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CRYSTALLOGRAPHY

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CRYSTALLOGRAPHY

A TREATISE ON THE

MORPHOLOGY OF CRYSTALS

BY

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PREFACE

THE establishment of Crystallography as a science at the hands of Haüy dates back a hundred years from the present time; and midway in that period (in 1839) Professor W. H. Miller published his now classical Treatise.

The notation by index-symbols, the representation of the relative positions of crystal faces by the distribution of their 'poles' on a sphere, the stereographic projection of the sphere and of the poles lying upon it, and some of the methods that Miller employed in dealing with the symbols of crystal faces, may indeed be shown to have been introduced in various memoirs by earlier geometers. Thus Bernhardi (1809), Neumann (1823), Frankenheim (1826), and Grassmann (1829) had all represented crystal faces by their normals; the three latter had projected the poles of faces and of zones on the great circles of a sphere: Neumann further represented a face by a symbol that was a first approximation to the simple form suggested by Whewell (1824–5) and afterwards by Justus G. Grassmann (1829) before it became in the hands of Miller an effective implement by which to rear the structure of modern Crystallography. Dr. Whewell, Miller's immediate predecessor in the Cambridge Chair, had indicated the method of deriving the symbol of a face from those of two other faces truncated by it.

But, after the labours of previous crystallographic writers have been thus recognised, the methods and systematic treatment of Crystallography in Miller's Treatise remain indisputably his own. The rationality of the anharmonic ratio of any four tautozonal planes was his, independently of a similar result afterwards published in the memoirs of Gauss; and to him belongs the merit of combining the methods of his predecessors with the stereographic projection into a complete and elegant system, in which the power of the indicial symbols in grouping the faces of the forms of crystals, and in colligating tautozonal faces as well as tautohedral zones, was manifested, and the foundation laid on which the laws of crystal symmetry have been well established.

The system thus identified with the name of Miller has met at length with an almost universal acceptance. Adopted forty years ago in Vienna by the school of crystallographic physicists in which Grailich and V. von Lang were pioneers, it has in recent times formed the crystallographic language employed in such complete treatises as those of Liebisch and of Mallard, in the encyclopædic Mineralogy of Dana, and in that great storehouse of crystallographic information, the Zeitschrift of Groth.

The growth of Crystallography during the last sixty years has resulted, notably, from the recognition of the limitations imposed on its symmetry by the homogeneity of a crystal. By various methods, Hessel in 1829, Bravais, von Lang, and Gadolin in 1866–7, and, among later investigators, more especially Sohncke, Schönflies and Fedorow, have defined the varieties of symmetry which can be presented by crystals.

The present Treatise, dealing solely with the Morphology of crystals, represents the substance of courses of lectures in which it was often found necessary to treat Crystallography in the simplest form compatible with strict geometrical methods; the classes, in such cases, consisting of students from other departments of science in which familiarity with mathematics was not demanded. On other occasions students of physics with high mathematical training formed the class.

I trust that, while endeavouring to fulfil the requirements of readers of the former kind, this book will not be found lacking in demonstrations that may satisfy those of the latter class, for whom indeed it may be said to commence at Article **53**, p. 65. Several chapters, those for instance on Symmetry, are written with more detail than a geometrical student might recognise as needed; but this seemed unavoidable if the usefulness of the book was not to be confined to such readers.

Among my old students there have been some who have taken a great interest in the subject and in my book: to their relations towards it and towards myself I may only advert here in a few too inadequate words. To a remoter time belongs Professor Lewis, formerly my colleague at the British Museum, and now Professor Miller's successor in the Mineralogical Chair at Cambridge.

More especially are my cordial thanks due to the friend who succeeded me in the Keepership of the Mineral Department of the British Museum, Mr. L. Fletcher, for numerous and valuable suggestions and additions to the original MS. and to the proofs which he has so carefully revised.

To Mr. Miers' skilful hand I am indebted for numerous figures, the drawing of which I found, with failing eyesight, to be increasingly difficult.

Various features that I have adopted from the remarkable Treatise (1866) of my old friend Viktor von Lang, formerly, for a time, my colleague at the British Museum, justify my numbering the distinguished Vienna Professor among those to whom I am as much indebted for the matter of my book as I am for their sustained friendship.

It is intended in a future volume to deal with the physical problems necessary to the practical crystallographer. It being my purpose in that as in the present work to bring crystallography into more familiar use by students of chemistry, mineralogy, and petrology, it was necessary that the treatment of the subject should not demand the advanced mathematics needed for reading parts of the master-work of Liebitsch. That the principles of crystallographic optics can be thus treated without the surrender of exact geometrical method is evidenced by Mr. Fletcher's Tract on the *Optical Indicatrix*; and there are other parts of crystallographic physics capable of being represented with a like simplicity.

The greater part of the present Treatise was long ago written, and indeed in print; and for the delay in its publication its author is alone responsible. The causes have been numerous, partly arising from the distractions of other unavoidable and not unimportant duties, but mainly from a certain indecision habitual to the workman, who has felt how 'easy it is to begin but to finish how difficult.' For the acknowledgement of the longsuffering and kindly treatment he has experienced at the hands of that unique institution, the University Press, he has not words.

N. S.-M.

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CRYSTALLOGRAPHY.

CHAPTER I.

ON THE GENERAL PROPERTIES OF CRYSTALS.

1. The Crystalline Condition.

Most substances, whether chemical elements or compounds, assume the crystalline condition when they become solid under circumstances favourable to the gradual and unconstrained deposition of their particles. Under conditions conducive to the growth of separate individuals, crystals are polyhedra with plane faces and without re-entrant angles.

The processes by which substances pass into the crystalline condition belong generally to one or other of the following classes:—

(*a*). From *sublimation*: in this case the substance passes directly from the gaseous to the solid condition.

Iodine, arsenic and camphor are familiar examples of crystallisation from sublimation.

(*b*). From *fusion*: when the substance passes directly from the liquid to the solid condition.

Both bismuth and sulphur afford examples of crystallisation from a fused mass during cooling; and water, on freezing, assumes the crystalline condition, though the individual crystals of ice are in general difficult to distinguish.

(*c*). From *solution*: the deposition of crystals may result from diminution of the solvent capacity of a liquid holding the substance in solution: and this may arise from change of temperature, or of osmotic pressure; or from removal, by evaporation or otherwise, of the solvent liquid; or, again, from that liquid losing an ingredient that imparts to it higher solvent power, or receiving a fresh ingredient that diminishes that power.

These various methods of crystallisation from solution are universally employed in the processes of the laboratory, and, on a larger scale, in the operations of technical chemistry.

(*d*). By *change in the solid condition.* It sometimes happens that a substance passes from a non-crystalline to a crystalline condition, or from one crystalline type to another, without passing through the liquid or gaseous state.

Non-crystalline (vitreous) arsenious anhydride in lapse of time developes octahedral crystals; again, the transition of (tetragonal) crystals of the red mercuric iodide into those of the yellow (orthorhombic) type, and the reconversion from the latter type into the former, as the temperature is successively above and below 126° C., is a familiar example of a numerous class of changes. Of such transformations the remarkable optical researches of Mallard and others have contributed illustrations; prominent among these are the passage of boracite at a temperature of 265° C., and of leucite at about 500° C., in each case from a pseudo-cubic symmetrically grouped aggregate of crystals (orthorhombic in the case of boracite) to a single crystal of cubic type, and the reconstruction of aggregates of pseudo-cubic type below those temperatures (see Article **166**, p. 187).

Further, under any of the preceding processes, new compounds produced as a result of the mutual decomposition of two or more substances may assume the crystalline condition.

Many of the largest and finest crystals met with among minerals have resulted from natural processes involving mutual decomposition, and numerous illustrations of artificial processes of the same character are afforded by the microscopic crystals produced in the course of micro-chemical analysis.

2. The characters of crystals may be referred to two different classes:—

I. *Morphological characters*, which result from the distribution and geometrical relations of their plane faces;

II. *Physical characters*, which result from their homogeneity and the distribution of the physical properties.

3. I. Morphological Characters.

Conclusions drawn from observations on numerous crystals of one and the same substance are the following:—

On any individual crystal, faces presenting similar physical characteristics, superficially exemplified in the lustre, striation, hardness, &c., of their surfaces, may usually be recognised as recurring on different parts of the crystal, and, indeed, recurring in a certain ordered and symmetrical arrangement. In general the faces of a crystal are not all alike either in form or in the characters of their surfaces; they exhibit lustre that may vary from an adamantine or a metallic brilliance, through a glassy or a nacreous reflection, to a dulness that reflects no image: or, again, a striation, that in some cases consists of a fine linear tracing, in others of a coarse channelling of the surface; the direction of striation being generally parallel to certain edges of the crystal.

All the faces of the crystal fall into one or into several such groups or *forms*; each form comprising the *symmetrically recurring faces characterised by similar features and properties.*

4. *Crystallographic law.* All the faces belonging to any one crystal are connected by certain geometrical relations which obey a simple law known as the Law of *Rationality of Indices.*

Different crystals of the same substance may present an indefinite variety in their *forms* and *combinations* of forms: such crystals may even have no form in common, and therefore no faces that are directly comparable with each other. Yet, by virtue of the law just mentioned, it is practicable to determine from any one crystal of a substance all the faces that may possibly occur on the same or different crystals of that particular substance, and to establish a system of planes that shall be characteristic of it. By reference to this system of planes, it is possible to establish the morphological relationship of all crystals of the same substance, whatever faces they may exhibit.

The faces of crystals of a given substance, even when they belong to the same form upon the same crystal, are usually found to differ in size; but while they obey no law as regards their relative magnitude, the mutual inclination of every pair is the same as that of every corresponding pair, whether of the same or different crystals of the given substance, at the same temperature.

5. *Crystallographic systems.* Whereas the different forms upon crystals of the same substance differ not only in the mutual inclinations of their faces, but also in other characteristics, such as lustre, striation, &c., they will all be found to conform to the same type of symmetry.

Further, while in all crystals of the same substance the morphological features are distributed in accordance with one type of orderly arrangement or symmetry, it will be hereafter shown that there can be six and only six such types of symmetry to one or other of which every crystal whatsoever can be referred.

These are termed the *crystallographic systems* and are distinguished as:—

I. *The Cubic system.*

II. *The Tetragonal system.*

III. *The Hexagonal system.*

IV. *The Ortho-rhombic system.*

V. *The Mono-symmetric* (or *Clino-rhombic*) *system.*

VI. *The Anorthic system.*

6. *Crystallographic elements.* The geometrical relations connecting the faces of a given substance are expressed by means of certain constants termed the *crystallographic elements*, and, from what has been said in Article **4**, it follows that these are characteristic of the particular substance. The crystallographic elements, and, as following from them, the essential morphological characters of a crystal, thus depend, not on the relative dimensions of its faces or lengths of its edges, but on the relative *directions* of both, and therefore on the *dihedral angles* of the latter.

While the crystallographic elements of an individual substance are thus the same at a given temperature for all the crystals of that substance, those of the different substances which crystallise in any one of the systems present indefinite variety. Crystals of the Cubic system are an exception to this statement; for the crystallographic elements in this system are the same for all, so that analogous forms have identical angles on such crystals. A like constancy of angle attaches also to certain forms on crystals of the Tetragonal and Hexagonal systems: the faces of such forms are parallel to a particular direction, termed the *morphological axis*, peculiar to such crystals, around which, either singly or in pairs, they are symmetrically repeated in quadruple, sextuple, or triple recurrence.

7. II. **Physical Characters.**

Crystals are further distinguished from other matter in that while the physical characters of ordinary uncrystallised matter are generally the same, in crystals they are different, in different directions. At the same time it is generally true that the properties characteristic of any given direction in a crystal are found to characterise also other directions in it. But the different directions thus similarly endowed, which may be considered as if they were repetitions of the first direction, will be found to be repeated *symmetrically* and in general accordance with the principle of symmetry that controls the repetitions of the morphological features on the crystal.

A brief review of some of the distinctive physical properties that characterise crystals will serve to illustrate the above principle.

8. *Elasticity and cohesion.*

(*a*). The discussion of the conditions of elasticity in crystals involves very complicated mathematical expressions. In the most general case the determination of the stress requisite to produce a given homogeneous strain involves as many as twenty-one independent constants termed *coefficients of elasticity*: the constants, however, become reduced as the symmetry of the crystal assumes a higher type, and are reduced to three in the case of a cubic crystal. In terms of these constants the coefficients of *elasticity of volume* and the coefficients of *elasticity of figure*, or the *rigidity*, can be expressed.

It is found that directions which correspond in their morphological relations are endowed with the same characters as regards elasticity.

(*b*). When a solid body is acted upon by forces which produce deformation it may behave in one of various ways.

The deformation may be permanent or it may be only temporary: if it be wholly or in part permanent, the original arrangement of the particles is not restored when the stresses to which the strain was due have ceased to operate; a *ductile* body, for example, undergoing a permanent elongation under a tensile stress; a *malleable* body admitting of entire change of form by successive impacts from a hammer or the continuous pressure of a roller, in both cases without disruption. It is obvious that though a crystal may possess such qualities, the exercise of them is incompatible with its particles retaining their crystalline arrangement.

(*c*). *Glide-planes*. Certain remarkable changes, however, involving permanent deformation in a crystal without disruption have been effected by pressure at the edges or quoins (solid angles).

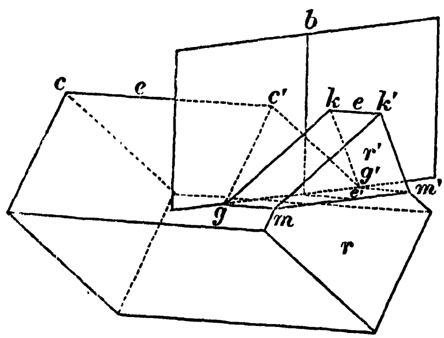


Fig. 1.

A simple mode of producing such a deformation in a cleavage-rhombohedron of Iceland spar is one first suggested by Baumhauer. Three edges of the rhombohedron meet on the morphological axis at an apex where the three plane angles are each obtuse, being 109° 8′ 12″. If a knife-blade represented by the plane *bgg′* be applied at a point *c′* not very far from that apex, with its faces perpendicular to one of the edges *e*, and be subjected to a steady pressure or a slight blow, a wedge-shaped fissure *c′gg′k* is readily formed, and the portion of the crystal between the blade and the original face *r* of the rhombohedron is moved to a new position.

A plane parallel to the edge *e*, and equally inclined to the faces forming that edge, is a not infrequent face of a calcite crystal. If *e′* be a plane parallel to such a possible face, meeting the two sides of the fissure in the line *gg′*, the block included between the plane *e′* and the edge *e* has been sheared, the sheared part of the face *r* becoming *r′*; the particles of the line *e* between the blade and the apex have glided until the displaced portion becomes *kk′*; the particles similarly aligned on all lines parallel to the edge *e* have in the same way glided into new positions in the crystal-block, rolling over, doubtless, through a half-turn during the deformation.

The *glide-plane* *e′* bisects the angle between the original face *r* and the sheared face *r′* along the line *mm′*. The *shear-plane* passing through the edge *e* cuts the glide-plane *e′* and the plane of the blade perpendicularly. The angles *kk′m* and *kk′m′* originally obtuse are now acute, while the obtuse angle *mk′m′* has remained unaltered in magnitude; *k′* in the deformed part of the crystal is thus no longer an apex situated on the morphological axis but a lateral quoin similar to *c*. It will be easily seen that the deformation has been effected without any change of volume of the crystal.

(*d*). *Cleavage.* A characteristic property of most crystals is that of their being more or less easily fissile along certain directions which are termed *cleavage-planes* or *cleavages*. The cleavages (and also the glide-planes) are usually parallel to prominent or crystallographically important faces of the crystal.

The facility with which the cleavages on the same crystal may be effected will be found to be different where the faces to which they are parallel have a different morphological significance: where, on the other hand, they are crystallographically equivalent (i.e. where they belong to the same form) the cleavages will be equally facile.

The relative facility of two or more cleavages presented by a crystal is frequently of assistance in determining the morphological symmetry.

The glide-planes, in the case of deformed crystals, are also planes along which disruption can be easily effected.

(*e*). *Impact-figures.* When the cohesion is suddenly overcome as by the impact of a pointed instrument on a plane surface of a crystal, linear cracks radiating from the point are produced, giving rise to what is known as an *impact-figure.* These cracks may be parallel to cleavages when such intersect the surface, but they are often due to the production of glide-planes as described in paragraph (*c*).

Since the directions of the cracks conform to the symmetry of the crystal, the impact figures are often serviceable in determining the character of that symmetry.

9. (*f*). *Hardness.* Attempts have been made to measure the cohesion of bodies by determining their so-called hardness.

The hardness of crystals in different directions has been estimated by means of an instrument termed a sclerometer. A pointed fragment of hard material, one of diamond being employed in the case of very refractory bodies, is loaded with a certain weight and drawn over the surface a certain number of times along lines parallel to a given direction. The loss of weight experienced by the crystal during this process is taken as a measure of the hardness in that direction. The operation is repeated for other directions in the same surface, and for other plane surfaces of the crystal; the values so obtained are always found to be symmetrically repeated in accordance with the morphological symmetry of the crystal.

Such a method is, however, obviously a rude one, and is subject to mechanical sources of error that interfere with the accuracy of the results obtained by it even as a means of determining hardness, and it is therefore far from affording a true measure of cohesion in a crystal.

The relations of hardness to cleavage have been carefully studied, though by necessarily imperfect methods, with the results that differences of hardness are found to be greater in crystals conspicuous for cleavage; that the hardness is more uniform in different directions in the absence of distinct cleavages; that the planes of cleavage are the surfaces of easiest abrasion; and that hardness increases as the direction is more nearly coincident with the normal of a cleavage-plane.

(*g*). *Results of erosion.* Differences in cohesion may also be made apparent by other than purely mechanical agencies. Thus a crystallised substance frequently manifests unequal solubility, or different degrees of resistance to chemical action, along different directions in the crystal, and this is probably universally true.

Lavizzari by the action of nitric acid reduced a sphere of calcite into the form of a crystal with pyramidal faces.

The action of a solvent on the faces of a crystal depends both upon the nature and strength of the solvent and the directions of the faces, one solvent acting more energetically on some forms, another solvent so acting on others; but all faces belonging to any one form are always affected in a similar way.

The action of an eroding agent on a single face of a crystal is a valuable method of probing its symmetry. It generally commences with the production of a number of minute hollows; these are sometimes small inverted or negative crystal forms, sometimes grooves or mere lines; but they infallibly indicate the symmetry of the crystal face, and, where other morphological evidence has been wanting, they have in important cases afforded the means of determining the merosymmetrical character of a crystal, that is to say have revealed a partial development of symmetry in the crystal in accordance with a law to be hereafter enunciated (in Art. **141**, p. 161).

10. *Light.*

The important subject of crystallographic optics will only be treated here so far as to place the optical behaviour of a crystal in general correlation with its other physical characters. In the case of crystals belonging to the Cubic system, as in the case of isotropic substances, along every direction a plane-polarised ray may generally be transmitted having its plane of polarisation in any direction whatever, and the velocity of the ray within the crystal is the same whatever may be the direction of the ray or of its plane of polarisation. In crystals belonging to other systems, though a plane-polarised ray may be transmitted along every direction, the plane of polarisation of that ray can in general have only one or other of two mutually perpendicular positions, and these are definitely related to other directions in the crystal.

The velocity of the ray within the crystal depends both upon the direction of the ray and that of its plane of polarisation, as enforced by the last-mentioned condition.

The direction, the velocity and the plane of polarisation of all rays of a mono-chromatic light within a given crystal can be geometrically represented by the aid of a single surface, the *optical indicatrix*, whose form and position completely define the refractive properties of the crystal for light of that particular colour. In the Anorthic, Mono-symmetric and Ortho-rhombic systems the indicatrix is an ellipsoid; in the Hexagonal and Tetragonal systems a spheroid; and in all of these the crystals are doubly refracting with respect to light. In the Cubic system the indicatrix is a sphere, and, by reason of the velocity being the same whatever the direction and plane of polarisation of the ray, the refraction on emergence cannot be other than single.

If the light be supposed to emanate in every direction from a point anywhere within the crystal, it will after a small interval of time reach a surface which is known as the *ray-surface* or *wave-surface*. In the case of a cubic crystal it is evident from what has been said that this is a sphere: but in doubly refracting crystals the surface will consist of two sheets, since in any direction rays can be transmitted with either of two velocities according to the position of the plane of polarisation. Though the transmission of light along any direction in a doubly refracting crystal may thus be mathematically regarded as consisting in the separate and simultaneous transmission of two independent rays which arrive at different points on the line after the lapse of the stated interval, the physical process must be one of a more complex nature.

