in 1842, $3\frac{1}{2}$ in 1843, $3\frac{3}{4}$ in 1844, and 4 per cent in 1845.

Required the amount as at the 1st November 1845.

A man lent, on the 1st January 1801, £754 to a butcher at $6\frac{1}{4}$ per cent, £15 to a grocer at 4 per cent, £387.3 to a baker at $9\frac{1}{8}$ per cent, what did the interest amount to on the 1st June 1849?

A man going abroad lent his fortune at the same time to the following persons, ordering the interest to be deposited in a bank: to an oil-merchant he lent £1790 at $4\frac{1}{4}$ per cent, to a farmer £580.5 at $2\frac{7}{8}$ per cent, to a tailor £299.4 at $4\frac{3}{8}$ per cent, and to a railway company £7960 at $3\frac{1}{2}$ per cent. On his return after $15\frac{1}{2}$ years what sum does he find to his credit in the bank?

A bank on the 1st May 1793 lent a mining company £57035. The company paid the interest as it fell^d due, but becoming rich they also began to pay off their debt; on the 1st March 1799 they returned to the bank £175.5, on the 1st January 1801, £2058.4, on the 1st May 1823, £2970.3, on the 1st February 1828, £5476.8, on the 1st December 1831, £4160, on the 1st June 1839, £4985.5, and on the 1st April 1845 the rest, how much interest did the company pay in all?

A bank lent money to the following companies: on the 1st December 1807, £5684 to a steam-ship company at $4\frac{3}{8}$ per cent, on the 1st November 1808, £2890 to a railway company at $3\frac{1}{2}$ per cent, on the 1st July 1811, £12957 to a canal company at $4\frac{1}{4}$ per cent, on the 1st January 1823, £10732.5 to a foreign government at $6\frac{1}{2}$ per cent, and on 1st

March 1825, £6320 to a fishing company at $3\frac{1}{2}$ per cent, how much interest has been paid to the bank up to the 1st January 1829?

D. The bankers in Scotland allow interest for the money lodged with them in deposit accounts. The merchant draws out or pays in money on these accounts as may suit the requirements of his business, and thus the interest has to be computed for the varying sums and for the periods during which they are left in the banker's hands. Now it would be exceedingly tedious to calculate the interest on each separate payment; we therefore seek to lessen the labour by considering that the interests of (say) £30 for 1 day, of £1 for 30 days, or of £5 for 6 days, are all alike. We therefore multiply each sum by the number of days which it has remained at interest, and add all these products together. In this way we obtain the sum of money which, if lent for one day, would gain an interest equal to the interests of the various sums that have been lent for different times.

For example, if on Jan. 1 a merchant had £387 in bank, and if he drew out and paid in according to the accompanying note, the calculation of the interest would be carried on as shown in the example.

				Days	
Jan.	1.		£387	37	14319
Feb.	7.	drew £53	334	23	7682
Mar.	2.	lodged 128	462	76	35112
May	17.	lodged 57	510	17	8823
June	3.	drew 274	245	79	19355
Aug.	21.	lodged 137	382	17	6494
Sept.	7.	lodged 153	535	69	36915
Nov.	15.	drew 98	437	47	20539
Dec.	31.		437	365	149236
			Interest,	12.26	
					449.26

From which it appears that the interest of £149236 for one

day is equal to the interest on the account-current. Supposing that the banker allows 3 per cent per annum, the amount comes out £12.26, and thus the sum at the credit of the merchant on the first of January is £449.26.

A merchant had on the 1st of May 1837 £3659 in a bank, of which on the 7th June 1837 he drew £569

29th June	...	he drew	190
3d July	...	he lodged	73
5th Aug.	...	he drew	1365
19th Aug.	...	he lodged	1050
5th Sept.	...	he drew	371
11th Nov.	...	he drew	732
25th Jan. 1838		he lodged	500
2d Feb.	...	he lodged	769
27th Feb.	...	he drew	970
5th April	...	he drew	155
28th April	...	he lodged	478

On the 1st of May 1838 how much money had he in the bank, interest being at 3 per cent ?

C. When a merchant has granted a note promising to pay a sum of money at a given time, the holder of that note may sell it to another party for ready cash. The buyer does not give the entire sum mentioned in the bill, but counts off, or *discounts* a portion for the use of the money. The quantity deducted is called the *discount*, and the remainder the *proceeds* of the bill. *Discount* is reckoned in per-centages like interest, and is counted on the entire sum for which the bill is drawn ; if the bill be at twelve months there is one year's discount, but if it be at three or at six months there is only a quarter or a half year's discount taken off : and in all cases the discount is computed for the interval between the date of the sale and that of the maturity of the bill.

There is no difference between the arithmetical operations for discount and for interest.

EXAMPLES.

- A merchant holds a promissory-note for £857 payable in 4 months ; which he discounts at a bank. What are the proceeds of the bill, discount being at $4\frac{1}{2}$ per cent per annum ?
- A man received a promissory-note for £1052 payable in 10 months which he wishes to discount, discount being at $3\frac{1}{4}$ per cent, what would he get for the bill ?
- A banker was offered a promissory-note for £7261.3 payable in 9 months, discount being at $5\frac{3}{4}$, what would he need to give for it ?
- The amount of purchase at a shop being £5671, the shopkeeper giving a discount of $3\frac{1}{2}$ per cent on ready-money transactions, how much must be paid ?
- A person holding a promissory-note for £1378 payable in 17 months, wishes to discount it, discount being at $4\frac{1}{8}$ per cent, what would he get for the bill ?
- When discount is at $5\frac{1}{4}$ per cent, what would be the value of a promissory-note for £7629.2 payable in $11\frac{1}{2}$ months ?
- If a merchant gave $5\frac{1}{2}$ per cent discount on ready-money purchases, what nominal price should be put on goods for which he intends to take $15\frac{1}{8}$ shillings per yard ?

We must be careful to distinguish between the rate of discount and the rate of interest. A person who lends money at 5 per cent does not make so much profit as he who discounts bills at 5 per cent. The interest is computed on the present sum, the discount on the future payment.

Suppose that A lends £7600 at 5 per cent for one year, he must receive at the end of the year £380 of interest and the principal, in all £7980. But if B discount a bill at 12 months for £7980 at 5 per cent, he will pay for the bill £7589, so that for £11 less than A he has purchased the right to receive £7980 at the end of the year.

A person having a bill for £3897 payable in twelve months, and having to pay up his railway shares, discounts the bill at

5 per cent, and pays the proceeds into stock. Now at the end of the year a dividend is declared of 5 per cent. What is his gain or loss by the change?

C. When bargains are made and business transacted through Middlemen, Brokers, or Agents, charges called brokerage and commission are made: these are estimated by per-centage, and the calculation of them is exactly like that of interest.

EXAMPLES.

What are the *nett* proceeds of a bill for £673 at nine months, allowing discount at the rate of $4\frac{1}{2}$ per cent per annum, and a commission of $\frac{1}{8}$ of a per-cent to the broker? *N.B.*—The commission is computed on the sum named without reference to the time.

If a person have a bill for £5840 at 18 months, and sell it through a Broker who charges a commission of $\frac{1}{8}$ per cent, discount being at $3\frac{1}{2}$ per cent, what will be the proceeds of the bill?

What will be the proceeds of 3 bills, the first being for £270 at 5 months, the second £585 for 3 months, and the third £192.5 at 1 month, discount being at $4\frac{1}{8}$ per cent and the Broker's charge being $\frac{1}{8}$ per cent?

When a partner who has contributed to the general funds of any company wishes to withdraw from it, he sells his share to some one who takes his place as partner. Now the concern may have been a profitable one, in which case his share is worth more than he paid for it; or the concern may have been a losing one, in which case the share is worth less; and the market value of the share thus varies according to circumstances. The market value is best stated as at so much per £100 of original stock, but it is often given as at so much per share. Very often there is a fictitious stock, as when the shareholders subscribe for shares of say £50 each, to be paid up if called for, while only a part has really been paid up: this makes

the distinction between *nominal* and *paid-up* capital; we often see the advertisements of joint-stock companies stating that they have one, two, or three millions of capital: a cautious person inquires whether it be *paid-up* or merely *subscribed* capital.

EXAMPLES.

A person paid up at the first starting of a gas company £2750 for shares. The stock has now risen to 173; what is the present value of his shares?

Having expended £2500 in the purchase of railway stock when it was at 83, what is the value of my shares now that the stock has fallen to 77?

A person who has £1890 shares in railway stock wishes to exchange against canal-company shares: now the railway stock is at 58 and the canal stock at 72, what amount of canal stock should he receive?

A person bought £18360 worth of shares at $73\frac{1}{2}$. The stock having risen to $101\frac{1}{2}$, what is the present value of his shares?

A person having shares to the amount of £3672 in a mining stock wishes to sell out: now the stock is at $152\frac{1}{2}$, what should he get?

A person bought £265.3 worth of water-company's shares, the stock being at 99; but the stock having fallen to 82, what are his shares worth?

The public *Funds* are analogous to stock, but differ from it in this respect that a certain annual payment is made. The nation borrowed money for the war expenses, agreeing to pay to the lenders so much per cent by way of interest, and the stock was made transferable by sale. Thus a person who had lent £100 at 3 per cent receives £3 every year: he may, however, wish to sell his stock; now if at the time of the sale interest be at 4 per cent nobody will give him £100, and he must submit to a loss; he may take £75 for it: in that case 3 per cent stock is said to sell at 75, the estimate being made in per-centages.

EXAMPLES.

What is the value of £200 of 3 per cent stock when the market price is $92\frac{1}{2}$?

How much stock can I purchase for £125 when the market price is 79?

What will I get for £558.3 of 3 per cent stock when the market price is $75\frac{7}{8}$?

How much stock can I purchase for £8765 when the market price is $98\frac{1}{8}$?

What is the value of £1780 of 3 per cent stock when the market price is $87\frac{1}{4}$.

How much stock can I purchase for £307 when the market price is 89?

If I invest £4908 in 3 per cent stock at $81\frac{1}{8}$, how much stock can I purchase, and what is the annual return for my money?

A person had £9650 of 3 per cent stock, and wanted to sell out, the market price being $78\frac{1}{2}$, what does he get?

A person having purchased £18470 worth of foreign stock at $5\frac{1}{2}$ per cent, the market price being at 67, how much stock does he get, and what interest does it produce?

A company having bought £5023 worth of 4 per cent stock when at $73\frac{1}{2}$, sold out when it was at $77\frac{1}{8}$, what gain had they?

Having considered the proportionality of one pair of magnitudes to another pair, we are able to examine those cases in which several magnitudes of one class are proportional to as many magnitudes of another class. Thus in a case of partnership the several partners have contributed various sums to the general stock of the concern, and their shares of the profit are proportional to their contributions. It is clear that the whole amount contributed must be to one of the contributions as the entire profit is to the corresponding share; and thus the matter is brought into the form of an ordinary proportion.

Suppose that A, B, C, D, E, and F, have contributed £300, £150, £200, £500, £450, and £250, to the capital of a mercantile firm, and that the profit £300 has to be distributed amongst them. Here the whole capital is £2150, and therefore to compute A's share we have the proportion $2150 : 300 :: £300 : £41\frac{2}{3}$; and we may proceed in the same way for each of the others.

A	£300	£41 $\frac{2}{3}$
B	150	
C	200	
D	500	
E	450	
F	250	
		£2150

However, this process may be considerably abbreviated, for if £2150 have yielded a profit of £300, the fiftieth part of £2150, that is £43, must have yielded £6 of profit: wherefore £1 of capital must have yielded $£\frac{6}{43}$ of profit: and thus to find the share belonging to each partner we have only to take six forty-thirds of his capital.