In crystals belonging to the Hexagonal and Tetragonal systems the ray-surface consists of a sphere and a concentric spheroid that generally touch each other at the ends of a common axis; in the remaining systems neither sheet is spheroidal or ellipsoidal in form, but, for a given monochromatic light the ray-surface possesses, in common with the ellipsoidal indicatrix from which it may be geometrically derived, three perpendicular planes of symmetry.

Although in form and position both the indicatrix and its derived wave-surface may vary for monochromatic lights of different colours, they always do so in such a manner as to accord with the morphological symmetry of the crystal—a symmetry-plane in the latter being invariably identical in direction with one or other of the symmetry-planes of the ray-surface.

11. *Thermal conductivity.*

The degrees of facility with which heat is conducted in different directions through the substance of a crystal have been investigated by cutting thin sections with parallel plane surfaces along different directions in the crystal, covering a face of such a section with a thin layer of wax or paraffin, and observing the form ultimately taken by the fused portion of the wax on the continued application of a heated wire to a point on the surface. As the form is invariably found to be either circular or elliptical, the *continuous isothermal surface* which would result from the maintenance of a given temperature at a point inside a crystal must be either a sphere, a spheroid, or an ellipsoid.

In each case the isothermal surface is found to be symmetrical to the planes of morphological symmetry, taking the form of a spheroid when the crystal possesses a morphological axis, and becoming a sphere when the symmetry is that of the Cubic system.

12. *Thermal dilatation.*

The nature of the dilatational changes resulting from variation of temperature in a crystal will be considered in a future chapter in Part II, for they have an important bearing on the character of crystal structure.

In general the amount of dilatation for a given change of temperature varies with the direction in the crystal; indeed, expansion in one direction is sometimes simultaneous with contraction in other directions. The mutual inclinations of the faces of the crystal are thus in general dependent on the temperature: but even for the largest attainable variations of the latter, the changes of angle actually observed have never exceeded a few minutes of arc.

If we conceive of a crystal-mass worked into a form which at a particular temperature is a sphere, it may be asserted that at a second temperature this will either remain a sphere, or will undergo a deformation whereby it becomes a spheroid or an ellipsoid; and it will have the one or the other form according to the symmetry of the system; planes of morphological symmetry being in every case symmetry-planes to the dilatational changes wrought in the crystal by changes of temperature.

13. *Electricity.*

While a crystal of tourmaline is being heated it becomes electrified, and oppositely so at parts of the crystal situated at opposite ends of the morphological axis (*pyro-electricity*): during cooling the polarity is reversed. This may be conveniently shown, as was suggested by Kundt, by dusting the crystal during change of temperature with a mixture of powdered red lead and sulphur which become electrified during the process by mutual friction. The positively electrified particles of red lead are attracted to the negatively electrified end of the crystal and the negatively electrified particles of yellow sulphur to the opposite end; the distribution of the electrification is thus indicated by the distribution of colour. A crystal which exhibits this character is said to be *pyro-electric.*

Compression of a crystal of tourmaline along its morphological axis also produces electrification (*piezo-electricity*); the distribution of the electrification is the same as that caused by the particular change of temperature, namely cooling, which, like compression, produces a contraction of the crystal along the axis. Conversely, it has been shown that electric induction causes expansion or contraction of a crystal of tourmaline along its axis.

There are many other substances in which opposite electrifications are developed at opposite ends of the crystal by change of temperature or pressure; others, for example quartz, which possess more than one axis of symmetry, may exhibit opposite electrifications at the extremities of more than one such axis.

In all these respects the electric properties of the crystal conform to its morphological symmetry, since in the above cases the axis of symmetry along which electric polarity is manifested is also dissimilar at its two ends as regards the disposition of the faces and edges of the crystal.

Magnetic induction. Unmagnetised bodies if brought near a magnetic pole are either attracted or repelled by it, and are said to be magnetised by induction; being described in the former case as paramagnetic and as diamagnetic in the latter case. When the body is a crystal, whether it be of para- or of dia-magnetic substance, the intensity of the induced magnetism varies, in general, with the position of the crystal relatively to the lines of magnetic force: and if the magnetic field be uniform the induced magnetism has the same character. In the general case, for a given set of physical conditions, there are only three positions of the crystal such that the direction of the induced magnetisation is coincident with that of the inducing magnetic force. These directions are mutually perpendicular; and the directions in the crystal which in these several positions coincide with that of the inducing magnetic force are called *axes of magnetic induction*; for one of them the intensity of the induced magnetism is a maximum, for another a minimum. Their positions in the crystal and the corresponding intensities of induced magnetisation accord with the symmetrical requirements of the crystal. The properties of the crystal as regards magnetic induction under the given conditions may be geometrically represented by means of an ellipsoid known as the *ellipsoid of magnetic induction*; for a uniform magnetic field of a different intensity, the ellipsoid may be different both in form and position in the crystal, but the variation will always be such as to conform to the morphological symmetry.

14. *Summary.*

On comparing the results obtained on the one hand by geometrical investigation, and on the other by experimental observations belonging to the domain of physics, of which latter an outline has been here given, one cardinal fact will hereafter be prominently brought out in regard to the physical and morphological characters of a crystal. It will be seen, namely, that these characters are not localised in particular portions of the crystal, but are attributes belonging to different *directions* in it, the properties of the crystal in respect to any given direction being the properties exhibited by every part of the crystal in respect to any line parallel to that direction. In a word, the crystal though æolotropic is a *homogeneous* body.

It will be found necessary for geometrical purposes to assume lines and points fixed, not merely in direction but in position, within and without the crystal, with a view to the comparison of these its diverse characters in different directions, and in order to establish the laws of crystal symmetry; but it cannot be too clearly understood at the outset that these form but the auxiliary scaffolding by the aid of which the ideal structure of crystallography as a science is reared, and that in Nature they have no reality or significance.

CHAPTER II.

MODES OF EXPRESSING AND REPRESENTING THE RELATIONS BETWEEN THE PLANES OF A SYSTEM.

SECTION I.—**Elementary considerations as to the methods of estimating the mutual inclinations of the Planes of a System.**

15. *A System of Axes.* The faces of a crystal have been described in Article **4** as forming a system of planes whereof the mutual inclinations or relative directions, but not the respective magnitudes, nor therefore their relative distances from any arbitrarily chosen point within the system, have to be investigated. The fundamental principle of the different methods by which geometry enables us to determine and compare the directions of any number of such planes consists in the employment of certain lines or axes fixed in their directions as regards the system of planes and intersecting in a point within the system, termed the origin.

Any three or more, but usually only three such axes, which of course must not all lie in the same plane, but in the general case are otherwise arbitrarily taken as regards their directions, are supposed to exist within the system of planes; and to these axes the planes are by various geometrical expedients referred. Where, as will be the case throughout this treatise, the axes are three in number, and intersect in a point, they are designated severally by the letters X, Y, Z, magnitudes measured from the point of intersection or *origin* O along any axis being deemed positive or negative according as they are reckoned in one arbitrarily chosen direction or its opposite. Each pair of axes, then, lies in a separate plane; and these planes, YZ, ZX, XY, divide the space round the origin into eight hollow quoins or *octants*, which may be distinguished by the signs of the axes which contain them. To each octant there will correspond three *adjacent* octants, which have each one axial plane in common with the original octant; three *attingent* octants, in contact with it only along an axis; and one *opposite* octant, meeting it only at the origin. Adjacent octants differ in only one of their signs, attingent octants differ in two, and opposite octants in all their signs.

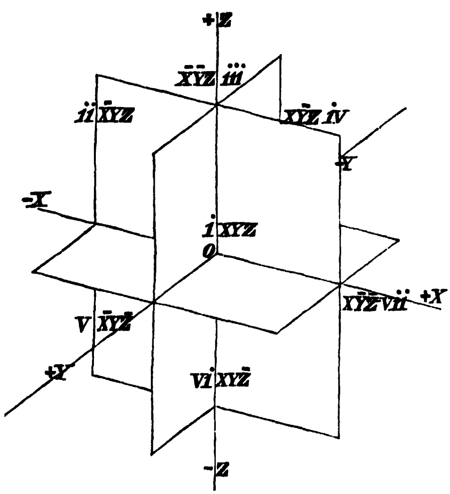


Fig. 2.

It is often convenient to denote the several octants by numbers, as indicated in Fig. 2. Those adjacent and opposite to the first octant ($+X$, $+Y$, $+Z$) are represented by even numbers; octants attingent to it by odd numbers.

The eighth octant, $\bar{X}\,\bar{Y}\,\bar{Z}$, is not lettered in the figure.

The angles at which the axes are inclined on each other will be designated—

the angle YOZ or YZ as ξ,
,, ZOX or ZX as η,
,, XOY or XY as ζ.

Each plane of the system must, if produced, intersect one and may intersect two or three of the axes. The length along any axis intercepted between the origin and the point in which a plane of the system is met by that axis is the *intercept* of the plane on that axis.

If several planes be supposed parallel to each other, their intercepts on the several axes will only differ by a common factor: where this factor is negative, the planes lie in opposite octants or on opposite sides of the origin, being represented by opposite signs.

Where two or more planes are not parallel to each other, they must differ either in the relative magnitudes of their intercepts on one or more of the axes, or in the signs of these, or in both of these respects.

16. *Normals to Planes.* If a perpendicular from the origin be drawn to any plane of the system which must be supposed to be extended if necessary for the perpendicular to meet it, then the *direction in space* of this plane and of any planes parallel to it is known when the direction of this perpendicular is determined in reference to the axes. A perpendicular so drawn through the origin to a plane is called the *centro-normal,* or briefly the *normal,* of that plane.

In this treatise the relative positions of planes will be represented by expressions denoting the relative positions of their normals, and normals will therefore be supposed to be drawn to all the planes of the system.

We proceed to discuss the mode in which the relative positions of planes can be thus simply represented.

17. If a plane P cut the axes X, Y, Z in the points A, B, C, and for simplicity be supposed to lie in the first octant; i.e. to intersect with all three axes belonging to that octant; and if the intercepts of the plane be $OA = a$, $OB = b$, $OC = c$, and if OP (Fig. 3) be the normal of the plane ABC; then

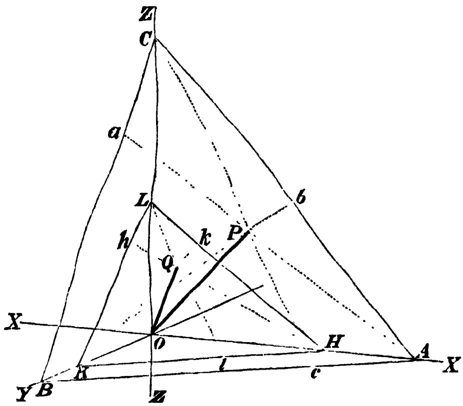
$$\cos PX, \text{ i.e. } \cos POA = \frac{OP}{OA};$$

$$\cos PY = \frac{OP}{OB}, \quad \text{and} \quad \cos PZ = \frac{OP}{OC}.$$

Whence $OP = a \cos PX = b \cos PY = c \cos PZ$, expressions which give the direction of OP and of the plane to which it is the normal, whatever be the distance of that plane from the origin:

this direction being thus represented by the *direction cosines* of the normal *OP* expressed in terms involving the intercepts of the plane *P*.

Let now a second plane *Q* inclined to the first plane and lying also for convenience in the first octant intersect the axes *X*, *Y*, *Z* in the points *H*, *K*, *L* respectively; then the intercepts *OH*, *OK*, *OL* of this plane may be expressed by values involving those of



the intercepts of the plane *P*; as for instance by taking factors *h*, *k*, and *l*, such that

[illegible] h [illegible] h, [illegible] k [illegible] k, [illegible] l [illegible] l [illegible]

And, if *OQ* be the normal of the plane *Q*,

$OQ = OH \cos QX,$

[illegible] $\frac{c}{l} \cos QZ, \ldots \ldots \ldots \ldots \ldots$ (A)

Similarly, $\frac{a}{p} \cos RX = \frac{b}{q} \cos RY = \frac{c}{r} \cos RZ$

would indicate the direction of a third plane *R* of the system, the intercepts of which would be in the ratios $\frac{a}{p} : \frac{b}{q} : \frac{c}{r}$ on the several axes taken in the order *X*, *Y*, *Z*.

18. The ratios $a : b : c$ of the intercepts of some one plane chosen as a standard or parametral plane are termed the *parametral ratios* or *parameters* of the system as referred to the axes X, Y, Z, and these ratios are evidently two in number.

The literal symbols hkl, pqr, &c., or any numbers in the ratios of hkl, &c., and generally the simplest numbers which represent these ratios, are termed the *indices* of the planes Q, R, &c.

The indices of a plane placed between brackets, e.g. (hkl), (pqr), (321), &c., &c., form a compact symbol, which is *the symbol of the plane.*

The parametral plane P evidently has for its symbol (111), since its intercepts are $\frac{a}{1} : \frac{b}{1} : \frac{c}{1}$.

N. B. The brackets are in practice frequently omitted where the use of the unbracketed symbol may involve no ambiguity.

In the cases so far considered the planes in question were supposed to lie in the positive octant, i. e. to intersect, either actually or if extended, the three positive axes. If however a plane will intersect two only of the axes, it cannot but be parallel to the third. Its intercept then on this last axis will be indefinitely great, as its point of intersection with it is infinitely remote. The index for the particular axis will thus become zero; since, for instance, an intercept $\frac{AO}{0}$, i.e. $\frac{a}{0}$, is infinitely great. Hence $(hk0)$, $(pq0)$, (110) are the symbols of planes intersecting with the axes X and Y but parallel to the axis Z; $(0kl)$, $(0qr)$, (011), (023) represent planes intersecting the axes Y and Z but parallel to X; and $(h0l)$, (101) are symbols of planes intersecting X and Z and parallel to Y. Furthermore, if a plane intersect only one axis, it must have one of the symbols (100), (010), or (001) if it intersect the axis on the positive side of the origin; any other value than unity for the non-evanescent index being without significance, since it is the direction, not the origin-distance, of the plane that is to be expressed by the symbol.

Where a plane lies otherwise than in the first or positive octant its position is at once defined by the signs of its several indices.

An index is taken as positive unless a negative sign is placed over it; and the signs of the indices so inscribed will denote the directions on the axes in which the intercepts of the plane are taken. Thus ($21\bar{3}$) would be the symbol of a plane lying in the sixth octant of Fig. 2.

Eight planes might therefore coexist having the same values in their respective indices but differing in their signs; one plane lying in each octant. Where, in a rectangular axial system, the intercepts of such a group of planes have the same magnitude on each axis, the solid is the regular octahedron of Geometry; and other solids more or less resembling the octahedron in form result from the coexistence of planes intersecting with the axes in each octant with the same intercepts on corresponding axes, but with intercepts that differ on axes of different denomination. A plane of such a group, whatever be the axial system, has been designated as an *Octaïd,* or, more correctly, an *Octahedrid* plane; it must have three indices that are not zero in its symbol.

For analogous reasons, planes parallel to a single axis but meeting the other two axes, and having therefore one zero in their symbol, will be termed *Prismatoid* planes; and the term *Pinakoid* planes (from πίναξ, 'a slab') will be given to such planes as, intersecting with only one of the axes, have two zeros in their symbol; namely the planes (100) and ($\bar{1}00$), (010) and ($0\bar{1}0$), (001) and ($00\bar{1}$).

Each prismatoid plane may be said to lie in or belong to two adjacent octants; its zero index refers to that axis for which the sign is different in the two octants. A pinakoid plane belongs equally to four octants; it is parallel to an axial plane, and intersects with the axis which does not lie in that plane.

In the event of a polyhedron referred to a certain system of axes not presenting octahedrid planes, parametral ratios may be determined by the ratios of the intercepts of two prismatoid planes which intersect with different pairs of axes; their symbols belonging to two of the three groups, (011) or ($0\bar{1}1$), (101) or ($\bar{1}01$), (110) or ($\bar{1}10$).

It is convenient to assume that the two planes thus chosen cut that axis with which they both intersect in a common point. The intercept on that axis thus becomes the same for both planes, and their other intercepts are directly comparable.

Let the plane LH parallel to the axis of Y cut the axes of X and Z with intercepts $OH = \frac{a}{h}$, $OL = \frac{c}{l}$, and let the plane BC parallel to the axis of X cut those of Y and Z at distances $OB = b$ and $OC = c$. Then LH and BC being taken for parametral planes, let a plane parallel to the plane LH cut the axial plane XZ in CA. The intercepts of the new plane will be OC on Z and OA on X; and their ratio $\frac{OA}{OC} = \frac{OH}{OL}$ that of the intercepts of the original plane LH; and the two planes AC, BC have their intercepts on the Z axis in common, and their X and Y intercepts are comparable.

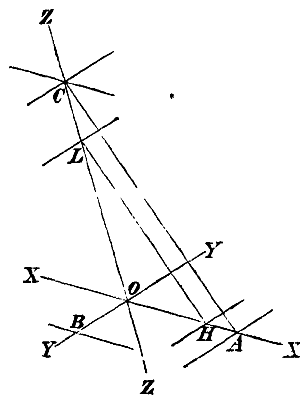


Fig. 4.

The intercepts of the two planes are, on axes X and Z,

$$OA,\ OC;$$

and on axes Y and Z, $\quad OB,\ OC$:

whence the parameters are

$$OA : OB : OC = a : b : c;$$

where $$a = OA = OC \cdot \frac{OH}{OL} = \frac{h}{l} \cdot c.$$

19. The angle contained between the normals to two planes is the supplement of the internal angle included between the planes themselves, i.e. that angle within which the normals meet.

If OM, ON be the normals to two planes ME and NE, and the plane containing these normals is the plane of the figure; then will this plane be perpendicular at E to the edge formed by the two planes; and

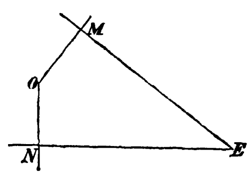


Fig. 5.

ME, *NE* are the edges formed by the normal plane with the two planes, and the angle *NEM* measures the inclination of the two planes; *MENO* therefore is a quadrilateral figure, whereof *OME* and *ONE* are right angles. Therefore $MEN + MON = 2$ right angles, and each is the supplement of the other.

Section II.—Expression for the direction of the Edge formed by two Planes.

20. Let *HKL*, *PQR* be two planes assumed for simplicity to lie in the same octant and having for their symbols (*hkl*), (*pqr*).

The most general case is that in which the two planes have no intercept in common on either axis.

Let the two planes lie in the octant *XYZ* and intersect with the axes in *HKL*, *PQR* respectively. In this case there will be common points of intersection of the two planes with each other

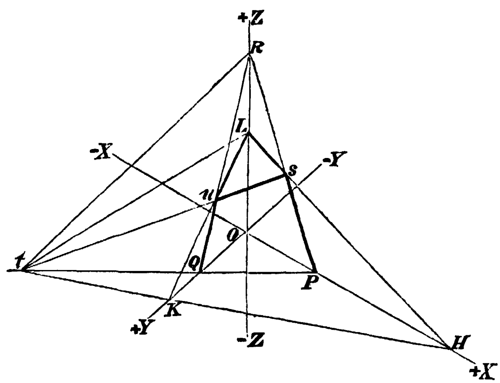


Fig. 6.

and with each of the axial planes: let *s* be the common point in the axial plane *XZ*, and *u* be that in the plane *YZ*. With the third axial plane they will intersect also in a common point *t*, in an adjacent octant. Then the points *t*, *u*, *s* will lie in one and the same straight line, which will be the line of intersection or edge of the planes *HKL*, *PQR*.

The direction of this line will be parallel to that of all edges formed by planes parallel to the original planes. Let then a plane parallel to PQR be drawn through K intersecting with the axes X and Z in E and G. The ratios of its intercepts will thereby remain unaltered, and that on Y is common to the two planes EKG, HKL.

By similar triangles, which need not be drawn,

$$OE = OP \cdot \frac{OK}{OQ} \quad \text{and} \quad OG = OR \cdot \frac{OK}{OQ}.$$

The planes will now have only two points of intersection with the axial planes; namely, a point K on the axis Y common to the two planes XY and YZ, and a point M situate in the plane ZX.

A line KM is therefore the edge of the planes HKL and EKG. Through M draw MD parallel to the axis Z and meeting the axis X in D. And on the axis Z take $OM' = DM$: finally, through the origin O draw ON parallel and equal to KM.