This example well illustrates the meaning of the words *ratio* and *proportion*. The literal meaning of the Latin word *ratio* coincides with our word *reason*, and the ratios of the several contributions to the entire capital are the reasons why the partners should get certain portions of the profit; and the operation is that of portioning out the profit for these reasons, or of *proportioning* it.

EXAMPLES.

Two persons P and Q contributed £320 to a business, P giving £176 and Q the rest: the profits of the business amounted in 16 months to £57.3, what should each get?

A firm is composed of three persons, A, B, and C, with a capital of £5622, of which A had contributed £2040, B £1859, and C the rest: at the end of a year the profit was £1628.5, what is each person's share?

A company of bakers, composed of 4 persons A, B, C, and D, with a capital of £9726, of which A had contributed £5560, B £3800, C £251, and D the rest, found that at the end of 4 years their profits amounted to £684.5, what share should each get?

A railway company composed of twenty persons with a capital of £75628, of which A contributed £760, B £5070, C £3164, D £3921, E £3956, F £1844, G £2060, H £1763, I £3209, J £2700, K £9627, L £8300, M £761, N and O the same as J, P £1394, Q £4107, R £754, S £1921, and T the rest: found that after 25 years their profits had amounted to £21569.3, during that time what did each receive?

D. Sometimes we are unable to compare one magnitude directly with another, and have to make the comparison by help of one or more intermediates. Thus we may have had to compare A with B, B with C and C with D, in order to discover the ratio of A to D. For example, I may have had no opportunity of comparing the English pound with the Turkish oka, but may find that the ratio of the English pound to the French gramme is known, while those of the French gramme to the German pfund, of the pfund to the Greek drachma, and of the drachma to the oka, are also known: then by help of these ratios I may find out the ratio of the pound to the oka. The ratio of the first to the last of such a series is said to be *compounded* of the intermediate ratios. The ratio A : D is then compounded of the three ratios A : B ; B : C ; C : D.

To take an example ; suppose that A is to B as 3 to 7, while B is to C as 5 : 4, and let it be required to compute the ratio of A to C. The ratio 3 : 7 may also be put 15 : 35 ; and the ratio 5 : 4, 35 : 28 wherefore A : B :: 15 : 35 and B : C :: 35 : 28, wherefore the 35th part of B is contained 15 times in A, and the same 35th part of B is contained 28 times in C, so that A : C :: 15 : 28 ; and thus we see that the ratio compounded of two ratios is the ratio of the products of the numbers which express the component ratios.

To take a business example let the question be proposed, £735 gained £87 in 5 months, what should £387 embarked in the same concern have gained in 7 months ?

Here if the times had been alike the profits would have been

proportional to the capitals, and if the capitals had been alike the interest would have been proportional to the times, so that the ratio of the one profit to the other is compounded of the ratio of the principals and the ratio of the times : or as it is usually written—

$$\left. \begin{array}{l} £735 : £387 \\ 5 \text{ mos.} : 7 \text{ mos.} \end{array} \right\} :: £87 :$$

The compound ratio is 3675 : 2709, or simplifying 525 : 387 and the result is $£64\frac{181}{1225}$ or 64.10693853.

EXAMPLES.

P : Q :: 7 : 15 and Q : R :: 4 : 11 what is P : R ?

A : B :: 16 : 21 and B : C :: 6 : 41 required A : C.

O : P :: 153 : 29 and P : N :: 12 : 73 and N : L :: 57 : 125 what is O : L ?

E : X :: 9 : 1 and X : Y :: 32 : 33 and Y : A :: 154 : 69 and A : D :: 13 : 41 and D : F :: 17 : 94 required E : F.

A embarked £37 in a business for two months and his profit was £3 ; B embarked £52 for 3 months in the same business, what should his profit be ?

A person contributed £155 to a business for $7\frac{1}{2}$ months ; another person contributed £376.5 to the same business and his profits for 17 months amounted to £29.3, what was the first person's profit ?

A joint-stock company composed of three members A, B, and C, entered into business with a capital of £15721, of which A had given £6120, B £4735, and C the rest. After $2\frac{1}{2}$ years partnership B withdraws, his profits having amounted to £197, 4 years after B had withdrawn C withdrew, and $2\frac{1}{2}$ years after that A gave up business, what was C and A's profits, the business having yielded a uniform profit ?

C. There remains for me now to notice what is called *Inverse Proportion* ; its nature may be best explained by an example.

If 7 men be employed to do a piece of work and finish it in 39 days, in how many days would 9 men have finished it? Here it is clear that with twice the number of men the work would be finished in half the time, and that our proportion must be $9 : 7 :: 39 : 30\frac{1}{2}$.

EXAMPLES.

3 men dug a well in 16 days, in how many days would 5 men have done it?

2 horses carted home the corn from a field in $29\frac{1}{2}$ hours, in how many hours would 7 horses have done it?

If 257 men took $6\frac{1}{2}$ months to build a bridge, in how many days would 410 men have built it?

If 15 men be able to dig a field of potatoes in $15\frac{1}{2}$ days, in what time would 51 men do it?

27 women were able to weed 156 acres of ground in $25\frac{3}{4}$ days, in what time would 120 women have done the same work?

It was found that a steam-engine of 25 horse power was able to thrash 1576 bushels of corn in $15\frac{1}{3}$ days, in what time would an engine of 125 horse power have done the same?

243 men were able to quarry 10449 tons of stone in $23\frac{1}{4}$ days, in what time would 365 men do the same work?

If 15 weavers were able to weave a certain quantity of cloth in 56 days, how many weavers would be required to weave as much in $12\frac{1}{2}$ days?

If 132 men took $73\frac{1}{2}$ days to dig a certain length of a canal, how many men would be required in order to dig the same length in 27 days?

If 19 horses be able to carry 367 tons of coal in 17 hours, how many horses are required to carry 1695 tons in $11\frac{1}{2}$ hours?

58 men were able to quarry 122 tons of granite in $113\frac{1}{4}$ hours, how many men would be required to quarry 579 tons of granite in $56\frac{1}{2}$ hours?

There are many ratios which are constant or determinate, and as these cannot all be kept in mind we make tables of them.

Thus the weights and measures as well as the monies of one country have known ratios to those of another country : and a knowledge of these ratios is essential to the merchant who buys in France, say, goods at so many francs per kilogramme, and sells them in England at so many shillings a pound.

Again, the ratio of the circumference to the diameter of a circle, that of the side to the diagonal of a square, and many other ratios, are needed by the geometer and mechanician ; and these fall to be considered under their proper heads.

But there is one class of ratios of which a knowledge is useful in ordinary business, and which, therefore, I shall here notice, although the subject belongs properly to mechanics : I mean those ratios to which the name *Specific Gravities* is given. Every one knows that lead is much heavier than stone, and stone much heavier than wood. Now when we say that lead is heavier than stone we do not mean that a certain stone may not be heavier than a certain piece of lead : what we mean is that a piece of lead is heavier than a piece of stone of the same bulk ; or that, if the two weigh alike, the stone would be bulkier than the lead. If we make a piece of lead to have exactly the same bulk with a piece of stone, and then find that it weighs twice as much as the stone does, we would say that lead is twice as heavy as stone ; but then different kinds of stone are not all equally heavy, and it would be a long affair to compare each kind with every other kind so as to know the ratios of their weight.

For the sake of ready comparison one substance is fixed on and all others are compared with it. The substance which is chosen as the standard of comparison is pure water, and the weights or rather *densities* (heavinesses) of all other substances are compared with that of an equal bulk of pure water ; and the ratios resulting from this comparison are called the *specific gravities* of the substances. Thus lead is somewhat more than eleven times heavier than an equal bulk of water, and the specific

gravity of lead is said to be 11. The specific gravities of many substances have been carefully ascertained ; the following is a list of the more important :—

Alabaster,	2.875	Jasper,	2.359
Ashwood,	.800	Lead,	11.325
Antimony,	4.064	Lignum vitæ,	1.327
Agate,	2.590	Limestone,	3.179
Beeswax,	.935	Marble,	2.840
Beechwood,	.852	Mercury,	13.568
Boxwood,	1.030	Maple,	.755
Brass, hammered,	8.350	Mahogany,	1.063
... cast,	8.100	Mulberry,	.897
... wire,	8.544	Oil, cloves,	1.036
Barytes,	4.230	... lintseed,	.936
Copper,	8.843	... olives,	.908
Cork,	.240	... lavender,	.894
Cowswood,	1.040	... cinnamon,	1.044
Cherry tree,	.715	... fennel,	.929
Coal,	1.370	... almond,	.917
Ebony,	1.177	... cod,	.923
Elm,	.600	... whale,	.923
Flint,	2.570	Oak,	.925
Fluor spar,	3.170	Platinum, rolled,	22.069
Firwood, male,	.550	... wire,	21.042
... female,	.498	... hammered,	20.337
Gold, fine,	19.640	... pure,	19.500
... standard,	18.888	Pewter,	7.471
... guinea,	17.793	Pear tree,	.661
Glass, flint,	3.310	Poplar tree,	.383
... bottle,	2.733	Plum tree,	.785
Granite,	2.737	Portland stone,	2.496
Honey,	1.450	Quartz,	2.655
Iron, bar,	7.788	Silver, fine,	11.090
... cast,	7.207	... standard,	10.535
Juniper tree,	.556	Steel,	7.850

Sea water,	1.030	Walnut tree,	.617
Spelter,	7.065	Willow tree,	.585
Sulphur, fused, . . .	1.991	Water,	1.000
Tin, pure,	7.471	Yew tree,	.788
... common,	7.320	Zinc, hammered, . . .	7.191

The numbers given in this list show the ratio of the weight of any bulk of each substance to the weight of the same bulk of water: thus if a certain bulk of water weigh 1 oz. an equal bulk of cast brass will weigh 8.1 oz. Or again if a certain bulk of water weigh 1000 lb, an equal bulk of mercury would weigh 13568 lb. Thus although these numbers be given in decimals to three places, they may be read as integers.

As an example of the use of such a table I shall suppose that a piece of furniture when made of fir is found to weigh 73 lb: and that the weight of a similar piece of furniture made in walnut wood is required.

The specific gravity of fir is .550, that of walnut .617, and therefore we have the proportion $550 : 617 :: 73 \text{ lb} : 81.89273$, the weight of the work in walnut.

EXAMPLES.

- A cask holds 3741 ounces of water, what weight of olive oil is required to fill it?
- A bottle holds 12937 grains weight of water, how many grains weight of oil of almonds will fill it?
- A pattern made in fir weighs 193 ounces, what will be the weight of its casting in iron?
- A bottle which holds 4873 grains of water holds 9072 grains of sulphuric acid: what is the specific gravity of the acid?
- A cubic inch of water weighs $252\frac{1}{2}$ grains, what is the weight of a cubic inch of granite?
- A gallon can holds 70000 grains of water: into such a can 63000 grains of broken marble are put, and the can is then filled up with water: query, how much?

A cask held 6.359 pounds of honey, what weight of mercury would it hold?

A ship built of fir weighed 1627.3 tons, what would it have weighed if it had been built of oak?

A bottle held 3689 grains of sea-water, after putting in 2175 grains of mercury, what weight of pure water will be required to fill it?

A pattern made of oak weighed 31.798 pounds, what weight would the casting be if made of iron, tin, copper, brass, or pewter?

CHAPTER XIII.

ON COMPOUND QUANTITIES.

C. IN the preceding chapters I have treated very fully on the method of counting by tens or decads. Before this method had become general very many different and incongruous systems had been in use, and, although the prodigious facilities attending the use of one simple and regular scheme be abundantly manifest, a goodly assortment of the antique numeration scales remains to perplex us.

Each kind of quantity has its own scheme, if scheme it may be called, in which we can trace no design. Our farthings we count in fours, our pennies in dozens, our shillings in scores. We reckon ounces by sixteens, pounds by fourteens, stones by fours, quarters by fours, and hundredweights by twenties. We divide the yard into sixteen nails for the haberdasher, but into thirty-six inches for the artisan. In fact our whole system of weights and measures is a mass of bungling and absurdity ; still that system is in use, and we must needs become acquainted with it.