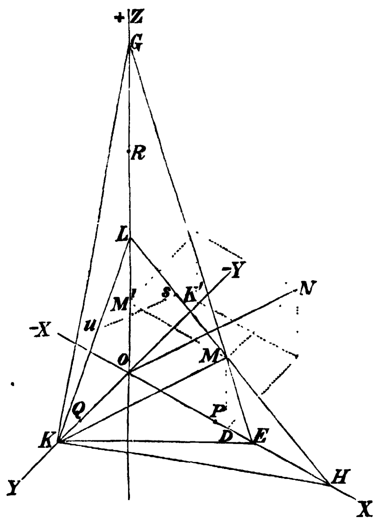


Fig. 7.

The direction of the line ON passing through the origin is therefore that of the edge formed by the original planes or by any planes parallel to them; and the coordinates of N are

$$OD, \ -OK, \text{ and } OM' = DM;$$

ON being the diagonal of a parallelepiped with OK', OD, DM for its sides.

In order to find the value of these coordinates, we have in the similar triangles HDM and HOL,

$$OM' = DM = \frac{OL}{OH} \cdot (OH - OD), \quad \text{(i)}$$

and $$OM' = DM = \frac{OG}{OE} \cdot (OE - OD). \quad \text{(ii)}$$

Dividing (i) by OL and (ii) by OG, and subtracting, we obtain

$$OM' \left(\frac{1}{OG} - \frac{1}{OL}\right) = OD \left(\frac{1}{OH} - \frac{1}{OE}\right).$$

Therefore $$\frac{OM'}{\frac{1}{OH} - \frac{1}{OE}} = \frac{OD}{\frac{1}{OG} - \frac{1}{OL}},$$

$$\frac{OM'}{\frac{1}{OH} - \frac{OQ}{OK} \cdot \frac{1}{OP}} = \frac{OD}{\frac{OQ}{OK} \cdot \frac{1}{OR} - \frac{1}{OL}},$$

and $$\frac{OM'}{\frac{1}{OH.OQ} - \frac{1}{OK.OP}} = \frac{OD}{\frac{1}{OK.OR} - \frac{1}{OL.OQ}};$$

and, by the symmetry of the problem,

$$= \frac{OK}{\frac{1}{OL.OP} - \frac{1}{OH.OR}}.$$

Here $$\frac{1}{OH} = \frac{h}{a}, \text{ \&c.};$$

$$\frac{1}{OP} = \frac{p}{a}, \text{ \&c.}$$

Therefore $$\frac{1}{OH.OQ} - \frac{1}{OK.OP} = \frac{hq - kp}{ab},$$

&c., &c., = &c.

Substituting these values in the ratios and dividing by the common factor abc, we obtain

$$\frac{OD}{a(kr - lq)} = \frac{OK}{b(lp - hr)} = \frac{OM'}{c(hq - kp)};$$

which expression gives the ratios of the coordinates of any point in a line through the origin parallel to the edge of the planes hkl and pqr, in terms of the indices of the two planes.

These ratios

$$a(kr - lq), \quad b(lp - hr), \quad c(hq - kp)$$

may be written briefly as

$$a\mathbf{u},\ b\mathbf{v},\ c\mathbf{w},$$

if
$$\mathbf{u} = kr - lq,$$
$$\mathbf{v} = lp - hr,$$
$$\mathbf{w} = hq - kp.$$

DEF. A plane parallel to any plane of a system but passing through the origin will be termed an *origin-plane*, and any line through the origin parallel to an edge of two planes of the system will be termed an *origin-edge*, and hereafter a *zone-line* or *zone-axis*; and in speaking of the planes of a plane-system the term 'face' will be more particularly applied to them when considered as the faces that bound a polyhedron.

SECTION III.—**A Crystalloid System. Principle of Rationality of Indices.**

21. In considering the general character of the expressions for the relations of the planes in a system as referred to axes of coordinates, the indices of a plane were not limited to any special kind of values, integral or fractional, rational or irrational, and the axial system might be arbitrarily chosen. If however the nature of the system of planes be limited by the condition that for a plane to belong to the system its indices must be rational, that is to say, capable of being represented by integral numbers, or one or two of them by zero, it will be obvious that some limitation must also be imposed on the selection of the axes to which the planes are referred.

22. It will be shown in Art. **75** that the axes must, in the case supposed, be themselves possible zone-lines of the system. A special case, confined to a single zone, will however serve here to illustrate this important principle.

Let us suppose that OX, OZ are axes arbitrarily taken to which two planes are referred, that may for convenience be prismatoid planes, cutting only these two axes, and parallel to the third. Let the lines AC, AL be the intersections of these planes with the axial plane XOZ, which is that of the figure; let $OA = a$, $OC = c$ be the intercepts on X and Z of one of the planes; and let a and c

be taken as the parameters: and of the other plane let OA and $OL = \frac{h}{l}c$ be the intercepts. Then $\frac{l}{h}$ is the index on Z of the plane AL, and if the ratio of OC to OL be capable of being expressed by whole numbers, $\frac{h}{l}$ is rational.

If we arbitrarily choose another axis OZ' elsewhere in the plane XOZ, and take OC' and OL' as the intercepts on that axis of the two planes; it is evident that the ratio of these will vary with the direction of OZ' and is not necessarily rational. Wherefore the axial system may not be arbitrarily chosen in the case of a system of planes of which the indices are rational.

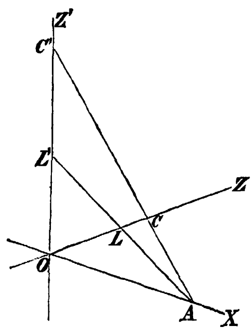


Fig. 8.

If, again, we change the axial system in such a manner that C becomes the origin and CA, CO the new axes; and if AO and AL be the lines of intersection of two planes with the axial plane COA; then CA and CO, the intercepts of the plane OX, may be taken as parameters for the system; and the intercepts of the other plane on the new axes are

$$CA = a', \quad \text{and} \quad CL = \frac{l-h}{l} . c,$$

so that its indices in respect to these axes are $l-h$ and l, and are obviously rational.

Hence, in the special case supposed, any planes, the origin-edges of which form axes to which it is possible to refer a system of planes with rational indices, must themselves fulfil the condition required for all planes of the system.

23. By the term a *crystalloid system of planes* we shall understand an assemblage of planes parallel to the faces of a polyhedron finite in number and presenting such mutual inclinations that if they be referred to an axial system formed by three different origin-edges parallel to edges of the system, and with parametral ratios determined by the intercepts of any plane of the system, it shall be a necessary condition in order for a plane to belong to the system that its indices be rational.

An axial system with axes and parameters so chosen will be termed a crystallographic *axial system.*

The *elements* of such an axial system are five in number; viz. the three axial angles ξ, η, and ζ, and the two parametral ratios $\frac{a}{b}$ and $\frac{c}{b}$.

Section IV.—On the Sphere of Projection, and the principles of its Stereographic Representation.

24. A convenient means of representing and comparing the relations of a system of planes forming a polyhedron is afforded by treating their normals as radii of a sphere. A sphere of arbitrary radius termed the *Sphere of Projection* is supposed to be described round any point within the system taken as its centre and as the origin of a system of centro-normals, or briefly of normals, perpendicular to the faces of the polyhedron. The point in which any such normal meets the sphere is termed *the pole* of the plane to which the particular normal is perpendicular. A pole may therefore also be defined as the point of contact of the sphere and a tangent-plane parallel to a plane of the system on the same side of the origin with the plane.

Where the system of planes is also referred to an axial system, with the centre of the sphere for its origin, the point in which an axis penetrates the surface of the sphere will be called its *axial point.*

It is evident that if the poles of planes be connected by great circles, their distances and relative positions, and therefore the inclinations and relations of the planes themselves, may be measured and investigated by the methods of spherical trigonometry: for in the plane of the great circle, thus connecting the poles of any two planes, the normals of these planes must lie; and the arc between their poles as measured on a great circle is that subtended by the angle contained by the normals, and is therefore the supplement of the angle of inclination of the planes themselves.

Another great advantage of this method of representing the positions of all the planes of a system by points distributed on a sphere is, that by a simple process of laying down such points in a projection of the sphere upon a plane we do not need the somewhat elaborate process of drawing a crystal by projecting its edges, in order to give a complete *conspectus* of all that crystallography seeks to represent; that is to say, of the general symmetry of the polyhedron and the distribution and relative inclinations of all its faces. Fig. 9 represents in orthographic projection the faces and the poles of the cubo-octahedron, i.e. of the two regular solids the cube and octahedron united into a single figure in which the faces of the one figure truncate the solid angles (or quoins) of the other figure. Fig. 10 represents the poles of the same faces and the great circles passing through those poles in what is termed the stereographic projection. In the former case, which is that usually employed for the projection of the edges of crystals in crystallographic figures, the eye is at an indefinitely great distance (the crystal being seen as if

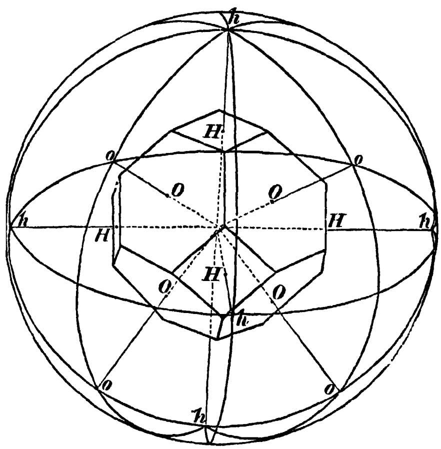


Fig. 9.

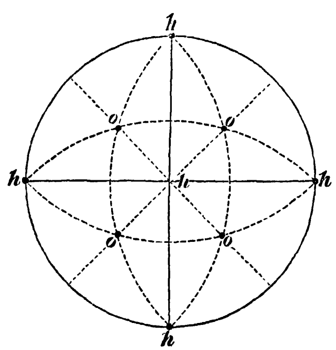


Fig. 10.

from a considerable distance through a telescope); the visual rays being parallel, and as a consequence all lines parallel in the object remaining parallel in the figure that represents it.

The representation of the faces of a polyhedron by the stereographic projection of its poles has the great advantage over a drawing that it does not aim at representing the relative magnitudes of the faces; while also by this method the poles of any number of planes may be laid down at their correct angular distances with speed and accuracy, whereas the number of faces admitting of representation in a drawing is necessarily limited. Furthermore, crystallographic problems may often be solved and calculation greatly simplified by its means.

25. In the Stereographic Projection, which is the simplest form of projection of a sphere with the above view (Fig. 10), the eye is supposed to be at a point of the sphere's surface and to see such poles or great circles as are distributed on the hemisphere opposite to it projected (as on a screen) upon a plane passing through the centre of the sphere and cutting the sphere in the great circle at the pole of which the eye is situate. The *plane of projection* thus bounded by a great circle of the sphere is represented by the plane of the paper on which the circle is drawn, which latter will be termed the *circle of projection* or *primitive* circle. The advantage of this over other forms of projection of the sphere is that any great or small circles of the sphere on the hemisphere opposite to the eye are by it projected either as straight lines or as circular and not as elliptical arcs, and thus, by means of a protractor and compasses, all the great circles can be laid down on which the poles of planes are distributed.

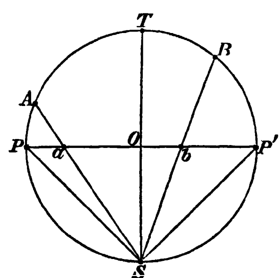


Fig. 11.

That this is the case follows at once from the properties of the oblique cone; but it may be otherwise proved thus: suppose S, Fig. 11, to be the point of sight on a sphere of which O is the centre and OS therefore the radius. Let PP'

represent the plane of projection. The circle of projection will then be a great circle of which *S*, the position of the eye, is one of the poles. The apparent position on the plane of projection of any point *A* on the sphere as seen by the eye at *S* will evidently be that point in which a straight line drawn from *S* to *A* meets the plane *PP′*. Thus the centre *O* of the circle of projection will be the point at which *T*, the pole of that circle opposite to *S*, will be projected. So the arc *PTP′* on the opposite hemisphere will be seen from *S* as a straight line coincident with *PP′*, and *any great circle passing through T will be projected as a diameter of the circle of projection;* such great circles or parts of them are therefore projected *as straight lines.* Thus a portion *AB* of the arc *PTP′* will be seen as *ab*, a portion of *PP′* limited by the points *a* and *b*, which are the projections of the points *A* and *B*. It will further be seen that while all points on the hemisphere opposite to the eye will be projected in points within the circle of projection, the projections of points lying on the same hemisphere with *S* will lie beyond the circumference of that limiting circle.

26. We proceed to establish further that *an arc of any circle on the sphere not passing through the point of sight is projected as a circular arc.*

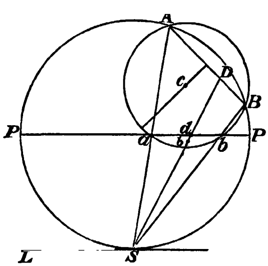


Fig. 12.

Let *ADB* be a circular section of the sphere, for convenience a section by a small circle; and let Fig. 12 represent a section through *S* and also through any two points *A* and *B* on this small circle. Let *PP′* be the trace of the plane of projection; *a*, *b* the points in which lines *SA* and *SB* cut *PP′*; they are therefore the projections of *A* and *B*.

Draw a tangent *LS*. Then

Therefore the four points *A*, *B*, *b*, *a* lie on a circle.

Now *A* and *B* in the section *ABS* being any points on the small circle, the locus of *ab* must be the section by the plane of projection of any surface containing all such circles as have been proved to contain any two points and their projections. That there is such a surface and that it is a sphere, and that consequently the projection *adb* of the circle *ADB* is itself a circle, will be evident from the following considerations.

Erect a perpendicular to the plane of the circle *ADB* from its centre; then every point in the *circle* is equidistant from any point in this perpendicular; and a sphere described round such a point *C* taken at equal distance from *B* and *b* will carry on its surface all the four points *A*, *B*, *D*, *b*.

And *d*, the projection of any other point *D* on the original circle, will also lie on the surface of this sphere. For if it do not, it will lie somewhere else in the line *SD*, as at a point δ.

But, as in the case of the projections *a*, *b* of the points *A* and *B*, we have proved $baS = ABS$; whence, by comparison of the triangles baS and *ABS*, we obtain the reciprocal proportion

$$Sa \,.\, SA = Sb \,.\, SB.$$

Similarly, we should obtain for the points *A*, *D*, *a*, δ

$$Sa \,.\, SA = S\delta \,.\, SD.$$

But also, as *SD* must cut the described sphere somewhere, as in a point *d*,

$$Sa \,.\, SA = Sd \,.\, SD$$

is also true, which can only be possible if *d* and δ are one and the same point. And this point is therefore at once the projection of *D* and situate on the surface of the described sphere. The locus of *ab* is therefore, as asserted, the section by the plane of projection of the described sphere; that is to say, is a circle.

27. The proof applies equally to the section of the sphere by a great circle: it may also be thus illustrated.

Let *AB*, Fig. 13, be a great circle intersecting with the plane of projection, of which the trace is *PP′*, in a line perpendicular at *O* to the plane of the figure; which plane passes in this case through the centre of the sphere as well as through *S*. From *S* draw a perpendicular to the plane *AB* meeting the plane of projection in *C*, and draw *SA* through *a* the projection of *A*, and *Sb* through

B to meet the plane of projection produced in b which will be the projection of B: then these lines will necessarily lie in the plane of the figure.

As in the previous case,

$$Sab = SBA,$$

whence $$Aab = bBA,$$

and the four points A, a, B, b lie on a circle; and it will follow, as in the previous case, that every point of the great circle AB will lie in the circular section in which the plane of projection will cut a sphere that shall be so described as to carry on its surface the four points A, a,

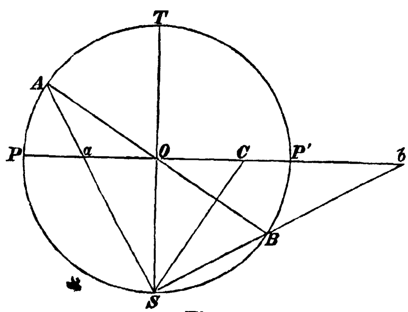


Fig. 13.

Cor. 1. The centre of the circle in which the original great circle is projected will be C, the point in which the perpendicular from S on the plane of AB meets the plane of projection; for, since

$$Ca = CS = Cb;$$

and since $aSbT$ lie on the same circle of which C is the centre, two points in which the circle of projection is cut by the plane through S and O perpendicular to the plane of projection will be the points of intersection of the circle of projection and the circular arc in which that circle, whose trace is A and B, is projected.

Cor. 2. Since ASb is a right angle, b may be at once obtained by drawing Sb perpendicular to AS and meeting the continuation of PP' in b.

For the practical applications of the Stereographic Projection, of which continual use will be made in this treatise, the following propositions will be found necessary.

28. Problem I.—*To determine the magnitude of the arc of a great circle which is represented by the projection of that arc.*

Let V and C be two great circles, Fig. 14, which intersect in the extremities of the diameter AA' of the sphere of projection, and let their poles be P and S.

Now, every circle on the sphere, whether great or small, passing through the points P and S—e.g. the great circle U or the small circle U'—will manifestly cut in a similar manner the two circles V and C; and moreover, any two circles on the sphere thus passing through the poles of two great circles on it must intercept on these two great circles arcs of the same magnitude; whence it is evident that the arcs intercepted on the great circles V and C by the two circles U and U', or those intercepted by U or U', and by a third plane, say U'', passing through A, must be equal; so that

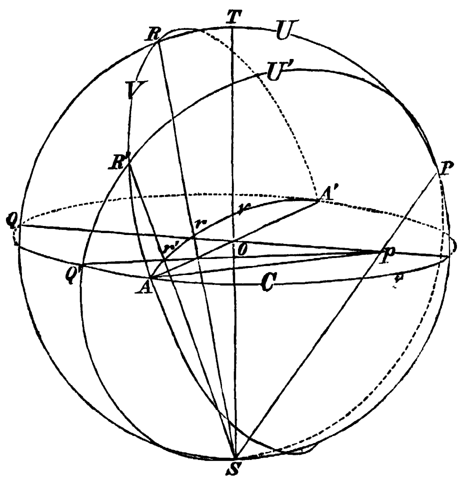


Fig. 14.

if Q, R be the points of intersection of a circle U with the great circles C and V, and Q', R' be those in which these great circles are intersected by the circle U'.

If now the plane of the great circle C be taken for the plane of projection, S being the position of the eye, every circle U (U', U'', &c.), whether great or small, will be projected as a straight line passing through p the projection of P. Indeed, since every such circle U lies in a plane passing through P, S, and a point R (or R') on the great circle V, straight lines drawn from S to points on the circumference of the circle U can only intersect with C in points upon a straight line passing through p, which is at once the projection of the circle U upon, and the intersection of its plane with, the plane C.

Consequently, while the great circle V will be projected in the circular arc ArA', every one of the circles U, U', &c. passing through the points R, R', &c. will be projected in a straight line pQ, (pQ',) &c., which will further cut ArA', the projection of the circle V, in a point r, (r', &c.) which will be the projection of the point R (or of R', &c.); r (or r', &c.) being in fact the point in which the planes of three great circles V, U (or U', &c.) and C intersect with each other.

It is thus that straight lines drawn from p, the projection of the pole of the great circle V, through r and r' to the circumference of the circle of projection (which straight lines are the projections of two circles U, only one of which can be a great circle,) come to intercept on this primitive circle an arc QQ'; and this arc $QQ' = RR'$, the arc of which rr' is the projection, and QQ' therefore measures the arc represented by rr'.

Whence follows the rule:—To determine the value of any arc of a great circle as represented in projection, find the projection of the pole of this great circle (which may be done by the succeeding problem); then—

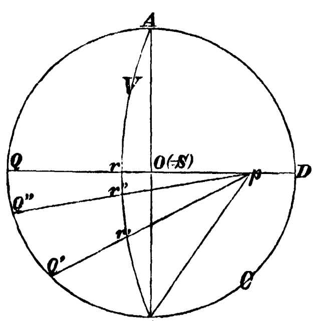


Fig. 15.

1. The value of an arc of which the projection is given may be measured by determining on the circle of projection the arc contained between two straight lines drawn from the projection of the pole through the extremities of the projected arc, Fig. 15; and,

2. An arc of a great circle of given magnitude is represented in the projection of that circle by the portion of the projection

determined by the intersection with it of two straight lines drawn from the projected pole to the circle of projection and intercepting on that circle an arc of the required magnitude.

If the point in which one of the straight lines intersects with the projection of the great circle be fixed (i. e. is the projection of a given point on the great circle), then the other extremity of the projection of the required arc will alone have to be determined by the latter of the two methods.

Cor. If the plane of a great circle, of which the projection is given, be perpendicular to the plane of projection, the great circle passes through S, the point of sight, and its projection becomes (by Art. **26**) a diameter of the circle of projection C; and the pole of the great circle will lie on the circle C, at the point of intersection with it of a diameter perpendicular to that in which the great circle is projected. And this pole and its projection manifestly coincide.

Hence arcs intercepted on the circle of projection by straight lines drawn from this pole through the diametral line in which the great circle is projected, will determine on that diameter the projection of an arc of any required angular magnitude.

Problem II.—*Given the projection of a great circle, to find that of its pole.*

29. Let DD', Fig. 16, be the diameter in which the given projection, and consequently also the original great circle, intersect with the circle of projection.

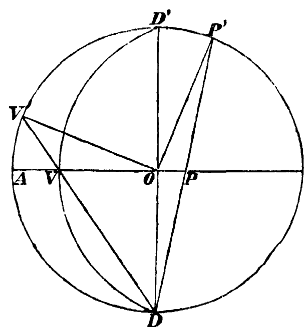


Fig. 16.

Since the pole required will necessarily lie on the diameter perpendicular to that in which the given projected circle intersects with the circle of projection, and must lie at a distance equivalent to a quadrant on that diameter from the point in which the given circular projection intersects with it; let V be the point of this intersection, V' that in which a line DV produced meets the circle of projection. Then, by Cor. Prob. I, if $V'P'$ be a quadrant, and a line DP' cut the diameter through V and O in P, VP would be the projection of $V'P'$ on the line VP as seen from D; and since a similar construction would hold good were the great circle $AD'P'$ to be drawn perpendicular to the plane of the figure, so as to be seen as AOP and to pass through S the point of sight, and since the true pole of V will lie on this great circle, the point P will be the projection of that pole.

Problem III.—*To draw the projection of a great circle, in which projection two points are given, that do not both lie on the circle of projection.*

30. Since only one great circle can pass through two points on the sphere not extremities of a diameter, the centre of the sphere and two such points suffice to determine the direction of the plane, of such a great circle. Hence the projections of two points and the centre of the sphere being given on the plane of projection, it should be possible to describe the circle which is the projection of the great circle on which the two points lie. The most convenient way of doing this is to find the projection of the point on the great circle which is the opposite extremity of the diameter on which one of the points lies of which the projection is given.

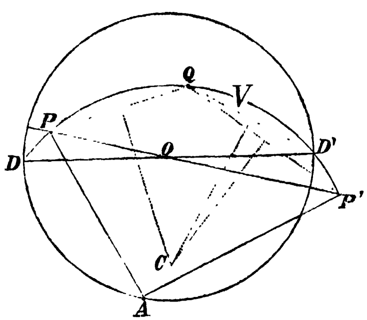


Fig. 17.

Thus, if two points on a great circle V' are projected in the points P and Q, Fig. 17—both of which do not lie on the circle of projection DAD'—and if it be required to draw V, the projection of V', it is necessary to find the projection of a third point in the circle V'; and the point diametrically opposite to either of the points P and Q is convenient for this purpose. If then we would project in P', the point diametrically opposite to that point of which P is the projection (and it is preferable for this purpose to select the more remote from the centre of the two projected points P and Q), we have to draw through the remoter point P a diameter of the circle of projection; and, where P does not lie on the circle of projection, from O, the centre of that circle, a perpendicular on PO is drawn to a point A in the circle of projection. A therefore is the projection of the pole of the great circle which is projected in the diameter through P; and P' will lie on the line PO at a distance equivalent in the projection to an arc π from P. From A draw a perpendicular to AP, meeting the prolongation of PO in P'; then, by article **27**, Cor. 2, P' and P are the projections of diametrically opposite points on the original great circle, since the construction would equally determine P' were the point A revolved round PP' till it became coincident with S the point of sight. Perpendiculars from the points of bisection of the straight lines joining Q, P, and P' will now meet in a point C; and from C as a centre a circle drawn through P and Q will be the required projection V of the original great circle V'. It frequently happens that the points P and P' coincide with the extremities of the diameter DD' in which the two circles V and C intersect.

Otherwise; where the distance from the circle of projection differs appreciably for the two points P and Q, an elegant mode of determining the position of the diameter DD' is the following. Through P and Q, Fig. 18, draw any circle intersecting the circle of projection in points p and q; (any circular disc laid on the figure will determine these points). Let the two straight lines PQ and pq be drawn to meet in M. Then DD', the diameter of the circle of

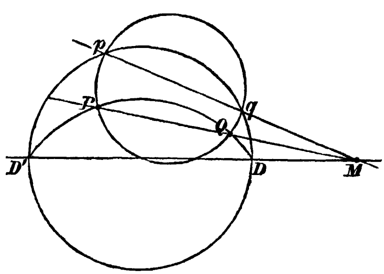


Fig. 18.

projection which would traverse M if continued, is the diameter required.

Through PQ and D describe an arc of a circle intersecting the line MD in a second point Δ. Then

$$MP.MQ = Mp.Mq = MD^2.$$

and Δ and D' are the same point. Hence, DD' is the diameter at the extremities of which the arc V (the projection of the arc V') intersects the circle of projection.

If PQ and pq are parallel they are parallel to DD'.

Cor. In order to draw the projection of a great circle where the projection of its pole is given: draw a diameter of the circle of projection through the point M in which the pole is projected; then, as in Cor. Prob. I, by aid of the lines Pm, Pq, where the arc $mq = \frac{\pi}{2}$, find the point Q on this diameter equivalent to a quadrant's distance from M, the projected pole. A second diameter PP', perpendicular to the first, will meet the circumference in the points of intersection with it of the required great circle; which may now be drawn by the help of Problem III.

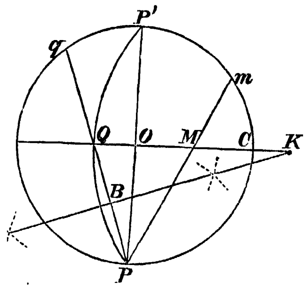


Fig. 19.

N.B. The centre K, from which this circle is to be drawn, will be the point in which a perpendicular on PQ from B, the point of bisection of PQ, will meet the diameter through QM. And the case in which this centre K falls on the circle of projection is that in which OM and OQ represents each an arc of $\frac{\pi}{4}$ when K coincides with C, and

$$CQ = CP = \text{chord } 90 = r.\sqrt{2} = 1.414r,$$

the radius of the required circle being CP. That in this case, where OM and OQ each represent an arc of $\frac{\pi}{4}$, the radius of the circle to be drawn is the chord CP is evident; for then,

and $$CQP = \frac{\pi}{2} - \frac{\pi}{8} = \frac{\pi}{4} + \frac{\pi}{8} = QPC;$$
so that CPQ is isosceles, and $CP = CQ$.

PROBLEM IV.—*To draw the projection of a great circle which intersects with another great circle at a given angle and in a given point; the projections of the latter great circle and of the point being given.*

31. Let a great circle V' intersect a second great circle W' in a point r' at an angle θ. Let V', Fig. 20, be projected in the arc V, and r' in the point r. It is required to draw the projection of the great circle W'. Project the pole of V' in P, and from r draw the line rPp to the circle of projection; on which circle let the points p and q intercept an arc equivalent to θ: draw U, the projection of the circle of which r would be the projected pole. U will therefore pass through P; and let Q be the point in which U intersects with qr.

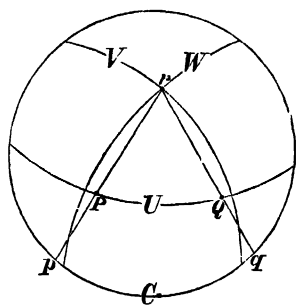


Fig. 20.

Draw the projection W of the great circle W', the pole of which would be projected in Q: it will be seen that the projection W will pass through r; and furthermore that it will be inclined to the projected arc V at the required angle θ. For the distance of the poles of two great circles, as measured on a great circle traversing them on the sphere, is necessarily equal to the angle at which the two circles intersect, and since r is the pole of the projected circle through P and Q, the arc PQ represents the arc on the great circle U measured by pq; that is to say, represents the angular magnitude θ.

Hence the circular arcs V and W are the projections of two great circles V' and W', which intersect in r' at an angle θ.

PROBLEM V.—*Given the projections T and P,* Fig. 21, *of two points on the sphere; to determine the position of a third point A which shall be the projection of a point on the sphere distant by an angular arc ϕ from the point of which T is the projection, and by an arc θ from the point of which P is the projection.*

32. If we suppose the triangle APT to be drawn, and if the angle at $P = \xi$, and that at $T = \zeta$, and the arc $TP = a$; then

$$\cos\xi = \frac{\cos\phi - \cos\theta\cos a}{\sin\theta\sin a},$$

$$\cos\zeta = \frac{\cos\theta - \cos\phi\cos a}{\sin\phi\sin a}.$$

Draw, by Prob. III, Cor., UV, WV, the projections of two great circles of which P and T are respectively the projected poles; and let them intersect in a point V. From P and T draw straight lines PVp, TVt meeting the circumference in p and t, and through a point x at a circumferential distance $\pi - \xi$ from p draw Px, intersecting with the great circle VU in U; and through z at a distance $\pi - \zeta$ from t draw Tz, intersecting with the great circle VW in W.

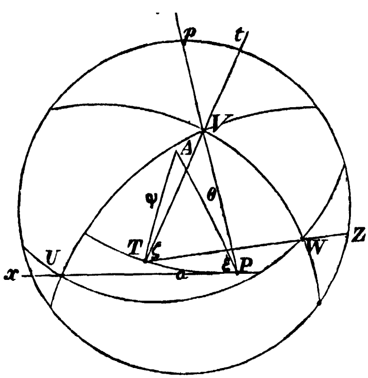


Fig. 21.

Through U and W draw the projection of a great circle (Prob. III), and find A the projection of its pole (Prob. II). Then will A be the point required. For, by construction, APT and UVW are projections of polar triangles. And since in UVW, $UV = \pi - \xi$, $WV = \pi - \zeta$, and the angle at $A = \pi - a$, the angle $U = \pi - \theta$, and angle $W = \pi - \phi$.

Whence in the triangle APT, AT represents an arc ϕ and an arc θ.

33. The manner in which the different varieties of crystals are represented in stereographic projection will be considered in future chapters. It is only necessary here to observe, that the plane of projection is always so selected as to contain one of the most important of the crystallographic planes, and is generally a plane which divides the crystal symmetrically.

In the stereographic projections employed in this volume the representation is such as would result from looking down on a plane of projection on which the poles of the hemisphere nearest the observer had been previously laid down, as seen by an eye at the opposite pole of the circle of projection; that is to say, at the nether pole of that great circle. It is often convenient to be able to represent the poles of both hemispheres within the same great circle of projection. In such case the poles of the upper hemisphere, as seen by an eye at the nether pole, will be represented by black dots; those belonging to the nether hemisphere, as seen by an eye placed at its upper pole, will be represented by small eyelets. Where two poles fall on the same spot the black dot is encircled by the eyelet.

Where literal or numerical symbols are employed to express the character of the faces to which these poles belong, the signs, or other values of these numerals, will usually serve to indicate to which hemisphere they are to be applied.

When the symbols of both sets of poles are introduced, those belonging to the nether poles will be the fainter and smaller in type upon the projection.

SECTION V.—**Expressions for determining the position of a pole on the sphere.**

34. Let X, Y, Z be the axial points, A, B, C the poles of the axial planes YZ, ZX, XY to which a crystalloid system of planes has been referred.

Hence ABC and XYZ, Fig. 22, are polar triangles, and A is 100, B is 010, C is 001. Let P be the pole of a plane the indices of which are $\pm h$, $\pm k$, $\pm l$. If however P lie on one of the great circles passing through two of the poles A, B, C, it is the pole of a face parallel to one of the axes, and one of its indices is zero; that is, the symbol is *okl* for any pole lying between two poles *B* and *C*, *hol* for a pole lying between two poles *C* and *A*, and *hko* when lying between two poles *A* and *B*: and in each case the index in question must change its sign as it passes from one to the other side of the respective great circles.

Whence the *poles* of all faces lying in any given octant will lie within the corresponding one of the eight spherical triangles into which the sphere is divided by the great circles passing through two of the poles $+A$, $+B$, $+C$ and the poles opposite to them $-A$, $-B$, $-C$: and the signs of the indices of the face *P* will correspond with those designating the particular spherical triangles within which its pole lies: e.g. the pole $h\bar{k}\bar{l}$ lies within the spherical triangle $+A-B-C$.

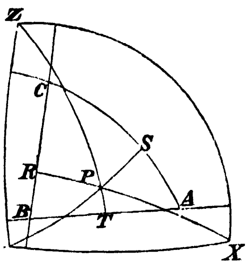


Fig. 22.

35. Let *P* be the pole of a plane of the system lying within the spherical triangle *ABC*. Through *P*, Fig. 22, draw the quadrantal arcs *XR*, *YS*, *ZT*.

Then,

$$\cos PX = \sin PR = \sin CP \sin BCP = \sin BP \sin CBP,$$
$$\cos PY = \sin PS = \sin AP \sin CAP = \sin CP \sin ACP,$$
$$\cos PZ = \sin PT = \sin BP \sin ABP = \sin AP \sin BAP.$$

But the symbol of *P* is *hkl*, and by the fundamental equation **A** in article **17**,

$$\frac{a}{h}\cos PX = \frac{b}{k}\cos PY = \frac{c}{l}\cos PZ$$

Substituting the above values for cos *PX*, cos *PY*, cos *PZ* we obtain six equivalent expressions,

$$\frac{a}{h}\sin CP \sin BCP = \frac{b}{k}\sin CP \sin ACP,$$

$$\frac{b}{k}\sin AP \sin CAP = \frac{c}{l}\sin AP \sin BAP,$$

$$\frac{c}{l}\sin BP \sin ABP = \frac{a}{h}\sin BP \sin CBP;$$

whence are obtained the equations

$$\frac{h}{k}=\frac{a}{b}\,\frac{\sin BCP}{\sin ACP},\quad \frac{k}{l}=\frac{b}{c}\,\frac{\sin CAP}{\sin BAP},\quad \frac{l}{h}=\frac{c}{a}\,\frac{\sin ABP}{\sin CBP},\quad \ldots \text{ D}$$

which give the relations between the parameters and the indices of the face P in terms of the angles which the arc joining its pole hkl to any of the poles 100, 010, 001 forms with the adjacent pairs of the arcs that unite these latter poles.

CHAPTER III.

ON ZONES AND THEIR PROPERTIES.

SECTION I.—**Expressions for a Zone.**

36. *Definitions.* If the centre of the sphere of projection coincide with the origin of the axes to which a system of planes is referred, the direction of any plane passing through this centre (origin-plane) is determinable when the positions are known of any two radii of the sphere not on the same diameter that lie in the plane in question. And if these radii be the normals of two planes of the system, the poles of these planes will be points on the great circle in which the origin-plane intersects with the sphere of projection.

Further, the two planes, and therefore also the edge in which they mutually intersect, will be at once perpendicular to the plane of the great circle containing their poles, and parallel to the diameter of the sphere which is the normal of that plane: this is true for all the planes the poles of which lie on the great circle, and for the edges in which each pair of them may intersect.

Thus, the direction of the plane of this great circle and that of its normal may be determined indifferently from any pair of these planes.

The great circle which contains two or more of the poles belonging to a system of planes will be called a *zone-circle*, and the *plane containing* a zone-circle its *zone-plane:* the planes or faces *perpendicular to the zone-plane* are the planes or *faces of the zone.* The diameter of the sphere normal to the zone-plane, and to which therefore the edges formed by the intersections of faces of the zone are parallel, is the *zone-axis*. Two or more poles (or their faces) are said to be *tautozonal* or *heterozonal* with a third, according as they lie in the same or different zone-circles (or zones) with it; and when two zones have a face in common, that is to say when their zone-circles intersect in a pole, they will be spoken of as *tautohedral* in that face or pole.

37. *Symbol for a zone.* That two different centro-normals, and therefore two faces not parallel, suffice for the determination of the position of the zone-plane to which the faces are perpendicular is further evident from this, that two points, and therefore two poles on a sphere, not extremities of the same diameter, can only be traversed by one great circle: and the direction of the zone-plane containing such a circle is known when that of its zone-axis is known.

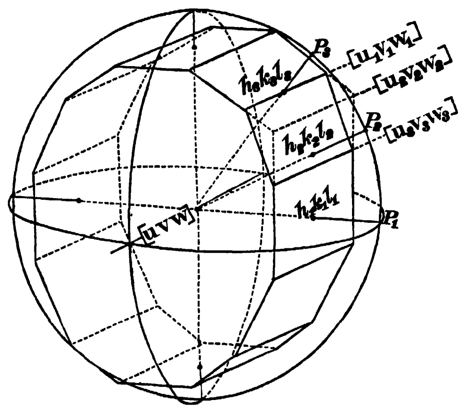


Fig. 23.

In fact, if two planes in the zone be P_1 or $(h_1 k_1 l_1)$, and P_2 or $(h_2 k_2 l_2)$, the coordinates for any point on its zone-axis are (Art. **20**) in the ratios,

A convenient symbol for representing a zone is formed by placing the letters or symbols representing the faces by which its direction is determined within square brackets, or so placing the coefficients of the parameters a, b, c in the above expression. Thus, $[P_1 P_2]$, $[h_1 k_1 l_1 ; h_2 k_2 l_2]$,

or

and, if P_3 be a third face of the zone,

$$[P_1 P_3] \text{ or } [u_2 v_2 w_2], \quad [P_3 P_2] \text{ or } [u_1 v_1 w_1],$$

are symbols that equally represent the zone $[P_1 P_2 P_3]$.

A zone, its zone-axis and zone-plane or zone-circle, will be represented by the same symbol.

It is however to be observed that the form in which this symbol is presented has not the same signification as the symbol for a plane of the system: moreover in the coordinates $a\mathbf{u}$, $b\mathbf{v}$, $c\mathbf{w}$ the indices in the symbol are seen to be integral, not fractional, coefficients of the parameters.

But although the two kinds of symbol thus have, even where their indices may be represented by the same numbers, entirely different significance, there are cases in which the symbol for a zone-plane or zone-axis comes to present identical indices with those in the symbol for a plane of the system parallel to the zone-plane.

These, and the conditions which have to be fulfilled by the axial systems to which it is possible to refer the system of planes, will be discussed in a future chapter.

Section II.—Relations connecting three tautozonal planes.

38. It results from the last article that if the different planes belonging to a zone be taken in pairs, a fresh expression for the zone-axis is obtained from each pair of the planes; as, for instance, in a zone containing the planes $(h_1 k_1 l_1)$, $(h_2 k_2 l_2)$, $(h_3 k_3 l_3)$, the symbols

$$[u_1 v_1 w_1], \ [u_2 v_2 w_2], \ [u_3 v_3 w_3]$$

may be obtained, where $[u_1 v_1 w_1]$ represents $[h_2 k_2 l_2 ; \ h_3 k_3 l_3]$, i.e. represents the symbol $[k_2 l_3 - l_2 k_3, \ l_2 h_3 - h_2 l_3, \ h_2 k_3 - k_2 h_3]$,

$$[u_2 v_2 w_2] \text{ being } [h_3 k_3 l_3 ; \ h_1 k_1 l_1],$$

and $$[u_3 v_3 w_3] \text{ being } [h_1 k_1 l_1 ; \ h_2 k_2 l_2].$$

And as these symbols equally represent the direction of an identical zone-axis, it is clear that their indices can only differ in their actual and not in their relative magnitudes; and these actual magnitudes can only differ, therefore, in the various symbols, by a series of common factors.