MONEY.

The pound sterling, marked £, is the value of a piece of gold alloy of a certain weight and containing a certain proportion of copper ; the weight and fineness being fixed by Act of Parliament 56 George III., chap. 68, section xi., to be according to an indenture with the Master of the Mint. The fineness of the gold is such that 24 parts of it contain 22 parts of pure gold

and 2 parts of alloy, and the weight is $123\frac{1}{2}$ grains troy ; or at the rate of £46, 14s. 6d. to the troy pound weight. Such a piece of gold when stamped or *coined* in the Mint is called a *Sovereign*. The utility of this stamping is twofold : it certifies that the coin is of the proper fineness and that it was originally of the proper weight ; and, as the impression is all round, on both sides and on the edge, it enables us to know whether any portion have been fraudulently scraped off or whether it be much worn.

The pound sterling is divided into twenty equal parts called *Shillings*, and as it would be inconvenient to have such small pieces of gold, shillings are coined in silver, the weight and fineness of the silver being also established by the same Act of Parliament ; 40 parts contain 37 parts of pure silver and 3 parts of alloy, and the weight of the shilling is $87\frac{1}{4}$ grains troy, or at the rate of 66 shillings to a pound troy.

The shilling is divided into twelve pennies coined in copper of the weight $291\frac{1}{2}$ grains troy, and the penny into four farthings ; and thus our scale of money stands as shown in the subjoined table :—

Pound.	Shilling.	Penny.	Farthing.
1	20	240	960
	1	12	48
		1	4

Although accounts be kept in pounds, shillings, and pence, we have other coins in circulation : the half sovereign in gold ; the crown worth five shillings, the half-crown, the florin worth two shillings, the sixpenny, fourpenny, and threepenny pieces in silver ; as also the halfpenny in copper. Pounds are marked £, shillings s, and pennies d (from the Latin denarius); thus £15, 7s. 6d. is read fifteen pounds seven shillings and six-pence.

LINEAR MEASURE.

The standard measure of the United Kingdom was a brass rod into which two gold pins were inserted ; and the distance between the centres of these pins was declared by Act 5 Geo. IV. Chap. LXXIV., passed 17th June 1824, to be a true yard. However all substances expand on being heated and contract on being cooled ; brass more than gold, silver, or iron ; hence it became necessary to state at what temperature the brass had the true length of one yard : that temperature was 62 Fahrenheit.

Copies of this standard were made and sent to all the principal towns, so that a true measure may be found everywhere. All other linear measures are taken in relation to this standard, as shown in the subjoined table :—

Mile.	Fur-long.	Chain.	Pole.	Yard.	Foot.	Inch.	Line.
1	8	80	320	1760	5280	63360	760320
	1	10	40	220	660	7920	95040
		1	4	22	66	792	9504
			1	5½	16½	198	2376
				1	3	36	432
					1	12	144
						1	12

Carpenters and machine makers divide the inch into eight equal parts, but for scientific purposes it is divided decimally into tenths, hundredths, and thousandths. Surveyors use tenths, hundredths, and thousandths of feet for vertical, but use the chain divided into 100 *links* for horizontal measure. Drapers divide the yard into four *quarters*, and each quarter into four *nails* ; and seamen count the depth of the sea in *fathoms* of two yards each.

SUPERFICIAL MEASURE.

The unit of superficial measure is the square yard, and all other measures of surface are obtained from it.

Square mile.	Acre.	Rood.	Square pole.	Square yard.	Square foot.	Square inch.
1	640	2560	102400	3097600	27878400	4014489600
	1	4	160	4840	43560	6272640
		1	40	1210	10890	1568160
			1	$30\frac{1}{4}$	$272\frac{1}{4}$	39204
				1	9	1296
					1	144

Land-surveyors measure by the square chain and square link, ten square chains or 100000 square links making an acre ; in this way they obtain the benefit of the decimal division in their own work, but they are obliged to convert the decimals of an acre into roods and poles, when giving the results of their measurements.

SOLID MEASURE.

The natural unit for the measurement of *volume* is the cube of the linear unit ; from this we have—

Cubic yard.	Cubic foot.	Cubic inch.
1	27	46656
	1	1728

Besides this scale we have another, used for all substances that can be poured as liquids or stricken as grain.

Chal- dron.	Quar- ter.	Bushel.	Peck.	Gallon.	Quart.	Pint.	Cubic inch.
1	4	32	128	256	1024	2048	70982.103
	1	8	32	64	256	512	17745.526
		1	4	8	32	64	2218.191
			1	2	8	16	554.548
				1	4 ✓	8	277.274
					1	2 ✓	69.318

And still another scale for such goods as are heaped up in the measure.

HEAPED MEASURE.

Chaldron.	Sack.	Bushel.	Peck.	Gallon.	Cubic inch.
1	12	36	144	288	101357.496
	1	3	12	24	8446.453
		1	4 ✓	8	2815.486
			1	2 ✓	703.871

The bushel for heaped measure is made to hold exactly one stricken bushel; but so that the outside diameter may be $19\frac{1}{2}$ inches; and the heaping is directed (Sects. VII. and VIII.) to be made in the form of a cone of which the height is 6 inches.

By section iv. of the same Act the standard from which all weights are to be deduced was declared to be the troy pound of the year 1758, which was directed to be divided into 5760

grains, 7000 of which grains are to make a pound avoirdupois ; thus all our weights may be said to be derived from the troy grain.

Pounds (in weight) are marked lb (from *libra*), to distinguish them from pounds (in money).

WEIGHTS.

The old scale of troy weight was as under :—

TROY WEIGHT.

Pound.	Ounce.	Penny-weight.	Grain.
1	12	240	5760
	1	20	480
		1	24

But apothecaries divided the ounce into 8 drams, the dram into 3 scruples, and the scruple into 20 grains, thus :—

APOTHECARIES' WEIGHT.

Pound.	Ounce.	Dram.	Scruple.	Grain.
1	12	96	288	5760
	1	8	24	480
		1	3	60
			1	20

However, that system of weights which is in general use is called Avoirdupois (French, *to have weight*); it is arranged as under :—

AVOIRDUPOIS WEIGHT.

Ton.	Hundred-weight.	Quarter.	Stone.	Pound.	Ounce.	Dram.	Troy grains.
1	20	80	160	2240	35840	573440	15680000
	1	4	8	112	1792	28672	784000
		1	2	28	448	7168	196000
			1	14	224	3584	98000
				1	16	256	7000
					1	16	437.5
						1	27.34375

The standard of weight may be obtained from the unit of linear measure in some such way as this : A box is made exactly one foot cube, and this is filled with pure water ; the weight of this quantity of pure water is made the standard of weight. According to Act of Parliament 5 Geo. IV. Chap. LXXIV., Sec. V., the grain must be so taken that the cubic inch of distilled water when at the temperature of 62° Fah. shall weigh 252.458 grains, when the barometer is at 30 inches. And it is necessary to attend to the temperature, because water varies in volume according to its warmth, and thus the fill of a bottle of warm water weighs less than the fill of the same bottle of cold water ; as also to the barometer, because the density of the air has a slight influence on the weight.

From the standard of weight we get that of liquid measure ; the gallon measure should hold exactly ten avoirdupois pounds weight of pure water at the temperature 62° of Fahrenheit's thermometer.

It thus appears that a cubic foot of distilled water at the temperature 62° Fah. should weigh 997.137 ounces avoirdupois.

In case of the loss or destruction of these standards it was decreed (Sec. III.), that the measure of length should be derived from the length of a simple pendulum vibrating seconds of mean solar time at the level of the sea at London ; the length of such a

pendulum had been carefully ascertained, and was declared to be 39.1393 inches.

Both standards were destroyed by fire in the year 1834, and the Commissioners appointed to restore them found it to be so difficult to obtain accurate conclusions from the measurement of the pendulum or from the weighing of water, that they had recourse to the expedient of restoring the standards of measure and weight from the best copies that had been made of them.

The Unit of TIME is what is called the Mean Solar Day ; it is reckoned from noon on one day till noon on the next ; but as the sun's motion is not equable, the days are not all alike : for that reason the average duration of all the days in one complete year or in several years is taken as the standard, and is called the mean solar day. The true year consists of 365.242217 of such days, and the Julian year of $365\frac{1}{4}$: this Julian year is divided very irregularly into 12 months, of which the names are—

January.....of 31 days.	July.....of 31 days.
February,.....of 28 ...	August.....of 31 ...
in leap-years...of 29 ...	September.....of 30 ...
March.....of 31 ...	October.....of 31 ...
April.....of 30 ...	November.....of 30 ...
May.....of 31 ...	AND
June.....of 30 ...	December.....of 31 ...

March used to be the first month of the year, and the names September, October, November, December (7th, 8th, 9th, 10th months), are derived from this old custom. The dates of the year have reference to the seasons, and it is important that the same month should always fall at the same season, in order that our agricultural operations may be readily regulated.

The name *month* is derived from moon, and properly means the time from new moon to new moon ; now this interval of time is 29.53058857 mean solar days, and twelve such months make nearly one year, accurately 354.36706286 days ; hence arose the custom of dividing the year into twelve months.

The Mahomedans reckon by the actual month, twelve of which

go to make their year ; and thus the Mahomedan year constantly precedes the actual solar year by 10.875154 days ; so that the month of Ramazan, during which the faithful eat only between sunset and the morning dawn, falls sometimes in winter, when the fast is easy, and sometimes in summer, when the length of the day renders it very severe.

The Julian year of $365\frac{1}{4}$ days was long in use, and is still used by the Russians and Greeks ; but as it is a little longer than the true year, the festivals and months were gradually displaced in respect of the seasons, and in 1582 Pope Gregory caused the calendar to be reformed, by leaving out three leap years in four centuries. Four hundred Gregorian years thus contain 97 intercalary days, so that the mean is 365.2425, which still somewhat exceeds the true length. The Gregorian is called the New Style, the Julian the Old Style of reckoning dates.

Astronomers use other kinds of days and years ; the year of the seasons is called by them the Equinoctial year ; and the ordinary day the solar day ; the one is the time in which the sun appears to return again to the equinox, the other is the interval between the sun's passages over the meridian. The time from the sun leaving any place among the stars until its return to the same place, is called the *Siderial year* ; and the time between the passages of a star over the meridian, the *Siderial day*. The principal clock in an observatory shows siderial time.

Year Julian.	Month.	Week.	Day.	Hour.	Minutes.	Seconds.
1	12	$52\frac{5}{8}$	$365\frac{1}{4}$	8766	525960	31557600
		1	7	168	8880	532800
			1	24	1440	86400
				1	60	3600
					1	60

ANGLES are measured by help of a circle, the circumference of which is divided into 360 degrees, the degree into 60 minutes,

and the minute into 60 seconds: the word *minute* is a contraction of *minuti primi*, and the word *second*, of *minuti secundi*; which may be translated division of the first degree and division of the second degree of minuteness. They form the two first descending steps of the sexagesimal system of numeration at one time in use among the Greeks, and bear to that numeral scale the same relation which decimal fractions bear to our usual arithmetical system. Degrees, minutes, and seconds, are marked thus, $23^{\circ} 15' 7''$; while hours, minutes, and seconds, are written $4^h 27^m 31^s$.

Having now given an account of the principal numeration schemes used in the United Kingdom, I proceed to show how a *compound quantity*—that is, a quantity composed of several denominations—may be expressed in terms of one denomination. This process is usually called *Reduction*.