It will thus be seen that it must be necessarily true that, for instance,

$$\lambda = \frac{u_3}{u_1} = \frac{k_1 l_2 - k_2 l_1}{k_2 l_3 - k_3 l_2},$$
$$= \frac{v_3}{v_1} = \frac{l_1 h_2 - l_2 h_1}{l_2 h_3 - l_3 h_2},$$
$$= \frac{w_3}{w_1} = \frac{h_1 k_2 - h_2 k_1}{h_2 k_3 - h_3 k_2};$$

where the value of λ may in general be determined from either of these ratios. It often happens, however, that from, for instance, certain indices being zero, one or two of these expressions will assume the indeterminate form $\frac{0}{0}$. Since, however, all the indices cannot equal zero, the value of λ can usually be got by inspection and selection of the indices; but it may always be found if the entire symbols be compared, that is, by determining the value of

$$\frac{[u_3\, v_3\, w_3]}{[u_1\, v_1\, w_1]}, \quad \text{viz.} \quad \frac{[\lambda u_1\, \lambda v_1\, \lambda w_1]}{[u_1\, v_1\, w_1]}, \text{ which is } \lambda.$$

39. *Notation for Factor-ratios.* This relation between the symbols of a zone as derived from those of different pairs of the planes lying in the zone will presently be seen to have much significance. A convenient notation for the representation of such a relation is afforded by confining the symbols of the planes in question between square braces or simply between vertical lines, the symbols or their equivalents being in the form of a fraction. Thus, if P or $(h_1 k_1 l_1)$, Q or $(h_2 k_2 l_2)$, R or $(h_3 k_3 l_3)$, be three tautozonal planes, their zone may be equally expressed as $[PQ]$, $[QR]$, or $[RP]$; but as these several symbols differ generally by a common factor, we should have, for instance,

$$\left[\frac{PQ}{QR}\right] = \left\| \frac{\left|\begin{matrix} h_1 k_1 l_1 \\ h_2 k_2 l_2 \end{matrix}\right|}{\left|\begin{matrix} h_2 k_2 l_2 \\ h_3 k_3 l_3 \end{matrix}\right|} \right\| = \left| \frac{k_1 l_2 - l_1 k_2,\ l_1 h_2 - h_1 l_2,\ h_1 k_2 - k_1 h_2}{k_2 l_3 - l_2 k_3,\ l_2 h_3 - h_2 l_3,\ h_2 k_3 - k_2 h_3} \right|$$
$$= \left| \frac{u_3\, v_3\, w_3}{u_1\, v_1\, w_1} \right| = \lambda,$$

as various ways of representing the relation of the symbols for the zone as deduced from two pairs of its planes; and indicating in this case that $\mathbf{u}_3 = \lambda \mathbf{u}_1$, $\mathbf{v}_3 = \lambda \mathbf{v}_1$, and $\mathbf{w}_3 = \lambda \mathbf{w}_1$. Where we would indicate that an expression for the zone is to be derived by the cross multiplication of the symbols of a particular pair of planes of the zone, the symbol may be written thus, $\begin{matrix} h_1 \, k_1 \, l_1 \\ h_2 \, k_2 \, l_2 \end{matrix}$ or where the ratio of two such expressions is to be indicated, as in the above example, the expression will take the form of double line therein represented.

Thus, if the planes P or (211), Q or (111), R or (100), S or $(0\bar{1}\bar{1})$, belong to the same zone, we may indicate this zone as $[PQ]$; i.e. as [211, 111], or $\begin{matrix} 211 \\ 111 \end{matrix}$ which gives its symbol as $[0\bar{1}1]$; or it may be designated from the planes PS, therefore, as $[211, 0\bar{1}\bar{1}]$ or $[02\bar{2}]$; or from R and Q as $[0\bar{1}1]$, or from R and S as $[01\bar{1}]$, &c.

And the ratio

$$\frac{PQ}{PS} \qquad \frac{211,\ 111}{211,\ 0\bar{1}\bar{1}} \qquad \frac{011}{02\bar{2}}$$

and the value of λ may be variously represented by $\lambda = -\frac{1}{2}$.

So the ratio

$$\left|\frac{RQ}{RS}\right| \text{ or } \left\|\frac{\left|\begin{matrix}100\\111\end{matrix}\right|}{\left|\begin{matrix}100\\0\bar{1}\bar{1}\end{matrix}\right|}\right\| = \left|\frac{0\bar{1}1}{01\bar{1}}\right| = -1.$$

And

$$\left|\frac{PQ}{PS}\right| : \left|\frac{RQ}{RS}\right| = \frac{1}{2}$$

represents the proportion of these ratios.

In expressions of the kind under consideration, the direction in which an arc joining two poles on the sphere is considered to be estimated, and therewith its sign, is indicated by the order of the letters or symbols which represent the poles; so that

$$QP = -PQ,$$

and

$$\frac{PQ}{QS} \qquad \frac{QP}{SQ} \qquad \frac{QP}{QS}, \text{ \&c.}$$

40. *Condition for a plane* $(h\,k\,l)$ *to lie in a zone* $[\mathbf{u}\,\mathbf{v}\,\mathbf{w}]$. It further results from the geometrical unity of a zone-axis and the consequent invariability of the ratios of the indices in the various symbols obtained for it as representing the edge formed by one or other of the different pairs of planes belonging to the zone, that, taking the symbols for any three tautozonal planes as before,

$$\frac{\mathbf{u}_1}{\mathbf{u}_2} = \frac{\mathbf{v}_1}{\mathbf{v}_2} = \frac{\mathbf{w}_1}{\mathbf{w}_2} = \frac{h_1\mathbf{u}_1 + k_1\mathbf{v}_1 + l_1\mathbf{w}_1}{h_1\mathbf{u}_2 + k_1\mathbf{v}_2 + l_1\mathbf{w}_2}$$

is evidently true.

Since the denominator of this last ratio is identically zero, as is seen on substituting the values of $\mathbf{u}_2\,\mathbf{v}_2\,\mathbf{w}_2$ in terms of $h_1\,k_1\,l_1$ and $h_3\,k_3\,l_3$, the numerator must also be so; and consequently,

$$h_1\mathbf{u}_1 + k_1\mathbf{v}_1 + l_1\mathbf{w}_1 = 0,$$

$$\text{i.e. } h_1(k_2 l_3 - l_2 k_3) + k_1(l_2 h_3 - h_2 l_3) + l_1(h_2 k_3 - k_2 h_3) = 0\,;$$

an equation establishing a relation between the symbols of any three planes in a zone.

41. *Symbol for a plane* $(h\,k\,l)$ *in which two zones* $[\mathbf{u}_1\,\mathbf{v}_1\,\mathbf{w}_1]$ *and* $[\mathbf{u}_2\,\mathbf{v}_2\,\mathbf{w}_2]$ *are tautohedral.* Since $(h\,k\,l)$ belongs to each of the zones in question,

$$h\mathbf{u}_1 + k\mathbf{v}_1 + l\mathbf{w}_1 = 0\,;$$

and

$$h\mathbf{u}_2 + k\mathbf{v}_2 + l\mathbf{w}_2 = 0:$$

whence

$$h(\mathbf{u}_1\mathbf{w}_2 - \mathbf{w}_1\mathbf{u}_2) + k(\mathbf{v}_1\mathbf{w}_2 - \mathbf{w}_1\mathbf{v}_2) = 0\,;$$

and

$$h(\mathbf{u}_1\mathbf{v}_2 - \mathbf{v}_1\mathbf{u}_2) + l(\mathbf{w}_1\mathbf{v}_2 - \mathbf{v}_1\mathbf{w}_2) = 0\,;$$

also

$$\frac{h}{k} = \frac{\mathbf{v}_1\mathbf{w}_2 - \mathbf{w}_1\mathbf{v}_2}{\mathbf{w}_1\mathbf{u}_2 - \mathbf{u}_1\mathbf{w}_2}, \qquad \frac{h}{l} = \frac{\mathbf{v}_1\mathbf{w}_2 - \mathbf{w}_1\mathbf{v}_2}{\mathbf{u}_1\mathbf{v}_2 - \mathbf{v}_1\mathbf{u}_2}$$

or,

$$\frac{h}{\mathbf{v}_1\mathbf{w}_2 - \mathbf{w}_1\mathbf{v}_2} = \frac{k}{\mathbf{w}_1\mathbf{u}_2 - \mathbf{u}_1\mathbf{w}_2} = \frac{l}{\mathbf{u}_1\mathbf{v}_2 - \mathbf{v}_1\mathbf{u}_2},$$

wherefore the indices of the plane $(h\,k\,l)$ have the ratios represented by the symbol

$$(\mathbf{v}_1\mathbf{w}_2 - \mathbf{w}_1\mathbf{v}_2,\quad \mathbf{w}_1\mathbf{u}_2 - \mathbf{u}_1\mathbf{w}_2,\quad \mathbf{u}_1\mathbf{v}_2 - \mathbf{v}_1\mathbf{u}_2).$$

And it will be seen that the form of the expression representing the indices of a plane in terms of the symbols of two zones tautohedral in it, is identical with that of the symbol of a zone as derived from the symbols of two of the planes tautozonal in it.

42. *Expedient for deducing the determinant symbols.* This *similarity in the form assumed by the symbol of a zone as derived from a pair of its planes and by that of a plane common* to two zones (as obtained in Art. **40**) leads to the adoption of a similar process for obtaining the symbol in either case. An expedient for performing this operation, useful when dealing with complicated symbols, is that of writing, one under the other, the two symbols to be operated upon; each symbol being once repeated on its own line and the first and last index on each of the lines being struck out. The differences of the cross products of each successive pair of indices on the two lines (these indices being connected in the subjoined example by an X) form the new indices, the product of the indices joined by the thin stroke being deducted from the product of those joined by the thick stroke. Thus if $h_2 k_2 l_2$ and $h_3 k_3 l_3$ be two planes in a zone, we have, for the value of $\left\| \begin{matrix} h_2 k_2 l_2 \\ h_3 k_3 l_3 \end{matrix} \right\|$,

$$\overset{\;\;1\qquad\quad 2\qquad\quad 3}{\left\| \begin{matrix} k_2 & & l_2 & & h_2 & & k_2 \\ & \times & & \times & & \times & \\ k_3 & & l_3 & & h_3 & & k_3 \end{matrix} \right\|}$$

1. $k_2 l_3 - l_2 k_3$ .... u_1,
2. $l_2 h_3 - h_2 l_3$ .... v_1,
3. $h_2 k_3 - k_2 h_3$ .... w_1;

where $[k_2 l_3 - l_2 k_3,\ l_2 h_3 - h_2 l_3,\ h_2 k_3 - k_2 h_3]$ or $[u_1 v_1 w_1]$ is the symbol of the zone sought.

Similarly, we get for the face $(h\,k\,l)$ in which two zones $[u\,v\,w]$ and $[u'\,v'\,w']$ are tautohedral, the symbol

$$(vw' - wv',\ \ wu' - uw',\ \ uv' - vu') \text{ or } (h\,k\,l).$$

The symbol for the zone containing the faces (342) and (324) is

$$\left\| \begin{matrix} 4234 \\ 2432 \end{matrix} \right\| \qquad \text{or} \qquad \left\| \begin{matrix} 2432 \\ 4234 \end{matrix} \right\|$$

$$\left.\begin{matrix} 12 \\ -6 \\ -6 \end{matrix}\right\} \text{ or } 12,\ \bar{6},\ 6, \qquad\qquad \bar{12},\ 6,\ 6;$$

which, reduced to their simplest forms, are $[2\bar{1}\bar{1}]$ or $[\bar{2}11]$, according to the order in which the two planes are taken.

The same symbol $[2\bar{1}\bar{1}]$ or $[\bar{2}11]$ is similarly obtained from the poles (111) and $(0\bar{1}1)$ which also belong to the above zone.

So again, the opposite poles in which two zone circles $[111]$ and $[0\bar{1}1]$ would intersect are $(2\bar{1}\bar{1})$ and $(\bar{2}11)$.

43. *Relations connecting the inclinations of the edges and of the normals of three planes.*

If *XOY*, *YOZ*, *ZOX*, Fig. 24, be three planes, *OX*, *OY*, *OZ* the zone-lines parallel to their edges, the great circles joining the axial points *X*, *Y*, *Z* of these zone-axes upon the sphere form a spherical triangle the sides of which evidently measure the plane angles at which each pair of edges meet in the solid angle at *O*; the angles at *X*, *Y*, and *Z* on the other hand are those of the inclinations of the planes.

If now *OA*, *OB*, *OC* be the normals of the planes *YZ*, *ZX*, *XY*; *A*, *B*, *C* being the poles of these origin-planes; *ABC* and *XYZ* are the angular points of two polar triangles, and the arcs *AB*, *BC*, *CA* are the supplements of the angles at which the planes *YZ*, *ZX*; *ZX*, *XY*; *XY*, *YZ* are inclined to each other; while the angles at *A*, *B*, and *C* are the supplements of the angles *YOZ*, *ZOX*, *XOY* at which the edges of the planes meet.

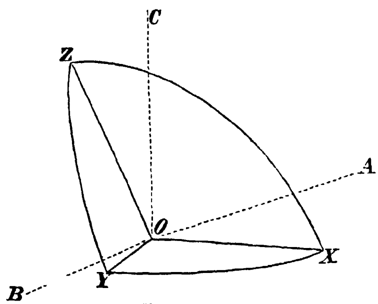


Fig. 24.

The arcs *AB*, *BC*, *CA* are generally capable of being determined by processes of measurement, so that from these the other values can be calculated.

Section III.—On the signs of the indices of a plane as determined by the position of the plane in respect to a given zone.

44. Two poles are said to be on the *same* or *on opposite sides* of a given pole lying in a zone with them according *as* *both* or *as only one* of them may lie within an arc-distance π from that pole as measured in the same direction on the zone-circle that contains the two poles.

Let [uvw] be a zone-circle containing the poles $p_1 q_1 r_1$, $p_2 q_2 r_2$; then the equation $\mathbf{u}e+\mathbf{v}f+\mathbf{w}g = 0$ gives the necessary condition for a plane efg to lie in the zone $[p_1 q_1 r_1, p_2 q_2 r_2]$.

Wherefore, for the plane hkl not belonging to the zone,

$$\mathbf{u}h+\mathbf{v}k+\mathbf{w}l > \text{ or } < 0;$$

so that, if the pole of this plane hkl lies on one of the hemispheres into which the sphere of projection is divided by the zone, this expression must have some definite value either positive or negative.

If, now, for any particular pole this value be found to be positive, it must be positive for all poles situate on the same hemisphere with that pole, since the expression cannot change sign except by the pole passing through zero, i.e. by its passing to the other side of the zone-circle.

DEF.—The *hemispheres* bounded by a zone-circle may be termed *positive* or *negative* in respect to that zone-circle according as the symbols of the poles lying in them give the above expression positive or negative; i.e. according as the determinant of the three symbols two of which belong to planes in the zone may be positive or negative.

45. The result of the process for determining the indices in the symbol of a face by the rule of zones is ambiguous, as regards the signs of the several indices, as is seen in the example in Article **42**; the ambiguity however can be removed by determining the position of the pole on the sphere relatively to two poles heterozonal to it, that is to say, relatively to the zone containing those poles. And other conclusions determining the distribution on the sphere of the different poles of a form, the symbols of which differ in the relative order as well as in the signs of their indices, can be arrived at by the aid of this principle when put into the following form.

Determination of the signs of the indices in a symbol. If R, Q_1, Q_2.

be three planes in a zone, the ratio $\left|\frac{RQ_1}{RQ_2}\right.$ will have a positive value if Q_1 and Q_2 are on the same side of R, and will be negative if they are on opposite sides of R.

If any pole K the symbol of which is $\lambda\mu\nu$ be taken external to the zone, while Q_1, Q_2, R are $e_1f_1g_1$, $e_2f_2g_2$, pqr respectively, then we have

$$(\mu r-\nu q)\,e_2+(\nu p-\lambda r)\,f_2+(\lambda q-\mu p)\,g_2 = C_2,$$

and $$(\mu r-\nu q)\,e_1+(\nu p-\lambda r)\,f_1+(\lambda q-\mu p)\,g_1 = C_1;$$

both positive or negative or one positive and the other negative according as Q_1 and Q_2 are on the same or opposite sides of R,—that is to say, transposing the expressions,

$$\lambda\,(qg_2-rf_2)+\mu\,(re_2-pg_2)+\nu\,(pf_2-qe_2) = C_2$$

and $$\lambda\,(qg_1-rf_1)+\mu\,(re_1-pg_1)+\nu\,(pf_1-qe_1) = C_1$$

present C_2 and C_1 with the same or with opposite signs according as

$$\frac{C_2}{C_1}, \text{ and therefore } \frac{\begin{array}{c}p\,q\,r\\ e_2f_2g_2\end{array}}{\begin{array}{c}p\,q\,r\\ e_1f_1g_1\end{array}} \quad \text{i.e. as } \frac{RQ_2}{RQ_1}$$

is positive or negative; which thus follows from Q_1 and Q_2 lying on the same or on opposite sides of R, that is, from their lying on the same or on the opposite hemispheres divided by the great circle $[KR]$.

Examples.

I. On a hexagonal crystal the zone-circle containing the poles $2\bar{1}\bar{1}$ and $11\bar{2}$, that is to say $[111]$, intersects with the zone-circle containing the poles 010, 001, that is to say $[100]$, in two poles which are $0\bar{1}1$ and $01\bar{1}$.

It is required to determine which of these poles is the nearer to the pole $11\bar{2}$, that is to say which will be on the same side with $11\bar{2}$, of the pole $2\bar{1}\bar{1}$ on the zone-circle $[111]$.

Here $\dfrac{\begin{array}{c}2\bar{1}\bar{1}\\ efg\end{array}}{\begin{array}{c}2\bar{1}\bar{1}\\ 11\bar{2}\end{array}} = \dfrac{m}{n}$ has a positive value; efg being the symbol of the face in question, where $e = 0$ and $f = -g$, while f and g are unity.

The expression becomes

$$\left|\frac{f-g,\ -(2g+e),\ e+2f}{3\qquad 3\qquad 3}\right| = \frac{m}{n}$$

$$= \left|\frac{2f\ \ 2f\ \ 2f}{3\ \ 3\ \ 3}\right| = \left|\frac{-2g\ \ -2g\ \ -2g}{3\ \ \ 3\ \ \ 3}\right|$$

and $\frac{m}{n} = \frac{2}{3}f$; $f = -g$; so that if $\frac{m}{n}$ is to be positive, f must be positive and g must be negative, and the symbol is $01\bar{1}$.

II. On a crystal of Calcite of which Fig. 25 is a projection the forms $r\{100\}$, $b\{2\bar{1}\bar{1}\}$, $v\{20\bar{1}\}$, $t\{310\}$ occur.

The poles v and v' are found by measurements of the angles vb'' and $v'b''$, vr and $v'r'$ to be symmetrically situated and to belong to the same form. Similarly, t_2 is found to belong to the same form with t' and so with t; and t and v are the faces 310 and $20\bar{1}$.

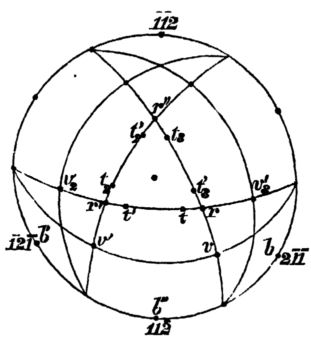


Fig. 25.

(1) Both faces t_2 and v' lie on a zone with r' and r'', i.e. with 010 and 001, and therefore on $[100]$.

Let then t_o be $0kl$ and v' be

(2) Also both lie on the same side with 010 of the zone-plane $[111]$, which is the plane of projection of the figure, and therefore $k+l$ and $f+g$ are both greater than zero, i.e. are positive.

(3) And they both lie on the same side with 010 of the zone-plane passing through $2\bar{1}\bar{1}$ and 100 (b and r), that is to say, of the zone $[01\bar{1}]$. Hence $f-g$ and $k-l$ are positive.

Hence from (1) k and l can only be 3 and 1 or $\bar{3}$ and $\bar{1}$, or else 1 and 3 or $\bar{1}$ and $\bar{3}$; and f and g can only be 2 and $\bar{1}$ or $\bar{2}$ and 1.

From (2) $k+l$ can only be $3+1$ or $1+3$, and $f+g$ can only be $2-1$ or $-1+2$.

But from (3) the second index must be greater than the third, or k must be greater than l and f than g: wherefore the symbols are of v', $02\bar{1}$; and of t_2, 031.

Section IV.—Relations connecting four tautozonal planes.