If there be a sum of money as thirteen pounds, seven shillings, and ninepence three farthings, and we wish to ascertain of how many farthings it consists; we may begin by considering that a pound contains 960 farthings, and that therefore £13 is equal to 12480 farthings: that a shilling contains 48 farthings, so that 7s. = 336 farth. and lastly that as a penny is four farth. 9d. must be 36 farthings; and then collecting all these together thus,

$$\begin{array}{rcl}
 \text{£}13 & = & 12480 \\
 7\text{s.} & = & 336 \\
 9\text{d.} & = & 36 \\
 \frac{3}{4} & = & 3 \\
 \hline
 & & 12855
 \end{array}$$

we find that the above sum of money contains 12855 farthings.

But it is more convenient to make the reduction by steps thus: multiplying the number of pounds by 20 and adding in the shillings we obtain the entire number of shillings; multiplying this by 12 and taking in the pence we obtain the entire number of pence; and lastly, multiplying this number by 4 and adding the farthings we obtain the number of farthings. The principle of this is quite simple, and may be applied to all kinds of quantity.

$$\begin{array}{r}
 \text{£}13 \quad 7 \quad 9\frac{3}{4} \\
 \underline{20} \\
 267 \\
 \underline{12} \\
 3213 \\
 \underline{4} \\
 12855
 \end{array}$$

A word, however, must be said about linear measure, for there we find that the pole contains $5\frac{1}{2}$ yards. Let us take an example: in 3 miles, 5 furlongs, 7 chains, 3 poles, 2 yards, 2 feet, and 11 inches, how many inches? In working this we find no trouble until we come to the poles; we have 1191 poles to be turned into yards at the rate of $5\frac{1}{2}$ yards for every pole. This gives 5955 and $595\frac{1}{2}$ yards, in all $6550\frac{1}{2}$ yards, or 6550 yards, 1 foot, 6 inches, so that we have—

Mile.	Fur.	Ch.	Pol.	Yds.	Ft.	In.
3	5	7	3	2	2	11
8						
	29					
	10					
		297				
		4				
			1191			
			$5\frac{1}{2}$			
			595	1	6	
			5955	2		
			6553	1	5	
			3			
			19660			
			12			
						235925 inches.

In all 6552 yards 3 feet and 17 inches, or 6553 yards 1 foot 5 inches: and with this we proceed in the usual way.

EXAMPLES.

In £1, 2s. 5d. how many pennies are there?

In £5, 17s. $11\frac{1}{2}$ d. how many farthings?

In £18, 19s. $0\frac{3}{4}$ d. how many farthings?

Reduce £79, 1s. $5\frac{1}{2}$ d. to farthings.

Reduce £647, 0s. 6d. to pennies.

Reduce £1276, 19s. 5d. to halfpennies.

Reduce £29476, 17s. 6d. to shillings.

How many inches are there in 7 yards 2 feet and 9 inches?

How many inches are there in 5 yards 1 foot and 3 inches?

Reduce 2 poles 2 feet $9\frac{1}{2}$ inches, to inches.

How many lines are there in 4 yards 2 feet and $11\frac{3}{4}$ inches?

How many yards are there in 3 poles 1 yard 2 feet 9 inches?

Reduce 5 chains 3 poles 1 yard and $6\frac{3}{4}$ inches to feet.

How many halfpennies are there in £76950, 17s. $11\frac{1}{4}$ d.?

Reduce 3 miles, 9 chains, 5 yards, 2 feet 7 inches, and 5 lines, to feet.

In 53 miles, 3 poles, 1 yard, 3 inches, how many feet are there?

How many square inches are there in 2 square yards, 1 square foot, and 11 square inches?

In 5 square yards and 7 square feet, how many square inches are there?

How many square inches are there in 17 square yards and 1 square foot?

In 5 miles, 7 furlongs, 3 poles, 4 yards, 5 inches, how many lines are there?

Reduce 11 square poles, 29 square yards, and 4 square feet, to square inches.

How many square feet are there in 27 square yards, 7 square feet, and $5\frac{1}{2}$ square inches?

In 32 poles, 19 square yards, and 9 square inches, how many square inches are there?

In 59 square roods, 31 square poles, 30 square yards, 1 square foot, and 110 square inches, how many square feet are there?

How many square yards are there in 5 square miles, 533 square acres, and 25 square poles?

In 3 cubic feet how many cubic inches are there?

In 20 cubic feet how many cubic inches are there?

In 54 cubic yards and 19 cubic feet how many cubic inches?

How many pints are there in 7 gallons and 1 quart?

In 3 quarters, 6 bushels, 1 peck, and 3 quarts, how many pints are there?

Reduce 12 ounces, 9 drams, and 15 troy grains, to troy grains.

In 7 chaldrons, 1 bushels, 1 peck, and 3 quarts, how many gallons are there?

In 8 pounds, 13 ounces, and 7 drams, how many troy grains are there ?

How many gallons are there in 1357 chaldrons, 7 bushels, 2 pecks, and 3 quarts ?

Reduce 3 quarters, 11 pounds, 4 ounces, 8 drams, and $3\frac{1}{2}$ grains, to ounces.

How many pounds are there in 19 hundredweight, 2 quarters, and 1 stone ?

In 23 tons, 16 hundredweight, 13 pounds and 15 ounces, how many grains are there ?

In 3 hours, 29 minutes and 31 seconds, how many seconds are there ?

Reduce 23 hours, 57 minutes, and 37 seconds, to seconds.

Reduce 5 days, 10 hours, 49 minutes, 59 seconds, to minutes.

How many minutes are there in 7 Julian years, 235 days, 13 hours, and 15 seconds ?

How many seconds are there in two degrees, 45 minutes, and 19 seconds ?

In 1394 Julian years, 331 days, 3 hours, and 17 minutes, how many seconds are there ?

Reduce 145 degrees, 16 minutes, 53 seconds, to seconds.

In 1857 Julian years, 143 days, 16 hours, 21 minutes, and 10 seconds, how many seconds are there ?

When it is required to change a number of measures of one denomination into measures of a higher denomination we reverse the above process. Thus if it were asked, "how many pounds are there in 73697 farthings?" we should obtain the answer by dividing this number by 4 to turn in into pennies, the number of pennies by 12 to find the shillings, and the number of shillings by 20 to get the pounds, noting the remainder at each division as in the annexed example: The result is £76, 15s. $4\frac{1}{2}$ d.; and the very same principles apply to all other cases.

$$\begin{array}{r|l}
 4 & 73697 & 1 \text{ far.} \\
 12 & 18424 & 4 \text{ pen.} \\
 20 & 1535 & 15 \text{ sh.} \\
 & \text{£76} &
 \end{array}$$

However, the very awkward division of the pole causes some trouble in the management of the work for linear measure. For example, if it be required to convert 155605 inches into miles, we proceed first to turn the inches into feet

$$\begin{array}{r|l}
 12 & 155605 & 1 \text{ inch.} \\
 3 & 12967 & 1 \text{ foot.} \\
 5\frac{1}{2} & 4322 & 4\frac{1}{2} \text{ yards.} \\
 & 785 &
 \end{array}$$

and thence into yards; in this way we find 4322 yards 1 foot, 1 inch; and the question now is, "how many poles does this contain?" On dividing we find 785 poles with $4\frac{1}{2}$ yards over; now the half yard is 1 foot 6 inches, wherefore the result is 785 poles 4 yards 2 feet 7 inches, after that there is no farther trouble.

EXAMPLES.

In 844 farthings how many shillings are there?

Convert 1453 farthings into shillings.

How many pounds are there in $12335\frac{3}{4}$ pennies?

How many pounds are there in 1039999 farthings?

In 334596761 pennies how many pounds?

In 2155 lines how many feet are there?

How many yards are there in 11111 lines?

How many poles are there in 56865 inches 7 lines?

In 808002 feet 6 inches how many miles are there?

In 1756 square inches how many square feet are there?

In 75312 square inches how many square feet are there?

How many square yards are there in 5976521 square inches?

How many square poles are there in 518364 square feet?

In 857614 square yards how many square poles are there?

How many square roods are there in 8674201 square feet?

In 1567169 square inches how many square roods are there?

How many cubic feet are there in 16614 cubic inches?

How many square miles are there in 40289562197 square inches?

Turn 4683169 cubic inches into cubic yards.
 In 97615372149 cubic feet how many cubic yards ?
 Turn 3598.251 cubic inches into pints.
 How many gallons are there in 14730 pints ?
 In 950 pints how many bushels are there ?
 How many chaldrons are there in 7651007 quarts ?
 In 286 gallons how many sacks are there ?
 Turn 6781.25 grains into ounces.
 In 3471 drams how many pounds are there ?
 How many stones are there in 841093.75 troy grains ?
 In 15582176 troy grains how many hundredweights are there ?
 Turn 4869277513 troy grains into tons.
 In 3574 seconds how many minutes are there ?
 In 86346 seconds how many hours are there ?
 How many days are there in 178173 minutes ?
 Convert 975609 seconds into degrees.
 In 3011709647 seconds how many years are there ?
 In 1227658 seconds how many degrees are there ?
 How many years are there in 93516723571617009 seconds ?

D. For many purposes, in calculation, it is essential to have the quantities with which we are operating expressed in one denomination ; and this we can always obtain by reducing them to the lowest name : thus we can count all our moneys in farthings, all our weights in grains ; but this is inconvenient when we have to do with large quantities : thus if instead of one thousand pounds we were to say nine hundred and sixty thousand farthings and so on, we would be incommoded by the immensity of the numbers. It is more convenient to express the smaller parts as fractions of the larger denomination, and I have now to show how this may be done.

Let it be proposed to express £23, 13s. $7\frac{3}{4}$ d. in pounds and fractions of a pound.

Here since a penny is the twelfth part of a shilling, $7\frac{3}{4}$ or $\frac{31}{4}$

of a penny must be $\frac{3}{4}$ of a shilling, and therefore we have $13\frac{3}{4}$ or $\frac{55}{8}$ of a shilling. Again one shilling is the 20th part of a pound, wherefore $\frac{55}{8}$ of a shilling is equal to $\frac{11}{8}$ of a pound, and thus the proposed sum may be written £23 $\frac{11}{8}$. It is quite apparent that this is only converting the 13s. 7 $\frac{3}{4}$ d. into farthings, and observing that the farthing is the 960th part of a pound, and perhaps this would be the most direct form of the operation. However, in many cases, the fractions may be simplified as we go: the above fraction, for instance, may be written $1\frac{3}{4}$, and in those cases the process as now given may be the shorter of the two.

EXAMPLES.

Express the following sums in pounds and fractions of a pound :—

£	s.	d.	£	s.	d.	£	s.	d.
0	2	1 $\frac{1}{2}$	19	7	9	1295	01	8
0	12	7 $\frac{3}{4}$	94	19	11 $\frac{3}{4}$	3900	16	11 $\frac{3}{4}$
1	15	11 $\frac{1}{4}$	157	12	7 $\frac{1}{2}$			
12	17	6	297	00	7 $\frac{1}{2}$			

Express in the same way the following :—

- 1 foot 7 inches.
- 3 feet 8 lines.
- 7 feet, 6 inches, and 3 lines.
- 12 yards, 2 feet, 4 inches.
- 3 poles, 4 yards, 1 foot, and 5 lines.
- 4 miles, 8 chains, 2 feet, and 3 inches.
- 1 chaldron, 3 quarters, 2 pecks, and 7 pints.
- 7 tons, 3 quarters, 8 pounds, and 15 drams.
- 157 days, 19 hours, 20 minutes, and 5 seconds.
- 189 degrees, 51 minutes, and 37 seconds.

In general it is most convenient to convert the smaller parts into decimal fractions of the larger. To turn 3 ton 1 cwt. 2 qrs. 27 lb. 11 oz. 13 dr. into decimals of a ton, we begin by dividing the 13 by 16, as there are 16 drams in an ounce; this gives us .8125, to which we must add the 11 oz. Again dividing by 16, we turn this into decimals of a pound, add the 27 lb, and divide by 28, so as to get the decimals of a quarter (the intermediate denomination a *stone* is not much in use), and continuing in this way we at last get 3.08738316, &c. for the weight expressed in decimals of a ton.

16	13.0000 0000	dr.
	248	
16	11.8125 0000	oz.
	6343 248	
	1 1	
28	27.7382 8125	lb.
	5184 850	
	2 11 2	
4	2.9906 5290	qrs.
20	1.7476 6323	cwt.
	3.0873 8316	tons.

When in linear measure we have to do with *poles* the divisor $5\frac{1}{2}$ may be treated as 5.5 or as $1\frac{1}{2}$, as the computer may think most convenient.

In performing these divisions we often break the divisor: thus instead of dividing at once by 16 we divide by 4, and the quotient again by 4: instead of dividing at once by 28 we divide by 4 and by 7. The student should try both methods for the sake of strengthening his powers.

4	13.0000 0000	dr.
4	3.2500 0000	
4	11.8125 0000	oz.
4	2.9531 2500	
4	27.7382 8125	lb.
7	6.9345 7031	
4	2.9906 5290	qrs.
20	1.7476 6323	cwt.
	3.0873 8316	ton.

The conversion of shillings and pence into decimals of a pound occurs in business so frequently that it is useful to have some compendious method of performing it mentally. Now the tenth part of a pound is 2 shillings, and therefore the shillings are convertible by halving; thus 12s. = £.600, 13s. = £.650; in this there is no trouble, and it only remains to convert the pence and farthings. But 960 farthings make one pound, and therefore

the farthing is a little more than £.001, so that $9\frac{1}{4}$ d. or 39 farthings must be somewhat more than £.039. The farthing is greater than the thousandth part of a pound in the ratio of 25 to 24: wherefore if to the number of farthings we add the 24th part of itself we shall have the thousandths; and, in business calculations, we seldom notice anything below thousandths. In this way $9\frac{1}{4}$ d = equal £.041, and 13s. $9\frac{3}{4}$ d. = £.691.

On the other hand, to turn decimals of a pound into shillings and pence, we first consider the tenths and half tenth if there be one; doubling these we obtain the shillings. Then from the remaining thousandths we deduct its twenty-fifth part: the remainder is the number of farthings. For example, to turn £.739 into shillings and pence, we double the 7 to obtain 14/, and from the remaining 39 subtract its twenty-fifth part (2 nearly) to obtain 37 farthings: whence £.739 = 14s. $9\frac{1}{4}$ d. Or to turn £.382 into shillings we take out .35, which is equivalent to 7s. while the remaining 32 give 31 farthings; in all 7s. $7\frac{3}{4}$ d.

EXAMPLES.

Express the following sums in pounds and decimals of a pound:—

£.	s.	d.	£.	s.	d.	£.	s.	d.
0	1	5	150	0	9	957	9	$7\frac{3}{4}$
1	7	$6\frac{1}{2}$	275	17	$1\frac{1}{2}$	1379	19	$0\frac{1}{4}$
15	14	$2\frac{3}{4}$	356	6	3	7690	13	$11\frac{3}{4}$
27	10	9	417	11	$10\frac{1}{4}$			

Express in the same way the following:—

2 feet, 7 inches, and 4 lines.

1 yard, 11 inches, and 10 lines.

2 yards, 1 foot, and 3 lines.

1 square yard, 5 square feet, and 19 square inches.

3 poles, 4 yards, 1 foot, 9 inches, and 6 lines.

3 chains, 1 pole, 3 yards, and 11 inches.

15 cubic feet and 976 cubic inches.

1 peck, 3 quarts, and 1 pint.

3 roods, 29 square poles, 8 square feet, and 75 square inches.

5 chaldrons, 7 bushels, 1 gallon, 3 quarts, and 1 pint.

35 miles, 9 chains, 3 poles, 4 yards, 6 inches, and 11 lines.

19 hundredweight, 1 quarter, 12 pounds, 5 ounces, and 7 drams.

147 tons, 3 quarters, 1 stone, 12 pounds, 8 drams, and 19 troy grains.

6 days, 21 minutes, and 32 seconds.

27 days, 58 minutes, and 3 seconds.

1 year, 19 days, and 59 seconds.

94 years, 47 days, 10 minutes, and 58 seconds.

1 degree and 15 seconds.

38 minutes and 17 seconds.

195 degrees, 9 minutes, and 11 seconds.

We have often also occasion to convert decimals of a high denomination into the lower denominations: thus we may have to turn the fractional part of 13.73268943 miles into furlongs, chains, &c. For this purpose we must multiply the number of fractions of a mile by 8 to get the number of furlongs and fractions of a furlong: but it is not necessary to multiply the number of whole miles, unless we wish to count entirely in furlongs. The fractional part of the furlongs we multiply by 10 to get the number of chains, and so on we proceed as shown in the margin. The ultimate result is 13 miles, 5 furlongs, 8 chains, 2 poles, 2 yards, 1 foot, 7 inches, and one-fifth of an inch.

Miles.	<u>13.73268943</u> × 8
Furl.	<u>5.86151544</u> × 10
Chains.	<u>8.61515440</u> × 4
Poles.	<u>2.46061760</u> × 5.5
	23030880
	<u>2.303088</u>
Yards.	<u>2.53339680</u> × 3
Feet.	<u>1.60019040</u> × 12
Inches.	<u>7.20228480</u>

EXAMPLES.

Convert the following decimals into compound quantities :

£2.6 £13.6394 £14.7102 £23.86875
 £39.49236 £45.0708333 £159.99895833
 £394.23333 £960.2916666 £3175.83629570342
 7.0875 feet. 16.7002570025 feet. 4.7361508 yards.
 27.6500473 yards. 9.735680713 chains.
 7.610032715 furlongs. 153.6054132 miles.
 3960.105370846502 miles. 27.693572 square yards.
 19.857306914 square yards. 37.99615247 square poles.
 59.6532704729 square poles. 2.756273572 roods.
 5.2439006201 roods. 549.17326598172 acres.
 490.263057392 acres. 9471.60513 acres.
 273.20615421983 square miles.
 10627.39039572654 square miles.
 6.72159 cubic feet. 973.60597321 cubic yards.
 7.6543218 bushels. 159.12756236 chaldrons.
 13.75826943 pounds. 94.73625049 tons.
 1592.50732984652 tons. 5.2046513 hours.
 19.612157365 hours. 27.9087325682 days.
 279.651432007 days. 236.50873462571 Julian years.
 297.80958614 Julian years. 57.29658314 degrees.
 79.3007621543 degrees.

CHAPTER XIV.

ON THE ADDITION AND SUBTRACTION OF COMPOUND QUANTITIES.

C. THERE is nothing in the addition or subtraction of compound quantities to call for any particular notice: we must be careful to add likes to likes, as it would not answer to add the number of pounds to the number of shillings; and whenever we have as many of one denomination as make one or more of the higher denomination, we carry just as in common arithmetic. Thus whenever we have twelve pence or more we carry one to the shilling column for each twelve pence, writing only the remainder in the pence column.

For example, if we have to add together the three weights given in the adjoining example, we begin by adding together the drams; we find 28 drams, now 16 drams make one ounce, therefore 28 drams make 1 ounce 12 drams, and we write the 12 in column for drams and carry 1 to that for ounces: and so on with the higher denominations.

Ton.	Cwt.	Qr.	Lb.	Oz.	Dr.
7	13	1	23	5	11
3	5	3	17	12	9
8	17	2	14	10	8
19	16	3	27	12	12

When the higher denomination contains fractional parts of the lower as in the cases of poles and square poles, we may have to change the numbers in the lower columns after they have been added up. It is difficult to imagine anything more clumsy and annoying than this arrangement of poles and yards.

Miles.	Fur.	Poles.	Yds.	Ft.	Inch.	Lines.
1	7	27	2	1	11	7
2	6	15	1	0	7	5
13	0	32	3	2	0	10
17	6	35	2	0	1	10

When performing subtraction we have often to borrow from the higher denomination, as in the following example—

$$\begin{array}{r} \text{From } £17 \ 13 \ 2\frac{1}{4} \\ \text{Take } \quad 11 \ 7 \ 5\frac{3}{4} \\ \hline £6 \ 5 \ 8\frac{1}{4} \end{array}$$

$$\begin{array}{r} £17 \ 12 \ 13\frac{1}{4} \\ \quad 11 \ 7 \ 5\frac{3}{4} \\ \hline £6 \ 5 \ 8\frac{1}{4} \end{array}$$

in which we change one penny into four farthings and one shilling into twelve pence, and operate as if the minuend had been £17, 12s. 13 $\frac{1}{4}$ d.; in this there is nothing new.

EXAMPLES.

Add—

- £1, 5s. 4d. to £2, 7s. 7d. £14, 6s. 9 $\frac{1}{2}$ d. to £7, 8s. 2 $\frac{1}{2}$ d.
 £25, 17s. 11 $\frac{3}{4}$ d. to £18, 13s. £49, 3s. 1 $\frac{1}{4}$ d. to £87, 16s. 10 $\frac{3}{4}$ d.
 £55, 14s. 4 $\frac{1}{2}$ d. to £47, 10s. 7d., to £97, 17s. 11 $\frac{1}{4}$ d.
 £196, 19s. 8 $\frac{3}{4}$ d. to £146, 8s. 7 $\frac{1}{4}$ d., to £370, 11s. 9 $\frac{1}{2}$ d., to
 £94, 00s. 10d., to £901, 18s. 5d.
 3 yards, 2 feet, 9 inches, and 4 lines, to 6 yards, 11 inches, and
 7 lines.
 7 poles, 1 yard, 10 inches, and 3 lines, to 31 poles, 2 feet, 11
 inches, and 8 lines.
 7 furlongs, 38 poles, 4 yards, 5 inches, and 1 line, to 3 fur-
 longs, 13 poles, 3 yards, 2 feet, 7 inches, and 11 lines, to 6
 furlongs, 27 poles, 4 yards, and 7 inches.
 5 miles, 1 furlong, 11 poles, 2 yards, 7 inches, and 2 lines, to
 27 miles, 5 furlongs, 25 poles, 4 yards, 2 feet, 3 inches, and
 9 lines, to 157 miles, 6 poles, 3 yards, 1 foot, and 10 inches.
 2 quarters, 15 pounds, 7 ounces, and 8 drams, to 3 quarters,
 9 pounds, and 12 ounces.
 5 tons, 17 hundredweight, 1 quarter, 27 pounds, 12 ounces,
 and 15 drams, to 58 tons, 10 hundredweight, 16 pounds, 11
 ounces, and 8 drams, to 193 tons, 3 quarters, 20 pounds,
 and 3 drams.
 27 square poles, 19 square yards, 6 square feet, and 53 square
 inches, to 11 square poles, 6 square yards, 7 square feet, and
 117 square inches.

- 3 roods, 31 square poles, 14 square yards, and 98 square inches, to 2 roods, 12 square poles, 18 square yards, 7 square feet, and 35 square inches, and to 593 acres, 3 square poles, 30 square yards, 5 square feet, and 7 square inches.
- 19 cubic yards, 10 cubic feet, and 84 cubic inches, to 54 cubic yards, 25 cubic feet, and 1713 cubic inches.
- 197 cubic yards, 23 cubic feet, 659 cubic inches, to 3972 cubic yards and 137 cubic inches, to 26 cubic feet and 298 cubic inches, to 13940 cubic yards, 19 cubic feet, and 907 cubic inches, and to 29 cubic yards, 3 cubic feet, and 1 cubic inch.
- 1 day, 23 hours, 15 minutes, and 48 seconds, to 7 days, 2 hours, 38 minutes, and 75 seconds.
- 12 days, 18 hours, 3 minutes, and 17 seconds, to 79 days, 35 minutes, and 58 seconds, and to 290 days 10 hours, and 41 seconds.
- 9 degrees, 39 minutes, and 5 seconds, to 235 degrees, 50 minutes, and 47 seconds, and to 217 degrees, 11 minutes, and 13 seconds.