46. Let $P_1\,(h_1\,k_1\,l_1)$, $Q_1\,(e_1 f_1\, g_1)$, $Q_2\,(e_2 f_2\, g_2)$ and $P_2\,(h_2\,k_2\,l_2)$ be four poles lying on the same zone-circle, and let X, Y, Z be

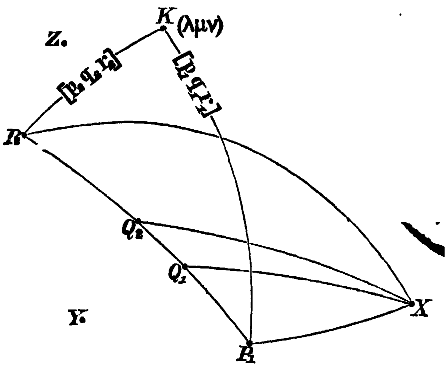


Fig. 26.

the axial points in which the axes to which the system of crystalloid planes is referred meet the sphere of projection. Then

$$\cos XP_1 = \cos XQ_1 \cos P_1Q_1 + \sin XQ_1 \sin P_1Q_1 \cos P_1Q_1X,$$

$$\cos XP_2 = \cos XQ_1 \cos P_2Q_1 + \sin XQ_1 \sin P_2Q_1 \cos P_2Q_1X;$$

$$\cos XP_1 \sin P_2Q_1 = \cos XQ_1 \sin P_2Q_1 \cos P_1Q_1 + \sin XQ_1 \sin P_2Q_1 \sin P_1Q_1 \cos P_1Q_1X,$$

$$\cos XP_2 \sin P_1Q_1 = \cos XQ_1 \sin P_1Q_1 \cos P_2Q_1 - \sin XQ_1 \sin P_1Q_1 \sin P_2Q_1 \cos P_1Q_1X;$$

whence

$$\cos XP_1 \sin P_2Q_1 + \cos XP_2 \sin P_1Q_1 = \cos XQ_1 \sin P_2P_1.$$

In the same way

$$\cos YP_1 \sin P_2Q_1 + \cos YP_2 \sin P_1Q_1 = \cos YQ_1 \sin P_2P_1;$$

therefore

$$\frac{\sin P_1 Q_1}{\cos XQ_1 \cos YP_1 - \cos YQ_1 \cos XP_1} = \frac{\sin Q_1 P_2}{\cos XP_2 \cos YQ_1 - \cos YP_2 \cos XQ_1}$$
$$= \frac{\sin P_1 P_2}{\cos XP_2 \cos YP_1 - \cos YP_2 \cos XP_1}.$$

Similarly, by considering the poles P_1, Q_1, Q_2, and again the poles P_1, Q_2, P_2, we may shew that the above expressions are equivalent to

$$\frac{\sin Q_1 Q_2}{\cos XQ_2 \cos YQ_1 - \cos YQ_2 \cos XQ_1} = \frac{\sin P_1 Q_2}{\cos XQ_2 \cos YP_1 - \cos YQ_2 \cos XP_1}$$
$$= \frac{\sin Q_2 P_2}{\cos XP_2 \cos YQ_2 - \cos YP_2 \cos XQ_2}.$$

From the fundamental equations (A),

$$\frac{a}{h_1} \cos XP_1 = \frac{b}{k_1} \cos YP_1 = \frac{c}{l_1} \cos ZP_1,$$

$$\frac{a}{e_1} \cos XQ_1 = \frac{b}{f_1} \cos YQ_1 = \frac{c}{g_1} \cos ZQ_1,$$

$$\frac{a}{e_2} \cos XQ_2 = \frac{b}{f_2} \cos YQ_2 = \frac{c}{g_2} \cos ZQ_2,$$

$$\frac{a}{h_2} \cos XP_2 = \frac{b}{k_2} \cos YP_2 = \frac{c}{l_2} \cos ZP_2.$$

By substituting values for $\cos XP_1$, $\cos XQ_1$, &c.

$$\frac{\sin P_1 Q_2}{\sin Q_1 Q_2} = \frac{f_1}{k_1} \frac{\cos YP_1}{\cos YQ_1} \frac{k_1 e_2 - h_1 f_2}{f_1 e_2 - e_1 f_2} = \&\text{c.} = \frac{p_1}{q_1} \frac{P_1 Q_2}{Q_1 Q_2}$$

$$\frac{\sin P_1 P_2}{\sin Q_1 P_2} = \frac{f_1}{k_1} \frac{\cos YP_1}{\cos YQ_1} \frac{k_1 h_2 - h_1 k_2}{f_1 h_2 - e_1 k_2} = \&\text{c.} = \frac{p_1}{q_1} \frac{P_1 P_2}{Q_1 P_2}$$

Hence

$$(1) \quad \frac{\sin P_1 Q_2}{\sin P_1 P_2} : \frac{\sin Q_1 Q_2}{\sin Q_1 P_2} = \frac{k_1 e_2 - h_1 f_2}{k_1 h_2 - h_1 k_2} : \frac{f_1 e_2 - e_1 f_2}{f_1 h_2 - e_1 k_2}$$
$$= \left|\frac{P_1 Q_2}{P_1 P_2}\right| : \left|\frac{Q_1 Q_2}{Q_1 P_2}\right| = \Lambda'. \quad \ldots \ldots \ldots \quad \text{C}$$

It may similarly be proved that

$$(2)\quad \frac{\sin P_1 Q_1}{\sin P_1 Q_2} : \frac{\sin P_2 Q_1}{\sin P_2 Q_2} = \frac{k_1 e_1 - h_1 f_1}{k_1 e_2 - h_1 f_2} : \frac{k_2 e_1 - h_2 f_1}{k_2 e_2 - h_2 f_2}$$
$$\frac{P_1 Q_1}{P_1 Q_2} \quad \frac{P_2 Q_1}{P_2 Q_2} = \Lambda''.$$

If Q_2 be external to P_2 the relation may be put into the form

$$(3)\quad \frac{\sin P_1 Q_2}{\sin P_1 Q_1} : \frac{\sin P_2 Q_2}{\sin P_2 Q_1} = -\frac{1}{\Lambda''}. \quad \ldots\ldots\ldots\ldots \quad \text{C}'$$

In the expression (2) the ratio is positive, in (3) it is negative, and these two relations, representing the cases where Q_2 is internal and where it is external to P_2, may be comprised under the single expression

$$\frac{\sin P_1 Q_2 \sin (P_1 Q_1 - P_1 P_2)}{\sin (P_1 Q_2 - P_1 P_2) \sin P_1 Q_1} = \left|\frac{P_1 Q_2}{P_2 Q_2}\right| : \left|\frac{P_1 Q_1}{P_2 Q_1}\right| = \Lambda$$

where Λ may be positive or negative.

By analogy, $\dfrac{\sin P_1 Q_2}{\sin P_2 Q_2} = \dfrac{p_1}{p_2} \left|\dfrac{P_1 Q_2}{P_2 Q_2}\right|$, or

$$\frac{p_2}{p_1} \frac{\sin P_1 Q_2}{\sin P_2 Q_2} = \left|\frac{P_1 Q_2}{P_2 Q_2}\right|$$

$$= \frac{\begin{vmatrix} h_1 & k_1 & l_1 \\ e_2 & f_2 & g_2 \end{vmatrix}}{\begin{vmatrix} h_2 & k_2 & l_2 \\ e_2 & f_2 & g_2 \end{vmatrix}} = \frac{\begin{vmatrix} k_1 & l_1 \\ f_2 & g_2 \end{vmatrix}}{\begin{vmatrix} k_2 & l_2 \\ f_2 & g_2 \end{vmatrix}} = \frac{\begin{vmatrix} l_1 & h_1 \\ g_2 & e_2 \end{vmatrix}}{\begin{vmatrix} l_2 & h_2 \\ g_2 & e_2 \end{vmatrix}} = \frac{\begin{vmatrix} h_1 & k_1 \\ e_2 & f_2 \end{vmatrix}}{\begin{vmatrix} h_2 & k_2 \\ e_2 & f_2 \end{vmatrix}}$$

$$= \frac{\lambda \begin{vmatrix} k_1 & l_1 \\ f_2 & g_2 \end{vmatrix} + \mu \begin{vmatrix} l_1 & h_1 \\ g_2 & e_2 \end{vmatrix} + v \begin{vmatrix} h_1 & k_1 \\ e_2 & f_2 \end{vmatrix}}{\lambda \begin{vmatrix} k_2 & l_2 \\ f_2 & g_2 \end{vmatrix} + \mu \begin{vmatrix} \lambda_2 & h_2 \\ g_2 & e_2 \end{vmatrix} + v \begin{vmatrix} h_2 & k_2 \\ e_2 & f_2 \end{vmatrix}}$$

$$= \frac{e_2(\mu l_1 - v k_1) + f_2(v h_1 - \lambda l_1) + g_2(\lambda k_1 - \mu h_1)}{e_2(\mu l_2 - v k_2) + f_2(v h_2 - \lambda l_2) + g_2(\lambda k_2 - \mu h_2)}.$$

Similarly for $\dfrac{p_2}{p_1} \dfrac{\sin P_1 Q_1}{\sin P_2 Q_1}$; and we have

$$\frac{\sin P_1 Q_2}{\sin (P_1 Q_2 - P_1 P_2)} \cdot \frac{\sin (P_1 Q_1 - P_1 P_2)}{\sin P_1 Q_1}$$
$$= \frac{e_2 \mathrm{p}_1 + f_2 \mathrm{q}_1 + g_2 \mathrm{r}_1}{e_2 \mathrm{p}_2 + f_2 \mathrm{q}_2 + g_2 \mathrm{r}_2} : \frac{e_1 \mathrm{p}_1 + f_1 \mathrm{q}_1 + g_1 \mathrm{r}_1}{e_1 \mathrm{p}_2 + f_1 \mathrm{q}_2 + g_1 \mathrm{r}_2},$$
$$= \Lambda; \quad \ldots\ldots\ldots\ldots \quad \text{D}'$$

where $[\mathrm{p}_1\,\mathrm{q}_1\,\mathrm{r}_1]$, $[\mathrm{p}_2\,\mathrm{q}_2\,\mathrm{r}_2]$ are $[h_1 k_1 l_1,\ \lambda \mu \nu]$, $[h_2 k_2 l_2,\ \lambda \mu \nu]$ respectively, $(\lambda \mu \nu)$ being the symbol of an *arbitrarily taken* pole K external to the zone-circle $[P_1 Q_2]$.

By selecting for K a pole presenting one or two zeros in its symbol, this symmetrical expression takes a very simple form.

47. It is seen from the form of the expressions leading to **C** and **D** that a fourth plane in the zone is needed in order to obtain a relation between the faces of a zone independent of the axial distances of the several poles.

It will moreover be seen that the expressions thus obtained for the relations of four faces belonging to a zone are in the form of one or other of the anharmonic ratios of the normals of the four faces; which form a sheaf of lines lying in the plane of the zone and meeting at the origin. And since these four normals are homographic with the traces on the zone-plane of the origin-planes parallel to the four tautozonal faces, it may be asserted alike of the normals and of the faces, that in a crystalloid system the anharmonic ratios of any four tautozonal planes must be rational; for it is obvious that the right-hand sides of the expressions **C** and **C′** as well as of **D** and **D′** are rational.

The anharmonic function is in each case equivalent to the ratios of the first minors of the determinants formed by the symbols of the faces taken in corresponding order. Since there are three anharmonic ratios resulting from the division of an angle or of a line, those of the four tautozonal planes $P_1 Q_1$, $Q_2 P_2$ are

$$(1)\qquad \frac{P_1 Q_2}{P_1 P_2}\ \frac{Q_1 Q_2}{Q_1 P_2} = \Lambda_1 = \frac{\left|\begin{matrix} h_1 & k_1 & l_1 \\ e_2 & f_2 & g_2 \end{matrix}\right|}{\left|\begin{matrix} h_1 & k_1 & l_1 \\ h_2 & k_2 & l_2 \end{matrix}\right|} \cdot \frac{\left|\begin{matrix} e_1 & f_1 & g_1 \\ e_2 & f_2 & g_2 \end{matrix}\right|}{\left|\begin{matrix} e_1 & f_1 & g_1 \\ h_2 & k_2 & l_2 \end{matrix}\right|}$$

$$(2)\qquad \frac{P_1 P_2}{P_1 Q_1}\ \frac{Q_2 P_2}{Q_2 Q_1} = \Lambda_2 = \quad \&c., \qquad \&c.;$$

$$(3)\qquad \frac{P_1 Q_1}{P_1 Q_2}\ \frac{P_2 Q_1}{P_2 Q_2} = \Lambda_3 = \quad \&c., \qquad \&c.$$

Solution of problems involving four tautozonal planes.

48. The expressions arrived at in the previous paragraphs afford the means of determining the position on a zone-circle of one of the four poles, or the symbol of one of them, when the remaining positions and symbols of the four poles are known.

Thus, first; let the position of the pole Q_2 be sought where the symbols of the four tautozonal poles P_1, Q_1, Q_2, P_2, that is, $(h_1\,k_1\,l_1)$, $(e_1 f_1 g_1)$, $(e_2 f_2 g_2)$, $(h_2\,k_2\,l_2)$ respectively, and the arcs $P_1\,Q_1$ and $P_1\,P_2$, i.e. the positions of the poles P_1 Q_1 and P_2, are known.

From the equation **D** we get

$$\frac{\sin P_1\,Q_2}{\sin (P_1\,Q_2 - P_1\,P_2)} = \Lambda \frac{\sin P_1\,Q_1}{\sin (P_1\,Q_1 - P_1\,P_2)} = \tan\theta,$$

where
$$\Lambda = \frac{P_1 Q_2}{P_2 Q_2} \quad \frac{P_1 Q_1}{P_2 Q_1}$$

and from this equation θ can be determined, since the second limb of the expression consists of known quantities. If the expression **D′** be employed, the value of Λ is obtained from the right-hand side of that equation.

In order to reduce the expression into a form adapted for the use of logarithms, we have,

$$\frac{1+\tan\theta}{1-\tan\theta} = \tan(45^\circ+\theta) = \frac{\sin (P_1\,Q_2 - P_1\,P_2) + \sin P_1\,Q_2}{\sin (P_1\,Q_2 - P_1\,P_2) - \sin P_1\,Q_2},$$

$$= \frac{2 \sin \tfrac{1}{2}(2\,P_1\,Q_2 - P_1\,P_2) \cos - \tfrac{1}{2}\,P_1\,P_2}{2 \cos \tfrac{1}{2}(2\,P_1\,Q_2 - P_1\,P_2) \sin - \tfrac{1}{2}\,P_1\,P_2},$$

$$= -\tan(P_1\,Q_2 - \tfrac{1}{2}\,P_1\,P_2) \operatorname{cotan} \tfrac{1}{2}\,P_1\,P_2;$$

and
$$\tan(P_1\,Q_2 - \tfrac{1}{2}\,P_1\,P_2) = -\tan\tfrac{1}{2}\,P_1\,P_2 \tan(45^\circ+\theta),$$

$$= \tan\tfrac{1}{2}\,P_1\,P_2 \tan(135^\circ - \theta). \quad \ldots \ldots \quad \mathbf{E}$$

49. Secondly, where the three angles between the four tautozonal planes and three of the symbols of the planes are known, and it is required to find the symbol of the fourth plane:—

Let the angles $P_1\,Q_1$, $P_1\,Q_2$, $P_1\,P_2$ be given, and also the symbols of P_1, Q_1, P_2, viz. $h_1\,k_1\,l_1$, $e_1 f_1 g_1$, $h_2\,k_2\,l_2$ respectively: it is required to find the symbol of Q_2, viz. $(e_2 f_2 g_2)$.

Putting the equation **D** into the form

$$\frac{P_1\,Q_2}{P_2\,Q_2} \quad \frac{P_1\,Q_1}{P_2\,Q_1}\,\Lambda = \frac{m}{-},$$

where Λ is to be obtained from the left side of either of the expressions **D** or **D′**, $\frac{m}{n}$ is rational and can be found from the known quantities constituting the right-hand side of the equation.

Then

$$\frac{P_1 Q_2}{P_2 Q_2} \quad \left|\frac{\begin{matrix} h_1 k_1 l_1 \\ e_2 f_2 g_2 \end{matrix}}{\begin{matrix} h_2 k_2 l_2 \\ e_2 f_2 g_2 \end{matrix}}\right. \qquad \frac{k_1 g_2 - l_1 f_2,\ l_1 e_2 - h_1 g_2,\ h_1 f_2 - k_1 e_2}{k_2 g_2 - l_2 f_2,\ l_2 e_2 - h_2 g_2,\ h_2 f_2 - k_2 e_2}$$

$$= \frac{m}{n},$$

and $$n(h_1 f_2 - k_1 e_2) = m(h_2 f_2 - k_2 e_2);$$

whence $$\frac{e_2}{nh_1 - mh_2} = \frac{f_2}{nk_1 - mk_2} = \frac{g_2}{nl_1 - ml_2},$$

by symmetry; and we have for the indices in the required symbol of Q_2,

$$\left.\begin{aligned} e_2 &= nh_1 - mh_2 \\ f_2 &= nk_1 - mk_2 \\ g_2 &= nl_1 - ml_2 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots \quad \mathbf{F}$$

Examples.

I. On a crystal of Diopside there occur in a zone four faces, their symbols, and the arcs between their poles, being

P_1 or $(h_1 k_1 l_1)$ or (100),

Q_1 or $(e_1 f_1 g_1)$ or (101), $P_1 Q_1 = 49^\circ\, 39'$,

P_2 or $(h_2 k_2 l_2)$ or (001), $P_1 P_2 = 73^\circ\, 59'$;

$\therefore\ \frac{1}{2} P_1 P_2 = 36^\circ\, 59'\, 30''$;

Q_2 or $(e_2 f_2 g_2)$ or $(\bar{3}01)$,

where the arc $P_1 Q_2$ is to be determined.

From expression **D**,

$$\left|\frac{P_1 Q_2}{P_2 Q_2}\right| \text{ or } \left\|\frac{\left|\begin{matrix}100\\ \bar{3}01\end{matrix}\right|}{\left|\begin{matrix}001\\ \bar{3}01\end{matrix}\right|}\right\| = \left|\frac{0\bar{1}0}{0\bar{3}0}\right| = \frac{1}{3}; \quad \left|\frac{P_1 Q_1}{P_2 Q_1}\right| = \left\|\frac{\left|\begin{matrix}100\\ 101\end{matrix}\right|}{\left|\begin{matrix}001\\ 101\end{matrix}\right|}\right\|$$

$$\therefore\ \Lambda' = -\frac{1}{3}.$$

In order to find the position of the pole Q_2 in the zone, the symbols of and arcs between the other poles being given as above, we have by equation **E**,

$$\tan\theta = -\frac{1}{3}\frac{\sin 49^\circ 39'}{\sin -24^\circ 20'}, \text{ and } \theta = 31^\circ 39' 17'';$$

$$\tan(P_1 Q_2 - \tfrac{1}{2} P_1 P_2) = \tan 36^\circ 59' 30'' \tan 103^\circ 20' 43'',$$
$$= \tan 107^\circ 28' 46'';$$
$$\therefore\ P_1 Q_2 - 36^\circ 59' 30'' = 107^\circ 28' 46'',$$
and
$$P_1 Q_2 = 144^\circ 28' 16''.$$

The value of Λ might have been equally obtained from the expression **D′**. Taking the pole (010) for K, i.e. for (λ, μ, ν); we have for $[p_1 q_1 r_1]$, $\left\|\begin{matrix}100\\010\end{matrix}\right\| = [00\bar{1}]$, and for $[p_2 q_2 r_2]$, $\left\|\begin{matrix}001\\010\end{matrix}\right\| = [100]$;

and
$$\frac{e_2 p_1 + f_2 q_1 + g_2 r_1}{e_2 p_2 + f_2 q_2 + g_2 r_2} : \frac{e_1 p_1 + \&c.}{e_1 p_2 + \&c.} = \frac{0+0-1}{-3+0+0} : \frac{0+0-1}{1+0+0},$$
$$\text{or } \Lambda = -\frac{1}{3}.$$

II. A zone on a crystal of Felspar presents the faces

P_1 or $(\bar{2}03)$,
Q_1 or $(\bar{1}11)$, $P_1 Q_1 = 31^\circ 12'$,
Q_2 or $(\bar{2}41)$, $P_1 Q_2$ is the arc required,
P_2 or $(\bar{1}30)$, $P_1 P_2 = 86^\circ 12'$, and $\frac{1}{2} P_1 P_2 = 43^\circ 6'$;

$$\left|\frac{P_1 Q_2}{P_2 Q_2}\right| = \left\|\begin{matrix}\left|\begin{matrix}\bar{2}03\\ \bar{2}41\end{matrix}\right| \\ \left|\begin{matrix}\bar{1}30\\ \bar{2}41\end{matrix}\right|\end{matrix}\right\| = \left|\begin{matrix}\bar{1}2 & \bar{4} & \bar{8}\\ 3 & 1 & 2\end{matrix}\right| = -4; \quad \left|\frac{P_1 Q_1}{P_2 Q_1}\right| = \frac{\overline{312}}{312} = -1,$$
$$\text{and} \quad \Lambda = 4;$$

$$\tan\theta = -4\frac{\sin 31^\circ 12'}{\sin 55^\circ}, \quad \text{and} \quad \theta = -68^\circ 25' 47''.$$

Whence $\tan(P_1 Q_2 - \frac{1}{2} P_1 P_2) = \tan 43^\circ 6' \tan 23^\circ 25' 47''$,
$$= \tan 22^\circ 4' 27'',$$
and
$$P_1 Q_2 = 65^\circ 10' 27''.$$

In order to determine the value of Λ by the use of expression **D′**, we may take for k the pole 001.