Subtract—

£3, 7s. 5d. from £7, 12s. 11½d.

£27, 15s. 9½d. from £30, 17s. 3d.

£158, 19s. 11¾d. from £301, 17s. 3d.

£69, 13s. 7¼d. from £752, 14s. 1½d.

£394, 7s. 2d. from £507.

£150, 8s. from £708, 5s. 6½d.

£1390, 18s. 7d½. from £2972, 15s. 7d.

£9742, 11s. 10¾d. from £29715, 10s. 3¼d.

£15427, 19s. 5d. from £75000, 00s. 4d.

£1942, 1s. 8½d. from £6531, 1s. 7¾d.

- 1 yard, 2 feet, and 5 inches, from 2 yards, 2 feet, and 6 inches.
- 4 yards, 1 foot, and 11 inches, from 1 pole, 2 yards, and 7 inches.
- 8 poles, 3 yards, 2 feet, and 3 inches, from 17 poles, 2 yards, and 7 inches.

- 3 furlongs, 27 poles, 2 feet, and 7 inches, from 1 mile, 6 poles, and 1 foot.
- 13 miles, 5 furlongs, 32 poles, 1 yard, 2 feet, and 11 inches, from 25 miles, 2 furlongs, 11 poles, 1 foot, and 8 inches.
- 13 pounds, 6 ounces, and 9 drams, from 17 pounds, 5 ounces, and 15 drams.
- 3 quarters, 5 pounds, and 7 drams, from 2 hundredweight, 25 pounds, 4 ounces, and 7 drams.
- 7 tons, 19 hundredweight, 3 quarters, 25 pounds, 2 ounces, and 6 drams, from 51 tons, 16 hundredweight, 19 pounds, 7 ounces, and 5 drams.
- 7 square yards, 8 square feet, and 98 square inches, from 1 square pole, 4 square yards, and 2 square feet.
- 253 acres, 2 roods, 39 square poles, 18 square yards, and 3 square feet, from 2 square miles, 123 acres, 1 rood, 5 square yards, and 139 square inches.
- 7 cubic yards, 3 cubic feet, and 957 cubic inches, from 8 cubic yards and 152 cubic inches.
- 341 cubic yards, 25 cubic feet, and 1599 cubic inches, from 790 cubic yards, 8 cubic feet, and 216 cubic inches.
- 3 days, 16 hours, and 41 seconds, from 7 days, 10 hours, and 39 minutes.
- 325 days, 23 hours, 6 minutes, and 57 seconds, from 2 years Julian, 66 days, 12 minutes, and 13 seconds.
- 27 degrees and 32 seconds, from 39 degrees and 27 minutes.
- 159 degrees, 45 minutes, and 25 seconds, from 360 degrees, 12 minutes, and 10 seconds.

CHAPTER XV.

ON THE MULTIPLICATION AND DIVISION OF COMPOUND QUANTITIES.

C. THE principles which guide us in the multiplication of a compound quantity by any number are just those which we have already used; but, on account of the mixture of various scales of progression, the manipulation is complex and troublesome.

When the multiplier is small the process is easy: thus if we wish to multiply 7 lb. 13 oz. 5 dr. by 8, we see that 40 drams make 2 ounces and 8 drams, and that 106 oz. make 6 lb. and 10 ounces, so that in all we have 62 lb. 10 oz. 8 dr., and all this can be done mentally.

Lb.	Oz.	Dr.
7	13	5
		8
62	10	8

But when the multiplier is large we have more trouble: thus if we have to multiply 13 cwt. 23 lb. 7 oz. 5 dr. by 573, we find no convenience in multiplying successively by the several parts of the multiplier as we did in common arithmetic, but rather a great deal of unnecessary labour. It is better to take the whole multiplier at once; and as it is impossible to keep in mind the numbers carried, we must have recourse to aside-work on our slates.

Cwt.	Qr.	Lb.	Oz.	Dr.
13	0	23	7	5
				573
		16	2865	
			179	

The student must bear in mind that the multiplier never can be anything else than a *pure number*. We never can have to

multiply pounds, shillings, and pence, by yards, feet, and inches, or by tons, quarters, and hundredweights. Although in some books the methods of performing such operations be explained, it is nevertheless true that to speak of multiplying so many feet and inches by so many feet and inches is pure unmixed nonsense, which has arisen from a misconception of the nature of certain propositions in regard to prices and surfaces. I can well enough understand what 7 times 13 shillings may be, but what can be meant by 7 yard times 13 shillings I have not the slightest idea.

Multiply—

£	s.	d.		£	s.	d.	
0	7	6	by 4.	76	15	1½	by 198.
4	12	2½	by 6.	295	11	6	by 307.
15	0	9¾	by 12.	690	3	10½	by 943.
29	17	10½	by 21.	1952	16	4½	by 7962.
58	6	8	by 92.				

- 7 yards, 2 feet, 7 inches, and 10 lines, by 5.
 17 poles, 4 yards, 10 inches, and 2 lines, by 13.
 37 poles, 3 yards, 2 feet, and 5 inches, by 29.
 6 furlongs, 39 poles, 4 yards, 1 foot, and 11 lines, by 193.
 159 miles, 25 poles, 2 feet, 11 inches, and 7 lines, by 695.
 17 pounds, 6 ounces, and 4 drams, by 37.
 7 hundredweight, 3 quarters, 14 pounds, and 5 drams,
 by 158.
 140 tons, 3 quarters, 18 pounds, 12 ounces, and 9 drams,
 by 7841.
 26 square yards, 8 square feet, and 75 square inches, by 47.
 16 square poles, 11 square yards, and 139 square inches,
 by 84.
 377 acres, 3 roods, 22 square yards, 7 square feet, and 57
 square inches, by 578.
 4 cubic yards, 18 cubic feet, and 196 cubic inches, by 245.
 1394 cubic yards, 8 cubic feet, and 1592 cubic inches, by 1904.

7 days, 15 hours, 27 minutes, and 32 seconds, by 125.
 154 days, 22 hours, 1 minute, and 59 seconds, by 3950.
 134 Julian years, 99 days, 13 hours, 49 minutes, and 11 seconds, by 9728.
 19 degrees, 14 minutes, and 25 seconds, by 381.
 187 degrees, 54 minutes, and 47 seconds, by 19762.

I have explained in Chapter V. that two very distinct operations are included in one name *DIVISION*, and when we apply these to compound quantities, their distinctive characters are well seen. The first of these two operations occurs when we seek to answer the question, *How often is one quantity contained in another quantity?* The second when we wish to divide a given quantity into a number of equal parts.

Suppose that we desire to know how often £2, 13s. 7d. is contained in £73860s. 11d.: we might begin by subtracting £2, 13s. 7d. once and again

until the whole of the larger	£2	13	7		£7386	0	11		2000
sum were exhausted, but					5358	6	8		
this process would mani-					2027	14	3		700
festly be a very tedious					1875	8	4		
one; we should have a great					152	5	11		50
many such subtractions to					133	19	2		
perform. We therefore make					18	6	9		6
a bold guess and say that as					16	1	6		
£2 is contained more than					2	5	3		

3000 times in £7000, our subtrahend may be contained 3000 in the minuend: but on multiplying £2, 13s. 7d. by 3000 we find the product to be £8037, 10s., so that 3000 is too much and we try 2000, which gives the product £5358, 6s. 8d.: subtracting this from the original sum we have £2027, 14s. 3d., which may contain our subtrahend some 700 times. In this way, by repeated trials, we discover that the dividend contains the divisor 2756 times with £2, 5s. 3d. over.

When the divisor consists of several units of its highest denomination it is not

very difficult to make our $\pounds 2\ 13\ 7$ | $\pounds 7386\ 0\ 11$
 guesses, but when it con- 53 | 147720
 tains, as in the above ex- 643 | 1772651 | 2756
 ample, a small number of 1286
 its highest name, we are 4866
 very apt to over-estimate 4501
 the quotient. It is there- 3655
 fore better to reduce both 3215
 the quota and the dividend 4401
 to lower terms, so as to get 3858
 rid both of this annoyance 543

and of the continual conversion of the products. That is to say it is better to count the quantities in tens, hundreds, and thousands of the lowest denomination, as in the adjoining work, which shows the same quotient with a remainder of 543 pence, or just $\pounds 2, 5s. 3d.$

D. We might also have converted the smaller parts into decimal fractions of the larger and treated the question $\pounds 2.67916\dot{6}$ | $\pounds 7386.0458\dot{3}\dot{3}$ | 2756
 as if we were seeking how 5358.333333
 often $\pounds 2.6791\dot{6}$ is con- 2027.712500
 tained in $\pounds 7386.0458\dot{3}$ 1875.416666
 as here shown, but on 152.295833
 account of the length of 133.958333
 decimals required this 18.337500
 method is seldom so short 16.075000
 as the usual one of reduc- 2.262500
 ing all to the lowest de-

nomination. However, when we compute by help of logarithms it is in general better to reduce the lower denominations to decimals of that denomination which is most in use, and to bring the higher denominations also to the same.

EXAMPLES.

How often does

£0, 7s. 11d. go in £56, 2s. 5d.?

£1, 6s. 2d. go in £372, 9s. 7d.?

£5, 13s. 6d. go in £570, 19s. 8½d.?

£17, 1s. 7d. go in £1925, 11s. 2¼d.?

£59, 18s. 6½d. go in £3149, 0s. 5¾d.?

£197, 15s. 10¼d. go in £4207, 3s. 11½d.?

£367, 5s. 11¼d. go in £7698, 19s. 2d.?

£190, 14s. go in £28065, 7s. 3½d.?

£3492, go in £962580, 16s. 4½d.?

2 feet, 8 inches, and 3 lines go in 3 yards, 1 foot, 6 inches, and 10 lines?

1 yard and 11 inches go in 2 poles, 2 yards, 2 feet, 5 inches, and 1 line?

4 yards, 2 feet, 10 inches, and 4 lines go in 1 furlong, 24 poles, 2 feet, and 1 inch?

12 poles, 5 yards, 9 inches, and 10 lines go in 7 miles, 7 furlongs, 23 poles, 1 yard, 2 feet, 8 inches, and 2 lines?

2 pounds, 4 ounces, and 3 drams go in 27 pounds and 15 ounces?

16 pounds, 11 ounces, and 10 drams go in 3 quarters, 21 pounds, and 5 drams?

27 pounds, 5 ounces, and 1 dram go in 5 hundredweight, 3 quarters, 15 pounds, and 9 ounces?

2 quarters, 14 pounds, and 7 ounces go in 6 tons, 17 hundredweight, 23 pounds, 11 ounces, and 8 drams?

2 square feet and 73 square inches go in 29 square yards, 6 square feet, and 91 square inches?

8 square feet and 101 square inches go in 2 square poles, 13 square yards, and 2 square inches?

3 square yards, 8 square feet, and 17 square inches go in 2 roods, 25 square yards, and 99 square inches?

37 square poles, 23 square yards, and 7 square feet go in 94

- square miles, 591 acres, 19 square poles, 28 square yards, 2 square feet, and 135 square inches?
- 3 cubic feet and 97 cubic inches go in 7 cubic yards, 19 cubic feet, and 970 cubic inches?
- 1 cubic yard, 25 cubic feet, and 3 cubic inches go in 159 cubic yards, 14 cubic feet, and 83 cubic inches?
- 5 hours, 19 minutes, and 3 seconds go in 1 day, 3 hours, 51 minutes, and 23 seconds?
- 23 hours, 59 minutes, and 59 seconds go in 18 days, 12 hours, 11 minutes, and 53 seconds?
- 6 days, 23 hours, and 15 seconds go in 2 years, Julian, 59 days, 1 hour, and 9 minutes?
- 2 degrees, 1 minute, and 7 seconds go in 29 degrees and 26 seconds?
- 23 degrees, 18 minutes, and 46 seconds go in 315 degrees, 47 minutes, and 1 second?

C. I have now to consider the second kind of division, that is the dividing of a quantity into so many equal parts. Let it be proposed, for example, to divide 496 gallons, 2 quarts, and 1 pint of oil equally among 23

	Gal.	Qr.	Pt.	Gal.	Qr.	Pts.			
persons. We at once find	23		496	2	1		21	2	0 $\frac{17}{23}$
that when each person has			46						
got 21 gallons, there will re-			36						
main 13 gallons and odds to			23						
be divided. We therefore			13						
turn these into quarts, and							54		
the 54 quarts afford 2 quarts							46		
to each, with 8 quarts over.							8		
We turn these into pints,									17

but the 17 pints do not afford so much as one pint to each, and therefore we write the fraction $\frac{17}{23}$ to signify that each party is to get the twenty-third part of seventeen pints, or seventeen twenty-third parts of one pint.

Divide—

EXAMPLES.

£	s.	d.		£	s.	d.	
16	1	7	by 9	760	9	1 $\frac{3}{4}$	by 111
47	14	6	by 18	917	13	0	by 145
159	17	11 $\frac{1}{2}$	by 26	1702	19	2 $\frac{1}{2}$	by 189
201	8	4 $\frac{1}{2}$	by 32	3644	2	10	by 1247
364	0	9 $\frac{1}{2}$	by 55	5841	6	5	by 7923
592	12	7	by 73	9325	12	3 $\frac{3}{4}$	by 3651

1 foot, 3 inches, and 7 lines by 29.

5 yards, 1 foot, 10 inches, and 1 line by 56.

2 poles, 1 yard, 5 inches, and 11 lines by 79.

13 poles, 3 yards, 1 foot, and 6 inches by 97.

3 furlongs, 27 poles, 2 feet, 4 inches, and 1 line by 247.

7 miles, 37 poles, 2 feet, 3 inches, and 10 lines by 1793.

3 pounds, 5 ounces, and 9 drams by 17.

13 pounds and 12 drams by 28.

17 hundredweight, 1 quarter, 15 ounces, and 8 drams by 51.

79 tons, 9 hundredweight, 3 quarters, 19 pounds, and 7 drams
by 3671.3 roods, 7 square poles, 15 square yards, 6 square feet, and 57
square inches by 34.4 square miles, 473 acres, 1 rood, 25 square poles, 11 square
yards, and 130 square inches by 397.

3 quarters, 4 bushels, 1 gallon, 3 quarts, and 1 pint by 198.

7 chaldrons, 7 bushels, 2 pecks, 3 quarts, and $\frac{1}{2}$ pint by 9721.

1 cubic foot and 674 cubic inches by 74.

26 cubic feet and 90 cubic inches by 397.

12 cubic yards, 11 cubic feet, and 1017 cubic inches by 57638.

21 hours, 17 minutes, and 42 seconds by 36.

5 days, 14 hours, 51 minutes, and 16 seconds, by 1368.

2 Julian years, 151 days, and 9 seconds by 39516.

1 degree, 25 minutes, and 32 seconds by 197.

159 degrees, 44 minutes, and 7 seconds by 7621.

We have often to multiply, as it is called, a quantity by a fractional number. When considered in regard to the real

nature of the questions from which such an operation seems to result, this is properly an operation in proportion. When we know the price of one yard of cloth, we find the price of a number of yards by repeating the price of one yard as often as there are yards, that is by multiplying the price of one yard by the number of the yards. And if we have to compute the price of a number of yards and fractions of a yard we still call the process by the name *Multiplication*, although it involve division also. For instance if we have to compute the price of $17\frac{5}{8}$ gallons of oil of cassia at £3, 17s. 5d. per gallon, we compute the price of the 17 gallons by multiplication, but that of the $\frac{5}{8}$ of a gallon by multiplication and division combined. Sometimes indeed, as I have already explained, this nominal multiplication is purely division.

$$\begin{array}{r} \text{£}3 \ 17 \ 5 \times 17\frac{5}{8} \\ 5 \\ 9 \overline{) \begin{array}{r} 19 \ 7 \ 1 \\ 2 \ 3 \ 0\frac{1}{8} \\ 65 \ 16 \ 1 \\ 67 \ 19 \ 1\frac{1}{8} \end{array}} \end{array}$$

EXAMPLES.

What is three-fifths of £2, 13s. 9d.?

Multiply 4 pounds, 3 ounces, 11 drams by $768\frac{7}{11}$.

What is sixteen twenty-firsts of £37, 19s. $6\frac{1}{2}$ d.?

Multiply £392, 12s. $10\frac{1}{4}$ d. by $1725\frac{9}{17}$.

£895, 1s. $2\frac{1}{2}$ d. by $\frac{37}{492}$.

£2165, 13s. $4\frac{3}{4}$ d. by $\frac{97}{88}$.

£93072, 0s. 11d. by $\frac{169}{211}$.

22 pounds, 5 ounces, and 7 drams by $31\frac{1}{8}$.

5 tons, 13 hundredweight, 2 quarters, 8 pounds, and 3 ounces by $197\frac{3}{8}$.

12 poles, 3 yards, and 10 inches by $21\frac{9}{8}$.

3 furlongs, 20 poles, 2 feet, and 11 lines by $384\frac{1}{8}$.

25 miles, 7 poles, 5 yards, 1 foot, 11 inches, and 9 lines by $1395\frac{7}{8}$.

10 hundredweight, 3 quarters, 17 pounds, and 3 drams by $37\frac{7}{11}$.

5 tons, 7 hundredweight, 19 pounds, 12 ounces, and 11 drams by $965\frac{1}{4}$.

5 square feet and 73 square inches by $15\frac{7}{8}$.

3 square yards, 8 square feet, and 101 square inches by $193\frac{7}{8}$.

3 roods, 32 square poles, 15 square yards, 7 square feet, and 15 square inches by $1954\frac{2}{3}$.

2 cubic yards, 18 cubic feet, and 356 cubic inches by $571\frac{2}{3}$.

630 cubic yards, 10 cubic feet, and 9 cubic inches by $7984\frac{3}{4}$.

4 hours, 25 minutes, and 10 seconds by $\frac{8}{7}$.

3 Julian years, 127 days, and 17 hours by $1\frac{2}{3}$.

9 degrees, 3 minutes, and 36 seconds by $5\frac{7}{12}$.

354 degrees, 21 minutes, and 49 seconds by $\frac{305}{403}$.

We have often also to divide a quantity by a fractional number: this occurs, for instance, when having given the price of a quantity of goods, we wish to ascertain the rate at which they have been sold; if the quantity of goods be expressed by an integer the process is one of division, but if the quantity be expressed by a fraction the process involves multiplication likewise. Thus if $13\frac{3}{11}$ gallons of olive oil weigh 1 cwt. 9 pounds, 7 ounces, 7 drams, the weight of one gallon is found by *dividing*, as we say, this weight by $13\frac{3}{11}$. In such a case we have our choice of two proceedings, each conducting to the same result. We may reason thus: if $13\frac{3}{11}$ gallons weigh so much, eleven times this quantity, that is

146 gallons of oil must weigh

eleven times as much, that is

11 cwt. 3 qr. 20 lb. 1 oz. 13

dr., and therefore one gallon

must weigh the one-hundred

and forty-sixth part of that, or 9 lb. 2 oz. $6\frac{1}{4}\frac{3}{8}$ dr. That is

to say we convert the mixed number $13\frac{3}{11}$ into the fraction $\frac{146}{11}$, multiply by the denominator and divide by the numerator.

Otherwise we may say $\frac{146}{11}$ of a gallon weighs so much, therefore one-eleventh must weigh the one-hundred-and-forty-sixth part of this, and the whole gallon eleven times the quota. In this way also we have 146 as a divisor, 11 as a multiplier. Properly it should be regarded as a case in proportion.

Cwt.	Qr.	Lb.	Oz.	Dram.
1	0	9	7	7

11

146	11	3	20	1	13
			9	2	$6\frac{1}{4}\frac{3}{8}$

EXAMPLES.

- If 1 yard 2 feet and 6 inches of cloth cost $2\frac{3}{4}$ shillings, what will I get for 1 shilling ?
- If 7 yards and 9 inches of silk cost $10\frac{1}{8}$ shillings, what will 1 yard cost ?
- I bought $12\frac{1}{2}$ barrels of fish for £1, 13s. 4d., how much did 1 barrel cost ?
- If 3 tons 15 hundredweight of coals cost £1, 17s. $7\frac{1}{2}$ d., what will 1 ton cost ?
- If a man charged $72\frac{5}{8}$ shillings for digging $21\frac{5}{8}$ acres of ground, what did he charge for digging 1 acre ?
- $3695\frac{3}{8}$ yards of rope cost £84, 13s. 10d., what is the price of 1 yard ?
- If $485\frac{1}{4}$ pounds of coffee cost £24, 0s. $7\frac{1}{2}$ d., what would 1 pound cost ?
- When you can get 692 eggs for $86\frac{1}{2}$ shillings, how many will you get for 1 shilling ?
- If $1362\frac{3}{4}$ pounds of cheese cost £59, 11s. $9d\frac{1}{2}$ d., how much will 1 pound cost ?
- If $743\frac{1}{2}$ pounds of sugar cost £19, 7s. $2\frac{3}{4}$ d., what will 1 pound cost ?
- If 39 gallons and $1\frac{1}{7}$ quarts of vinegar cost £1, what will be the price of 1 pint ?
- If a man can quarry 30 tons 10 hundredweight, 3 quarters, and 2 pounds, in $15\frac{1}{2}$ days, how much does he quarry per day ?
- If a pump draw 24528063 gallons of water in $359\frac{5}{7}$ days, how many gallons will it draw in 1 day ?
- A horse ran $69\frac{5}{7}$ miles in $7\frac{1}{8}$ hours, how many miles did it run per hour ?

CHAPTER XVI.

ON THE PROPORTION OF COMPOUND QUANTITIES.

C. THERE is nothing, as far as the principles are concerned, to distinguish the computation of proportion among compound quantities from that of ordinary proportion. If we reduce the quantities which are compared with each other to the same denomination, the question, whatever it may be, is immediately converted into one of common proportion. But the labour of this conversion is considerable, and the necessity for it is always felt as an irksome burden ; hence every contrivance by which we may avoid it is welcome. The aggregate of these contrivances forms what is very descriptively called *Practice*. A few examples may serve to give an idea of these various contrivances.

Let it be required to find the cost of 3 yards, 1 foot, 9 inches of gilded moulding, at 1s. $5\frac{1}{2}$ d. per yard.

1 yd. . . .	s. d.
3 yd. . . .	1 $5\frac{1}{2}$
1 ft. 6 in.	4 $4\frac{1}{2}$
3	$8\frac{3}{4}$
1 1 9	$1\frac{1}{4}$ $\frac{5}{8}$
	5 $2\frac{1}{4}$ $\frac{5}{8}$

Here since 1 yard costs 1s. $5\frac{1}{2}$ d. 3 yards must cost 4s. $4\frac{1}{2}$ d. The remainder 1 foot 9 inches is made up of 1 foot 6 inches, which is half a yard and 3 inches, which is the sixth part of that half yard : wherefore we halve 1s. $5\frac{1}{2}$ d. to get $8\frac{3}{4}$ d. the price of the half yard, and take the sixth part of that to get the price of the three inches. The calculation might also have been made by considering 1 foot as the third part of a yard, and 9 inches as made up of half a foot and the

half of that again ; and it affords good exercise to the student to seek out the different ways in which the computation may be made.

Again, let us propose to compute the price of 17 hundred-weight, 3 quarters, 15 pounds, 11 ounces, 7 drams of ivory, at £43, 13s. 9d. per cwt.