Then

$$p_1 q_1 r_1 = \left\|\begin{matrix}203\\001\end{matrix}\right\| = [020], \text{ and } p_2 q_2 r_2 = \left\|\begin{matrix}130\\001\end{matrix}\right\| = [310].$$

And by **D′**, $\Lambda = \frac{8}{-2} : \frac{2}{-2} = 4$;

the same result as by the other process.

Conversely, if the symbols in the above zone for the poles P_1, Q_1, P_2 are given and the arcs P_1Q_1 and P_1P_2 as before, but the arc P_1Q_2 be observed as $65° 20'$; and it be required to find the symbol of Q_2.

Then

$$\frac{m}{n} = \left|\frac{P_1Q_2}{P_2Q_2}\right| = \left|\frac{\bar{2}03,\ \bar{1}11}{\bar{1}30,\ \bar{1}11}\right| \cdot \frac{\sin 65° 20' \sin(31° 12' - 86° 12')}{\sin 31° 12' \sin(65° 20' - 86° 12')}$$

$$= -\frac{\sin 65° 20' \sin 55°}{\sin 31° 12' \sin 20° 52'} = -4{\cdot}034.$$

Assuming $\frac{m}{n} = \frac{-4}{1}$,

we have by equation **F** for the symbol of Q_2

$$(-2-4,\ 0+12,\ 3-0), \text{ i.e. } (\bar{6},\ 12,\ 3) \text{ or } (\bar{2}41).$$

It will be seen, however, that on the assumption that the symbol of Q_2 is $(\bar{2}41)$ and that the arcs P_1Q_1 and P_1P_2 had been correctly determined, the true value of the arc P_1Q_2 would not be $65° 20'$, as measured, but is, approximately, as calculated in the previous paragraph, $65° 10' 27''$.

Problems relating to four tautohedral zones.

50. The method of introducing a fifth pole external to the zone in which four poles lie, in order to represent the anharmonic ratios of these four poles in terms involving two zone circles intersecting with that of the original zone, may readily be extended so as to involve four zone-circles passing through the poles of the four tautozonal planes and also through an arbitrarily taken pole external to their zone.

If k be this arbitrarily chosen pole and P_1, Q_1, Q_2, P_2 be as before four poles lying on a zone to which k is external, then k will be the pole of an origin plane K or $(\lambda\mu\nu)$, and zone-circles passing through k and through P_1, Q_1, Q_2, P_2 will intersect with the great circle of which k is the pole in the points p_1, q_1, q_2, p_2 (Fig. 27).

Through p, the point in which the zone-circle P_1P_2 meets the great circle K, draw a straight line intersecting with radii of the sphere Op_1, Oq_1, Oq_2, Op_2 in a', b', c', p_2.

The zone-plane $[P_1 P_2]$ will intersect the plane kpp_2 in a line pd which will meet in $abcd$ lines drawn from O to P_1, Q_1, P_2, and Q_2, and from k to $a' b' c' p_2$: thus the pencil $OP_1 OQ_1 OQ_2 OP_2$ is homographic with the pencil $Op_1 Oq_1 Oq_2 Op_2$, which pencils are the traces of the four original zone-planes on the planes $OP_1 OP_2$ and $Op_1 Op_2$, and their anharmonic ratios are, as in the expression **D**,

$$\frac{\sin p_1 q_2 \sin (p_1 q_1 - p_1 p_2)}{\sin (p_1 q_2 - p_1 p_2) \sin p_1 q_1} = \frac{\sin P_1 Q_2 \sin (P_1 Q_1 - P_1 P_2)}{\sin (P_1 Q_2 - P_1 P_2) \sin P_1 Q_1}$$

$$= \left| \frac{P_1 Q_2}{P_2 Q_2} \right| : \left| \frac{P_1 Q_1}{P_2 Q_1} \right| = \Lambda,$$

where the arcs $p_1 q_1$, $p_1 q_2$, and $p_1 p_2$ measure the angles at which the zone-circles kP_1, kQ_1, kQ_2, kP_2 are inclined to each other.

It will be seen then from this construction that if there be two poles Q_1, Q_2 and the zone-circle denoted by them intersects two other zone-circles the symbols of which are $[\mathbf{p}_1 \mathbf{q}_1 \mathbf{r}_1]$, $[\mathbf{p}_2 \mathbf{q}_2 \mathbf{r}_2]$ which are tautohedral in a pole k; then the symbols $(h_1\ k_1\ l_1)$ and $(h_2 k_2 l_2)$ of the poles P_1 and P_2 in which the zone $[Q_1 Q_2]$ i.e. $[e_1 f_1 g_1, e_2 f_2 g_2]$ intersects the zone-circles $[\mathbf{p}_1\ \mathbf{q}_1\ \mathbf{r}_1]$ and $[\mathbf{p}_2 \mathbf{q}_2 \mathbf{r}_2]$ are known; and equally the symbol of the zone $[\mathbf{u}_1\ \mathbf{v}_1\ \mathbf{w}_1]$ passing

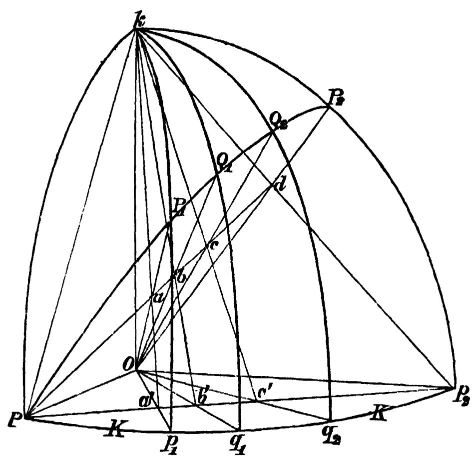


Fig. 27.

through $(\lambda\ \mu\ \nu)$, i.e. k and $(e_1 f_1 g_1)$, i.e. Q_1 and the symbol of the zone $[k Q_2]$, i.e. $[\lambda \mu \nu,\ e_2 f_2 g_2]$, i.e. $[\mathbf{u}_2 \mathbf{v}_2 \mathbf{w}_2]$, are known. There remain then the inclinations of the planes in the zone $[Q_1 Q_2]$

as measured by the arcs $P_1 Q_1$, $P_1 Q_2$, $P_1 P_2$, or the mutual inclinations of the four zone-planes

$$[\mathbf{p}_1 \mathbf{q}_1 \mathbf{r}_1],\ [\mathbf{u}_1 \mathbf{v}_1 \mathbf{w}_1],\ [\mathbf{u}_2 \mathbf{v}_2 \mathbf{w}_2],\ [\mathbf{p}_2 \mathbf{q}_2 \mathbf{r}_2]$$

as measured by the arcs $p_1 q_1$, $p_1 q_2$, $p_1 p_2$ to be determined. And where three of these are given, since the symbols are all known, the fourth may be found.

51. If then the arcs $p_1 q_1$, and $p_1 p_2$ be given the arc $q_1 q_2$ may be obtained by the method of Art. **48.** In fact

$$\frac{\sin p_1 q_2}{\sin(p_1 q_2 - p_1 p_2)} \cdot \frac{\sin(p_1 q_1 - p_1 p_2)}{\sin p_1 q_1} = \frac{\sin P_1 k Q_2}{\sin P_2 k Q_2} \frac{\sin P_1 k Q_1}{\sin P_2 k Q_1}$$

$$= \frac{\sin P_1 Q_2}{\sin(P_1 Q_2 - P_1 P_2)} \cdot \frac{\sin(P_1 Q_1 - P_1 P_2)}{\sin P_1 Q_1}$$

$$= \frac{e_2 \mathbf{p}_1 + f_2 \mathbf{q}_1 + g_2 \mathbf{r}_1}{e_2 \mathbf{p}_2 + f_2 \mathbf{q}_2 + g_2 \mathbf{r}_2} : \frac{e_1 \mathbf{p}_1 + f_1 \mathbf{q}_1 + g_1 \mathbf{r}_1}{e_1 \mathbf{p}_2 + f_1 \mathbf{q}_2 + g_1 \mathbf{r}_2} = \Lambda;$$

whence

$$\frac{\sin p_1 q_2}{\sin(p_1 q_2 - p_1 p_2)} = \frac{\sin p_1 q_1}{\sin(p_1 q_1 - p_1 p_2)} \cdot \Lambda = \tan\theta;$$

and as in **E**

$$\tan(p_1 q_2 - \tfrac{1}{2} p_1 p_2) = \tan\tfrac{1}{2} p_1 p_2 \tan(135^\circ - \theta). \quad \ldots\ \ldots\ \ldots \quad \mathbf{G}$$

52. Or again, if the arcs be given between two pairs of poles, the symbols of the zones $[k P_1]$ and $[k P_2]$ and of the poles Q_1, Q_2 being given as before, then the third arc may be found. Thus, if $P_1 Q_2$ be the arc sought, we have from **D′**, Art. **46**,

$$\sin P_1 Q_2 \sin(P_1 Q_1 - P_1 P_2) = \Lambda \sin P_1 Q_1 \sin(P_1 Q_2 - P_1 P_2),$$

$$2 \sin P_1 Q_2 \sin(P_1 Q_1 - P_1 P_2) = \cos(P_1 Q_2 - P_1 Q_1 + P_1 P_2) - \cos(P_1 Q_1 - Q_2 P_2),$$

$$2 \sin P_1 Q_1 \sin(P_1 Q_2 - P_1 P_2) = \cos(P_1 Q_1 + Q_2 P_2) - \cos(P_1 Q_1 - Q_2 P_2);$$

whence

$$\cos\{2 P_1 Q_2 - (P_1 Q_1 - Q_2 P_2)\} = (1 - \Lambda)\cos(P_1 Q_1 - Q_2 P_2) + \Lambda \cos(P_1 Q_1 + Q_2 P_2) \quad \ldots\ \ldots\ldots \quad \mathbf{H}$$

an equation in which every quantity is known but $P_1 Q_2$, and which therefore gives this required arc $P_1 Q_2$.

If $P_1 Q_1$ is a quadrant, the expression becomes

$$\sin(2 P_1 Q_2 + Q_2 P_2) = (1 - 2\Lambda) \sin Q_2 P_2. \quad \ldots\ \ldots\ \ldots \quad \mathbf{H'}$$

Section V.—**Analytical investigation of the zone-law.**

53. Expressions similar to those previously obtained may be deduced more briefly and with greater elegance by the methods of analytical geometry. Thus, if as before, *OX*, *OY*, *OZ* be any three axes, and

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

be the equation to a plane of the system, then, if the system be crystalloid in character, every other plane belonging to it will be parallel to one of the planes represented by the equation

$$\frac{h}{a}x+\frac{k}{b}y+\frac{l}{c}z=1\,;$$

where, as before, $\frac{a}{b}$ and $\frac{b}{c}$

are the parametral ratios, and

$$\frac{a}{h}:\frac{b}{k}:\frac{c}{l}$$

the ratios of the intercepts of the plane in question; h, k, l being integers, or one or two of them zero.

It is frequently convenient to consider the planes of a system as origin-planes: in which case the equation to the origin-plane, parallel to a plane

$$\frac{h}{a}x+\frac{k}{b}y+\frac{l}{c}z=d,$$

is

$$\frac{h}{a}x+\frac{k}{b}y+\frac{l}{c}z=0.$$

54. If the equations to two origin-planes are

$$\frac{h}{a}x+\frac{k}{b}y+\frac{l}{c}z=0,$$

$$\frac{p}{a}x+\frac{q}{b}y+\frac{r}{c}z=0,$$

the line in which they will intersect will be the zone-line or zone-axis

$$\frac{x}{\mathbf{u}\,a}=\frac{y}{\mathbf{v}\,b}=\frac{z}{\mathbf{w}\,c};$$

where $\mathbf{u}=kr-lq,\quad \mathbf{v}=lp-hr,\quad \mathbf{w}=hq-kp.$

And where the system of planes is crystalloid in character these values for **u**, **v**, and **w** are necessarily represented by whole numbers, or in the case of not more than two of them, by zero.

55. And if an origin-plane contain two lines passing through the origin which are possible zone-axes, that is to say, which have symbols involving only rational indices, this origin-plane is parallel to a possible face of the system; inasmuch as the equation to the plane containing the lines

$$\frac{x}{a\mathbf{h}} = \frac{y}{b\mathbf{k}} = \frac{z}{c\mathbf{l}} \quad \text{and} \quad \frac{x}{a\mathbf{p}} = \frac{y}{b\mathbf{q}} = \frac{z}{c\mathbf{r}}$$

is
$$(\mathbf{kr}-\mathbf{lq})\frac{x}{a} + (\mathbf{lp}-\mathbf{hr})\frac{y}{b} + (\mathbf{hq}-\mathbf{kp})\frac{z}{c} = 0,$$

and the symbol for this plane, viz. (**kr**−**lq**, **lp**−**hr**, **hq**−**kp**), contains indices which can only be integral, or equal to zero in the case of one or of two of them; since the indices in the symbols of the zones are only integers or zero.

56. Since the equations

$$\frac{p}{a}x + \frac{q}{b}y + \frac{r}{c}z = 0,$$

and
$$\frac{\lambda p}{a}x + \frac{\lambda q}{b}y + \frac{\lambda r}{c}z = 0,$$

are identical, representing one and the same origin-plane, the symbol of which may be written indifferently as (p, q, r) and $(\lambda p, \lambda q, \lambda r)$, the indices in the symbol for an origin-plane of a crystalloid system may be multiplied by any number positive or negative, or be divided by a common divisor. And the same is true for the symbol of a zone-axis, the equation to which is

$$\frac{x}{\mathbf{u}a} = \frac{y}{\mathbf{v}b} = \frac{z}{\mathbf{w}c}.$$

It may be written [**u v w**], or more generally [μ**u** μ**v** μ**w**].

57. If $(h_1\, k_1\, l_1)$, $(h_2\, k_2\, l_2)$, $(h_3\, k_3\, l_3)$ be three origin-planes intersecting in an identical zone-axis, their equations are

$$\left.\begin{array}{ll} (1) & h_1\frac{x}{a} + k_1\frac{y}{b} + l_1\frac{z}{c} = 0, \\ (2) & h_2\frac{x}{a} + k_2\frac{y}{b} + l_2\frac{z}{c} = 0, \\ (3) & h_3\frac{x}{a} + k_3\frac{y}{b} + l_3\frac{z}{c} = 0. \end{array}\right\}$$

Now x, y, and z have the same values in these three equations and are not all equal to zero, consequently the determinant

$$\nabla = \begin{vmatrix} h_1 & k_1 & l_1 \\ h_2 & k_2 & l_2 \\ h_3 & k_3 & l_3 \end{vmatrix} = 0;$$

that is to say,

$$\nabla = h_1(k_2 l_3 - l_2 k_3) + k_1(l_2 h_3 - h_2 l_3) + l_1(h_2 k_3 - k_2 h_3) = 0.$$

58. Conversely, the three planes intersect in the same zone-axis, if $\nabla = 0$.

For if the zone-axis in which, say, (2) and (3) intersect be the line $\mathbf{u}_1\, \mathbf{v}_1\, \mathbf{w}_1$ or $\left\| \begin{matrix} h_2 & k_2 & l_2 \\ h_3 & k_3 & l_3 \end{matrix} \right\|$, that is to say, the line the symbol for which is formed of the first minors of the determinant of the above three equations, the line is

$$\frac{x}{a\mathbf{u}_1} = \frac{y}{b\mathbf{v}_1} = \frac{z}{c\mathbf{w}_1},$$

and the plane $h_1\, k_1\, l_1$ will contain this line if

$$h_1 \mathbf{u}_1 + k_1 \mathbf{v}_1 + l_1 \mathbf{w}_1 = 0,$$

that is, if $\nabla = 0$.

59. And furthermore it will be seen that the necessary and geometrically sufficient condition for the plane hkl to contain a possible zone-axis $[\mathbf{u\, v\, w}]$ is

$$h\mathbf{u} + k\mathbf{v} + l\mathbf{w} = 0;$$

for the plane is

$$h\frac{x}{a} + k\frac{y}{b} + l\frac{z}{c} = 0, \quad \ldots\ldots\ldots \quad (1)$$

and the equation for the line must be

$$\frac{x}{a\mathbf{u}} = \frac{y}{b\mathbf{v}} = \frac{z}{c\mathbf{w}}; \quad \ldots\ldots\ldots \quad (2)$$

and any values for x, y, z satisfying (2) will satisfy (1); by substitution therefore in (1) of the values from (2) for the ratios $\frac{x}{a}, \frac{y}{b}, \frac{z}{c}$, we obtain $h\mathbf{u} + k\mathbf{v} + l\mathbf{w} = 0$.

Since a fresh expression for a zone-axis is obtained from each pair of planes in the zone, as, for instance, in the zone

$$[h_1 k_1 l_1,\ h_2 k_2 l_2,\ h_3 k_3 l_3],$$

the following symbols may be obtained for the zone-axis:-

$$[u_1\, v_1\, w_1],\quad [u_2\, v_2\, w_2],\quad [u_3\, v_3\, w_3];$$

where $u_1\, v_1\, w_1$ represent

$$\left\|\begin{matrix} h_2\, k_2\, l_2 \\ h_3\, k_3\, l_3 \end{matrix}\right\| \text{ or } k_2 l_3 - l_2 k_3,\quad l_2 h_3 - h_2 l_3,\quad h_2 k_3 - k_2 h_3;$$

and similarly for the others: but as these can only differ in their several actual, but not in their relative, magnitudes, the actual magnitudes of the different symbols can only differ by a series of common factors; that is, adopting the previous notation,

$$\left|\frac{u_1\, v_1\, w_1}{u_2\, v_2\, w_2}\right| \text{ or } \left\|\frac{\left\|\begin{matrix} h_2\, k_2\, l_2 \\ h_3\, k_3\, l_3 \end{matrix}\right\|}{\left\|\begin{matrix} h_3\, k_3\, l_3 \\ h_1\, k_1\, l_1 \end{matrix}\right\|}\right\| = \lambda = \frac{u_1}{u_2} = \frac{v_1}{v_2} = \frac{w_1}{w_2};$$

results identical with those previously obtained.

60. The equation $h_3 u_3 + k_3 v_3 + l_3 w_3 = o$, which represents the condition for a plane $(h_3\, k_3\, l_3)$ to belong to the zone $[u_3\, v_3\, w_3]$ or $[h_1\, k_1\, l_1,\; h_2\, k_2\, l_2]$, is indeterminate; the condition however which has to be fulfilled by the symbol of the plane may be put into another and more immediately applicable form.

Thus every origin-plane, the symbol of which has the form

$$(\lambda h_1 + \mu h_2\;\; \lambda k_1 + \mu k_2\;\; \lambda l_1 + \mu l_2),$$

lies in the same zone with the two planes $(h_1\, k_1\, l_1)$ and $(h_2\, k_2\, l_2)$, and is a plane of the system if $\frac{\lambda}{\mu}$ is rational.

For the determinant

$$\left|\begin{matrix} h_1 & k_1 & l_1 \\ h_2 & k_2 & l_2 \\ \lambda h_1 + \mu h_2 & \lambda k_1 + \mu k_2 & \lambda l_1 + \mu l_2 \end{matrix}\right| = \text{zero}.$$

And, conversely, if $(h_3\, k_3\, l_3)$ be the symbol of a plane belonging to a zone determined by the planes $(h_1\, k_1\, l_1)$ and $(h_2\, k_2\, l_2)$, its symbol —in general not its simplest, but some equivalent symbol—can be represented in the form

$$(\lambda h_1 + \mu h_2\;\; \lambda k_1 + \mu k_2\;\; \lambda l_1 + \mu l_2).$$

For if these two sets of indices represent the same plane,

$$\frac{\lambda h_1 + \mu h_2}{h_3} = \frac{\lambda k_1 + \mu k_2}{k_3} = \frac{\lambda l_1 + \mu l_2}{l_3} = -\nu,$$

relations which may also be expressed thus,

$$(1)\quad \lambda h_1 + \mu h_2 + \nu h_3 = 0,$$
$$(2)\quad \lambda k_1 + \mu k_2 + \nu k_3 = 0,$$
$$(3)\quad \lambda l_1 + \mu l_2 + \nu l_3 = 0.$$

By the reasoning in Article **57**, we have

$$\begin{vmatrix} h_1 & h_2 & h_3 \\ k_1 & k_2 & k_3 \\ l_1 & l_2 & l_3 \end{vmatrix} = 0;$$

wherefore the above equations are consistent, any pair of them giving the same values for λ, μ, ν.