1 cwt.		£43 13 9
17 cwt.		741 12 1
2 qr.	$\frac{1}{2}$	21 16 10 2
1 qr.	$\frac{1}{2}$	10 18 5 1
14 lb.	$\frac{1}{2}$	5 9 2 2 $\frac{1}{2}$
1 lb.	$\frac{1}{14}$	7 9 2 $\frac{13}{14}$
8 oz.	$\frac{1}{2}$	3 10 3 $\frac{13}{8}$
2 oz.	$\frac{1}{4}$	11 2 $\frac{131}{224}$
1 oz.	$\frac{1}{2}$	5 3 $\frac{131}{448}$
4 dr.	$\frac{1}{4}$	1 1 $\frac{1525}{1792}$
2 dr.	$\frac{1}{2}$	2 $\frac{3317}{3584}$
1 dr.	$\frac{1}{2}$	1 $\frac{3317}{7168}$

Here we first compute the cost of 17 cwts. : we then take half a cwt. and the half of that, which make the 3 quarters : half of a quarter gives 14 lb. and the fourteenth part of that gives one pound : and thus we proceed as is indicated by the fractions placed in the middle column. Having found the price of each of the component parts of the weight, we add all these up to get the price of the whole.

Although many prefer such a process as this, I am of opinion that it is, in the majority of cases, longer than if we were to reduce all the quantities to their lowest terms.

This method can only be used when the quantity of which the rate is given is expressed by a simple number. Here is a case of very common occurrence in the business of navigating a ship. In 3 hours the moon's motion is $1^{\circ} 29' 32''$, required its motion in $2^h 53^m 21^s$.

3 ^h		1°	29'	32"
1	$\frac{1}{3}$		29	50.666
2	2		59	41.333
45	$\frac{1}{4}$		22	23
8	$\frac{1}{15}$		3	58.755
20 ^s	$\frac{1}{24}$			9.948
1	$\frac{1}{20}$			.497
1 53 21		1	26	13.534

Here we first compute the motion in 1 hour, thence that in two hours. 53 minutes is made up of 45 and 8, the first of these is the quarter of 3 hours, the second is the fifteenth part of 2 hours.

After these examples the learner may easily perform the following calculations.

EXAMPLES.

What will 7 yards 2 feet 7 inches and 9 lines of silver wire cost, at 2s. 5d. per foot ?

If 1 yard of silk cost 5s. 2d., what will 12 yards 11 inches and 3 lines cost ?

1 pound of cheese cost 1s. 5 $\frac{3}{4}$ d., what will 10 hundredweight, 3 quarters, 14 pounds, and 10 drams of the same cheese cost ?

1 pound weight of cinnamon cost 4s. 10 $\frac{1}{4}$ d., what will 19 pounds, 12 ounces, and 5 drams cost ?

If 1 pound of sugar cost 7 $\frac{1}{4}$ d., what will 15 hundredweight, 2 quarters, 18 pounds, 9 ounces, and 7 drams cost ?

If 1 pole of road cost 18s. 7 $\frac{1}{2}$ d., what will 2 furlongs, 25 poles, 3 yards, 2 feet, and 6 inches of the same road cost ?

1 fathom of a cable was found to weigh 25 pounds, 3 ounces, and 8 drams, what will the weight of a cable 127 furlongs, 19 poles, 5 yards, and 2 feet long of the same kind be ?

A hundredweight of gold cost £6384, 7s. 4 $\frac{1}{2}$ d., what will 10 tons, 3 hundredweight, 1 quarter, 15 pounds, 5 ounces, and 12 drams cost ?

- A merchant bought 1 gallon of wine for £5, 2s. 10d., what will the price of a cask of the same wine, which held 2 tuns, 1 puncheon, 1 hogshead, 35 gallons, 2 quarts, and $\frac{1}{2}$ pint, be ?
- A cubic inch of water weighs 252.5 grains avoirdupois, how much will 2 cubic yards and 396 cubic inches weigh ?
- If 1 square pole of ground cost 5s. $7\frac{1}{2}$ d., what will the price of 3 acres, 19 roods, 10 square poles, and 8 square feet, be ?
- A hogshead of oil cost £33, 13s. 9d., what will 1 tun, 1 puncheon, 40 gallons, 3 quarts, and $\frac{1}{2}$ pint cost ?
- What will be the price of a piece of ground, consisting of 173 acres, 2 roods, 22 square poles, 20 square yards, 5 square feet, and 105 square inches, the price of an acre being £56, 13s. $6\frac{1}{2}$ d. ?
- If a man charged 2s. $7\frac{1}{2}$ d. for carting 1 ton of goods, what will the cartage of 159 tons, 15 hundredweight, 3 quarters, and 20 pounds of goods, at the same rate, be ?
- If a cubic foot of marble weigh 2704 ounces, what will 33 cubic yards, 25 cubic feet, and 562 cubic inches of the same marble weigh ?
- If I can get 17 feet, 4 inches, and 8 lines of brass cornice for £1, how much will I get for £25, 17s. $9\frac{1}{2}$ d. ?
- If I can get 4 tons, 13 hundredweight, 2 quarters, and 25 pounds of gravel for £1, how much will I get for £12, 10s. 6d. ?

When the quantity of which the rate is given is complex, decidedly the best method is to reduce it and its consequent to their lowest terms, so as to make the ratio one of common numbers ; and then the calculation does not differ in any respect from that of common proportion. A very extensive set of examples is subjoined.

EXAMPLES.

- If I can buy 4 pounds, 12 ounces, and 8 drams of tea for 16s. $7\frac{1}{2}$ d., what will I pay for 11 pounds, 7 ounces, and 2 drams of the same tea ?

- A banker lent £284, 16s. 6d. at £4, 5s. 6d. per cent per annum, what would the interest amount to in 3 years 6 months?
- A wheel revolves 16780.5 times in 3 hours, 23 minutes, and 14 seconds, how often will it revolve in 6 days, 1 hour, 54 minutes, and 2 seconds?
- A pump drew 40 gallons, 3 quarts, and $1\frac{1}{2}$ pints of water in 1 hour, 7 minutes, and 38 seconds, how much will it draw in 4 days, 12 hours, 31 minutes, and 15 seconds?
- A person paid £39, 16s. 1d. for 5 tons, 16 hundredweight, and 2 quarters of turnips, what would he pay for 17 tons, 2 hundredweight, and 3 quarters?
- If I paid £120, 10s. $10\frac{1}{2}$ d. for 2 ton, 17 hundredweight, 1 quarter, and 17 pounds of sugar, what would I pay for 20 tons, 5 hundredweight, 6 pounds, and 8 ounces?
- A man can plough 1 acre, 3 roods, 21 square poles, and 18 square yards of ground in 5 hours, 47 minutes, and 30 seconds, what time would he take to plough 15 acres, 38 square poles, and 10 square yards?
- A plate of copper 2 feet broad, $\frac{1}{4}$ inch thick, and 16 yards, 2 feet, and 8 inches long, cost £5, 9s. 4d., what will a piece 97 yards, 1 foot, and 3 inches long, of the same plate cost?
- A miller charged for grinding 4 quarters, 3 bushels, and 1 peck of flour, 14s. $7\frac{1}{2}$ d., what will he charge for grinding 7 tons, 4 quarters, and 1 bushel?
- A collier dug out 147 tons, 18 hundredweight, 13 quarters of coal in $8\frac{3}{4}$ months, how much will he dig in 3 years and $3\frac{1}{2}$ months?
- A mason charged £14, 5s. $6\frac{1}{2}$ d. for building a wall with 24 square yards and 1 square foot of facing, what should 175 square yards, 2 square feet, and 7 square inches of the same kind of wall cost?
- A company borrowed £5620, 10s. at £3, 12s. 7d. per cent, what interest will they have to pay in 7 years and 10 months?
- A man dug 2 acres, 1 rood, 11 square poles, and 20 square yards of ground in 3 days, 6 hours, and 37 minutes, in what

- time would he dig 37 acres, 2 roods, 35 square poles, and 5 square yards?
- A block of stone consisting of 5 cubic feet and 864 cubic inches weighed 9 hundredweights, 3 quarters, 11 pounds, and 11 ounces, what would a block of the same of 196 cubic yards, 10 cubic feet, and 785 cubic inches weigh?
- A farmer got £398, 16s. 7½d. of profit from a farm of 156 acres, 3 roods, and 35 square poles, in 2 years and 7 months, what should a farm of 3692 acres, 1 rood, and 20 square poles, yield in 7½ years at the same rate?
- A grocer borrowed £123, 16s. 5d. for 3 years and 7½ months at £3, 15s. 2d. per cent, what amount of interest will he have to pay?
- A planing-machine was able to plane 27 square yards, 2 square feet, and 25 square inches of iron in 5 days, 13 hours, 54 minutes, and 30 seconds, in what time will it plane 162 square yards, 8 square feet, and 7½ square inches of iron?
- A baker's oven was able to bake 1629354 4-pound loaves in 8 Julian years and 139 days, how many loaves will it bake in 23 Julian years and 210 days?
- 35 men were able to quarry 13482 tons, 16 hundredweight, and 1 quarter, in 321 days, 13 hours, and 40 minutes, how much will they do in 69 days, 19 hours, and 15 minutes?
- An ironwork was able to turn out 130 tons, 17 hundredweights, 3 quarters, and 10 pounds of iron in 57 days, 7 hours, and 35 minutes, how many tons would be turned out at the same rate in 12 Julian years, 320 days, 15 hours, 41 minutes, and 27 seconds?
- If you can get £73, 8s. 3d. of profit in 18½ months by trading with £376, 5s. 4d., what profit will you get by trading with £257, 19s. 1d. during 7 months?
- A carpenter charged £30, 13s. 6½d. for 3 furlongs, 28 poles, 4 yards, 2 feet, and 6 inches of railing, what will 2 miles, 7 furlongs, 35 poles, 5 yards, and 10 inches of the same railing cost?
- A gentleman has a fortune of £6725, 4s. which yields him an

income of £305, 12s. 6d. per annum, what is that per cent, and what would a fortune of £10621, 17s. 2d. yield at the same interest?

If you can get 1 ounce and 6 drams of gold for £5, 6s. 6½d. how much will you get for £1369540, 10s. 5½d.

A steamer sailed 1613 miles, 27 poles, 4 yards, 2 feet, and 3 inches, in 6 days, 13 hours, 24 minutes, and 30 seconds, how many miles will it sail in 120 days, 12 hours, 15 minutes, and 47 seconds, and how long will it take to sail 5624 miles, 6 furlongs, 5 poles, and 3 yards?

A pump was found to lift 3517 gallons, 3 quarts, and ½ pint, in 1 hour, 7 minutes, and 39 seconds, how long will it take to pump up 1653582 gallons, 2 quarts, and ⅓ pint?

A lump of stone of 4 cubic yards, 15 cubic feet, and 432 cubic inches, weighed 8 tons, 19 hundredweight, 3 quarters, 11 pounds, 10 ounces, and 12 drams, what was the size of a stone of the same kind which weighed 39 tons, 1 hundredweight, 16 quarters, and 9 pounds?

If a wire-drawing machine be able to draw 3 furlongs, 4 yards, 1 foot, and 2.7 inches of wire in 1 hour, 42 minutes, and 13 seconds, how much wire will it draw in 6 days, 5 hours, 31 minutes, and 1 second?

If 25 men be able to dig 17 acres, 2 roods, 27 square poles, and 19 square yards, in 18 hours, 14 minutes, and 42 seconds, how many men will I need to employ to dig 163 acres, 1 rood, and 7 square poles, in 2 days, 7 hours, 30 minutes, and 12 seconds?

An engine of 30-horse power was able to draw 103 tons and 10 hundredweight of coals in 3 hours and 27 minutes, how long will it take to raise 7620 tons, 18 hundredweight, and 2 quarters?