Thus (2) and (3) give

$$\frac{\lambda}{k_2 l_3 - l_2 k_3} = \frac{\mu}{k_3 l_1 - l_3 k_1} = \frac{\nu}{k_1 l_2 - l_1 k_2},$$

or $$\frac{\lambda}{\mathbf{u}_1} = \frac{\mu}{\mathbf{u}_2} = \frac{\nu}{\mathbf{u}_3};$$

(3) and (1) give $$\frac{\lambda}{\mathbf{v}_1} = \frac{\mu}{\mathbf{v}_2} = \frac{\nu}{\mathbf{v}_3};$$

(1) and (2) give $$\frac{\lambda}{\mathbf{w}_1} = \frac{\mu}{\mathbf{w}_2} = \frac{\nu}{\mathbf{w}_3}.$$

Having then found values from any one of these systems for λ, μ, ν, the ratios of which must clearly be rational, we may write for $(h_3\, k_3\, l_3)$ the equivalent symbol

$$(-\nu h_3, \quad -\nu k_3, \quad -\nu l_3),$$

or $$(\lambda h_1 + \mu h_2, \quad \lambda k_1 + \mu k_2, \quad \lambda l_1 + \mu l_2);$$

which was the expression to be arrived at.

Example.—To find the symbols of a series of planes lying in the same zone with the planes (111) and (320), i.e. in the zone $[2\bar{3}1]$.

Let $\lambda = 2$, $\mu = 1$, then the symbol is (542),

$\lambda = -2$, $\mu = 1$, „ „ $(10\bar{2})$,

$\lambda = 1$, $\mu = 2$, „ „ (751),

$\lambda = -1$, $\mu = 1$, „ „ $(21\bar{1})$;

all of which symbols belong to planes of the zone $[2\bar{3}1]$: thus from the symbol (542) we have

$$5 \times 2 - 4 \times 3 + 2 \times 1 = 0,$$

and so for the others.

61. COR. Where $\lambda=\mu=\pm 1$, the process is simply the addition or subtraction of the indices in the symbols $(h_1\, k_1\, l_1)$ and $(h_2\, k_2\, l_2)$.

Whence, whether we take the sums or the differences of the corresponding indices in two symbols, we equally obtain a symbol for a face belonging to the zone.

A face, the symbol of which is obtained by adding the indices in the symbols of two other faces, will cut off or *replace*, and in certain cases, where the faces are symmetrically disposed, will *truncate* their edge; that is to say, will, in the latter case, be equally inclined on the faces.

In cases where a face truncates an edge, symbols for faces that *bevil* the edge (that is to say, would be inclined each at the same angle on a truncating face) are obtained by adding the indices of the symbol for the truncating face to those of the symbols for any other pair of corresponding faces that meet in and bevil the edge. [See Article **133.**]

Thus, if (110) and (101) be the symbols of two faces the edge of which is truncated by the face (211), the same edge would be bevilled by the faces (321) and (312), as well as by the faces 532 and 523, &c.

The problems of four tautozonal planes.

62. Let there be a system of tautozonal planes; the equations to two of which are $\mathbf{u} = 0$ and $\mathbf{v} = 0$; where

$$\left.\begin{aligned} \mathbf{u} &= \frac{p_1 x}{a} + \frac{q_1 y}{b} + \frac{r_1 z}{c} = 0, \\ v &= \frac{p_2 x}{a} + \frac{q_2 y}{b} + \frac{r_2 z}{c} = 0. \end{aligned}\right\} \quad \ldots\ldots \quad (1)$$

Any plane passing through the same zone-axis $[\mathbf{u}\,\mathbf{v}]$ with these planes $p_1\, q_1\, r_1$ and $p_2\, q_2\, r_2$ must be represented by an equation of the form $\mathbf{u}+\mu\mathbf{v} = 0$, or

$$(p_1+\mu p_2)\frac{x}{a} + (q_1+\mu q_2)\frac{y}{b} + (r_1+\mu r_2)\frac{z}{c} = 0 \ldots\ldots\ldots \quad (2)$$

But if the plane be crystalloid and (hkl) its symbol, its equation may also be written in the form

$$h\frac{x}{a} + k\frac{y}{b} + l\frac{z}{c} = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

In order that these equations may be identical we must have

$$\frac{h}{p_1+\mu p_2}=\frac{k}{q_1+\mu q_2}=\frac{l}{r_1+\mu r_2}; \quad \ldots\ldots\ldots \quad (4)$$

whence μ must be rational if the remaining letters represent rational numbers; i.e. if the planes be crystalloid.

As in Article **46**, let P_1, Q_1, P_2, Q_2 be four planes of a zone $[\mathbf{u}\,\mathbf{v}]$, of which **u** and **v** are any two planes; and let

$$\left.\begin{array}{lll} \mathbf{u}+\mu_1\mathbf{v}=0 & \text{be the plane} & P_1 \text{ or } (h_1 k_1 l_1), \\ \mathbf{u}+\mu_2\mathbf{v}=0 & \quad ,, \quad ,, & Q_1 \text{ or } (e_1 f_1 g_1), \\ \mathbf{u}+\mu_3\mathbf{v}=0 & \quad ,, \quad ,, & P_2 \text{ or } (h_2 k_2 l_2), \\ \mathbf{u}+\mu_4\mathbf{v}=0 & \quad ,, \quad ,, & Q_2 \text{ or } (e_2 f_2 g_2), \end{array}\right\} \quad \ldots\ldots \quad (5)$$

where $\mu_1, \mu_2, \mu_3, \mu_4$ are, as above, rational.

Then the anharmonic ratio of these planes

$$\frac{P_1Q_2}{P_2Q_2}:\frac{P_1Q_1}{P_2Q_1}=\frac{\sin P_1Q_2}{\sin P_2Q_2}\cdot\frac{\sin P_2Q_1}{\sin P_1Q_1}=\frac{\mu_1-\mu_4}{\mu_3-\mu_4}\cdot\frac{\mu_3-\mu_2}{\mu_1-\mu_2}=\Lambda, \ldots \quad (6)$$

and thus must be rational.

In order to determine the value of the anharmonic ratio in terms of the indices of the planes; since **u** and **v** are perfectly arbitrary planes of the zone, we may take them such that each passes through one of the crystallographic axes: and evidently such origin-planes must be crystalloid, since each contains two zone-lines. Take, then, **u** as passing through axis Y; $\therefore q_1=0$:

v as passing through axis Z; $\therefore r_2=0$.

Hence, from equation (4),

$$\frac{h_1}{p_1+\mu_1 p_2}=\frac{k_1}{\mu_1 q_2}=\frac{l_1}{r_1}, \quad \text{and} \quad \mu_1=\frac{q_1}{q_2}\frac{k_1}{l_1}.$$

Similarly,

$$\mu_2=\frac{r_1}{q_2}\frac{f_1}{g_1}, \quad \mu_3=\frac{r_1}{q_2}\frac{k_2}{l_2}, \quad \mu_4=\frac{r_1}{q_2}\frac{f_2}{g_2}. \quad \ldots\ldots \quad (7)$$

Hence the anharmonic ratio is

$$\left.\begin{aligned} \Lambda=\frac{(\mu_1-\mu_4)(\mu_3-\mu_2)}{(\mu_3-\mu_4)(\mu_1-\mu_2)} &= \frac{\dfrac{r_1^2}{q_2^2}\cdot\left(\dfrac{k_1}{l_1}-\dfrac{f_2}{g_2}\right)\left(\dfrac{k_2}{l_2}-\dfrac{f_1}{g_1}\right)}{\dfrac{r_1^2}{q_2^2}\left(\dfrac{k_2}{l_2}-\dfrac{f_2}{g_2}\right)\left(\dfrac{k_1}{l_1}-\dfrac{f_1}{g_1}\right)} \\ &= \frac{(k_1g_2-l_1f_2)(k_2g_1-l_2f_1)}{(k_2g_2-l_2f_2)(k_1g_1-l_1f_1)} \end{aligned}\right\} \quad \ldots \quad (8)$$

whatever be the values represented by these letters.

In a similar way values may be obtained for Λ involving lh and ge, or hk and ef; and the anharmonic ratio may be written as

$$\left\| \frac{\begin{vmatrix} h_1 & k_1 & l_1 \\ e_2 & f_2 & g_2 \end{vmatrix}}{\begin{vmatrix} h_2 & k_2 & l_2 \\ e_2 & f_2 & g_2 \end{vmatrix}} \right\| : \left\| \frac{\begin{vmatrix} h_1 & k_1 & l_1 \\ e_1 & f_1 & g_1 \end{vmatrix}}{\begin{vmatrix} h_2 & k_2 & l_2 \\ e_1 & f_1 & g_1 \end{vmatrix}} \right\|$$

$$= \frac{\sin P_1 Q_2}{\sin (P_1 Q_2 - P_1 P_2)} \cdot \frac{\sin (P_1 Q_1 - P_1 P_2)}{\sin P_1 Q_1}$$

$$= \Lambda \ldots\ldots\ldots\ldots\ldots\ldots \quad (9)$$

the sign of an arc being indicated by the order of the letters.

The solution of problems involving the symbols and arc-distances of four poles on a zone-circle follows from the last formula, as in Articles **48** and **49**.

Thus where the symbols of four poles and two arcs are given, the position of the fourth pole Q_2 is found from the formula

$$\tan (P_1 Q_2 - \tfrac{1}{2} P_1 P_2) = \tan \tfrac{1}{2} P_1 P_2 \tan (135 - \theta),$$

where

$$\tan \theta = \Lambda \frac{\sin P_1 Q_1}{\sin (P_1 Q_1 - P_1 P_2)};$$

and where the arcs are given and three of the symbols, the symbol of the fourth pole may be otherwise obtained from the expressions in equations (7). For

$$\Lambda = \frac{\sin P_1 Q_2}{\sin P_2 Q_2} \cdot \frac{\sin P_2 Q_1}{\sin P_1 Q_1},$$

and by equations (7),

$$\Lambda = \frac{\begin{vmatrix} k_1 & l_1 \\ f_2 & g_2 \end{vmatrix} \cdot \begin{vmatrix} k_2 & l_2 \\ f_1 & g_1 \end{vmatrix}}{\begin{vmatrix} k_2 & l_2 \\ f_2 & g_2 \end{vmatrix} \begin{vmatrix} k_1 & l_1 \\ f_1 & g_1 \end{vmatrix}} \left\{ = \frac{\begin{vmatrix} l_1 & h_1 \\ g_2 & e_2 \end{vmatrix} \cdot \begin{vmatrix} l_2 & h_2 \\ g_1 & e_1 \end{vmatrix}}{\begin{vmatrix} l_2 & h_2 \\ g_2 & e_2 \end{vmatrix} \begin{vmatrix} l_1 & h_1 \\ g_1 & e_1 \end{vmatrix}} = \frac{\begin{vmatrix} h_1 & k_1 \\ e_2 & f_2 \end{vmatrix} \cdot \begin{vmatrix} h_2 & k_2 \\ e_1 & f_1 \end{vmatrix}}{\begin{vmatrix} h_2 & k_2 \\ e_2 & f_2 \end{vmatrix} \begin{vmatrix} h_1 & k_1 \\ e_1 & f_1 \end{vmatrix}} \right.$$

where such pairs of symbols may be chosen by inspection as will not result in an indeterminate value.

For the symbol $e_2 f_2 g_2$ of Q_2.

Let
$$e_2 = nh_1 - mh_2,$$
$$f_2 = nk_1 - mk_2,$$
$$g_2 = nl_1 - ml_2.$$

Then

$$\Lambda = \frac{\begin{vmatrix} k_1 & l_1 \\ (nk_1 - mk_2) & (nl_1 - ml_2) \end{vmatrix}}{\begin{vmatrix} k_2 & l_2 \\ (nk_1 - mk_2) & (nl_1 - ml_2) \end{vmatrix}} \cdot \frac{\begin{vmatrix} k_2 & l_2 \\ f_1 & g_1 \end{vmatrix}}{\begin{vmatrix} k_1 & l_1 \\ f_1 & g_1 \end{vmatrix}} = - \frac{m \begin{vmatrix} k_1 & l_1 \\ k_2 & l_2 \end{vmatrix}}{n \begin{vmatrix} k_2 & l_2 \\ k_1 & l_1 \end{vmatrix}} \cdot \frac{\begin{vmatrix} k_2 & l_2 \\ f_1 & g_1 \end{vmatrix}}{\begin{vmatrix} k_1 & l_1 \\ f_1 & g_1 \end{vmatrix}}$$

$$= \frac{m}{n} \frac{\begin{vmatrix} k_2 & l_2 \\ f_1 & g_1 \end{vmatrix}}{\begin{vmatrix} k_1 & l_1 \\ f_1 & g_1 \end{vmatrix}}$$

In the same way, or by symmetry,

$$\Lambda = \frac{m \begin{vmatrix} l_2 & h_2 \\ g_1 & e_1 \end{vmatrix}}{n \begin{vmatrix} g_1 & e_1 \\ l_1 & h_1 \end{vmatrix}} = \frac{m \begin{vmatrix} h_2 & k_2 \\ e_1 & f_1 \end{vmatrix}}{n \begin{vmatrix} e_1 & f_1 \\ h_1 & k_1 \end{vmatrix}}. \quad \ldots\ldots\ldots\ldots \quad (10)$$

The multiplier of $\frac{m}{n}$ is thus any one of the fractions obtained by taking as numerator any determinant formed by taking a pair of columns from $\begin{matrix} e_1 & f_1 & g_1 \\ h_2 & k_2 & l_2 \end{matrix}$ and dividing by the corresponding determinant taken from $\begin{matrix} h_1 & k_1 & l_1 \\ e_1 & f_1 & g_1 \end{matrix}$.

The anharmonic ratio of the four planes P_1, Q_1, P_2, Q_2 may be written

$$\frac{\sin P_1 Q_1 \sin P_2 Q_2}{\sin P_1 Q_2 \sin P_2 Q_1} = \frac{[P_1 Q_1][P_2 Q_2]}{[P_1 Q_2][P_2 Q_1]} = \frac{\begin{vmatrix} h_1 & k_1 \\ e_1 & f_1 \end{vmatrix} \begin{vmatrix} h_2 & k_2 \\ e_2 & f_2 \end{vmatrix}}{\begin{vmatrix} h_1 & k_1 \\ e_2 & f_2 \end{vmatrix} \begin{vmatrix} h_2 & k_2 \\ e_1 & f_1 \end{vmatrix}}$$

$$= \text{\&c.} = \lambda. \quad .. \ (11)$$

Since the zones $[P_1 Q_1]$, $[P_2 Q_2]$, $[P_1 Q_2]$, $[P_2 Q_1]$ are identical in position, corresponding indices in any pair of symbols must bear the same ratio: let $[ABC]$ be the *simplest expression* for the zone.

Let $[P_1 Q_1] = \begin{vmatrix} h_1 & k_1 & l_1 \\ e_1 & f_1 & g_1 \end{vmatrix} = [A_1 B_1 C_1] = [\alpha A \ \alpha B \ \alpha C]$,

$[P_2 Q_2] = \begin{vmatrix} h_2 & k_2 & l_2 \\ e_2 & f_2 & g_2 \end{vmatrix} = [A_2 B_2 C_2] = [\beta A \ \beta B \ \beta C]$,

$$[P_1 Q_2] = \begin{vmatrix} h_1 & k_1 & l_1 \\ e_2 & f_2 & g_2 \end{vmatrix} = [A_3 B_3 C_3] = [\gamma A \; \gamma B \; \gamma C],$$

$$[P_2 Q_1] = \begin{vmatrix} h_2 & k_2 & l_2 \\ e_1 & f_1 & g_1 \end{vmatrix} = [A_4 B_4 C_4] = [\delta A \; \delta B \; \delta C].$$

Then from (11),

$$\lambda = \frac{A_1 A_2}{A_3 A_4} = \frac{B_1 B_2}{B_3 B_4} = \frac{C_1 C_2}{C_3 C_4} = \frac{\alpha\beta}{\gamma\delta},$$

and, p, q, r being any three quantities whatever,

$$A_1 A_2 p^2 = \lambda A_3 A_4 p^2,\ B_1 B_2 q^2 = \lambda B_3 B_4 q^2,\ C_1 C_2 r^2 = \lambda C_3 C_4 q^2; \quad (12)$$

$$\left.\begin{aligned} &(B_1 C_2 + B_2 C_1) qr \{= 2\alpha\beta BC qr = 2\lambda\gamma\delta BC qr\} \\ &\qquad\qquad\qquad\qquad = \lambda (B_3 C_4 + B_4 C_3) qr. \\ \text{So,}\ &(C_1 A_2 + C_2 A_1) rp = \lambda (C_3 A_4 + C_4 A_3) rp, \\ &(A_1 B_2 + A_2 B_1) pq = \lambda (A_3 B_4 + A_4 B_3) pq. \end{aligned}\right\} \quad (13)$$

By addition of equations (12) and (13) and resolution into factors,

$$\lambda = \frac{(A_1 p + B_1 q + C_1 r)(A_2 p + B_2 q + C_2 r)}{(A_3 p + B_3 q + C_3 r)(A_4 p + B_4 q + C_4 r)}.$$

Let $[\mathbf{u}_1\, \mathbf{v}_1\, \mathbf{w}_1]$ be $[e_1 f_1 g_1, pqr]$ and $[\mathbf{u}_2\, \mathbf{v}_2\, \mathbf{w}_2]$ be $[e_2 f_2 g_2, pqr]$, pqr being the symbol of any heterozonal pole.

Then
$$A_1 p + B_1 q + C_1 r = p \begin{vmatrix} k_1 & l_1 \\ f_1 & g_1 \end{vmatrix} + q \begin{vmatrix} l_1 & h_1 \\ g_1 & e_1 \end{vmatrix} + r \begin{vmatrix} h_1 & k_1 \\ e_1 & f_1 \end{vmatrix}$$

$$= h_1 \begin{vmatrix} f_1 & g_1 \\ q & r \end{vmatrix} + k_1 \begin{vmatrix} g_1 & e_1 \\ r & p \end{vmatrix} + l_1 \begin{vmatrix} e_1 & f_1 \\ p & q \end{vmatrix} = h_1 \mathbf{u}_1 + k_1 \mathbf{v}_1 + l_1 \mathbf{w}_1.$$

So
$$A_2 p + B_2 q + C_2 r = h_2 \mathbf{u}_2 + k_2 \mathbf{v}_2 + l_2 \mathbf{w}_2,$$
$$A_3 p + B_3 q + C_3 r = h_1 \mathbf{u}_2 + k_1 \mathbf{v}_2 + l_1 \mathbf{w}_2,$$
$$A_4 p + B_4 q + C_4 r = h_2 \mathbf{u}_1 + k_2 \mathbf{v}_2 + l_1 \mathbf{w}_2;$$

and
$$\lambda = \frac{(h_1 \mathbf{u}_1 + k_1 \mathbf{v}_1 + l_1 \mathbf{w}_1)(h_2 \mathbf{u}_2 + k_2 \mathbf{v}_2 + l_2 \mathbf{w}_2)}{(h_1 \mathbf{u}_2 + k_1 \mathbf{v}_2 + l_1 \mathbf{w}_2)(h_2 \mathbf{u}_1 + k_2 \mathbf{v}_1 + l_2 \mathbf{w}_1)} \cdots\cdots \Delta$$

If $[\mathbf{p}_1\, \mathbf{q}_1\, \mathbf{r}_1]$ is $[h_1 k_1 l_1, pqr]$ and $[\mathbf{p}_2\, \mathbf{q}_2\, \mathbf{r}_2]$ is $[h_2 k_2 l_2, pqr]$, by inverting the expression Δ, $\frac{1}{\lambda}$ becomes Λ, and we have the form of the ratio D', as in Art. **46**; and similarly for any other form of the anharmonic ratio.

Section VI.—On Isogonal Zones.

63. *Harmonic division of a zone.* In investigating the relations which connect the planes belonging to a zone, and establishing the principle—which in fact is but a more philosophical form of enunciating the fundamental law of a crystalloid system—that the anharmonic ratio of four tautozonal planes is in such a system rational, we are brought into a position from which we may advance to discuss a problem which involves the whole principle of crystalloid symmetry.

This problem deals with the conditions under which the angle at which any two planes of a zone are inclined on each other may be repeated in the zone.

The simplest case to be considered will be that in which two planes of the zone are equally inclined on a third at an angle ϕ.

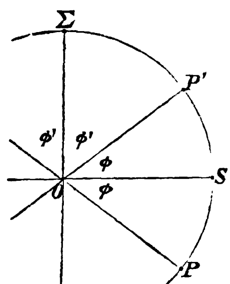


Fig. 28.

Thus, if P and P' be two origin planes the angle between which is bisected by the plane S, as in Fig. 28, where the lines OP, OP', &c. are the traces of these planes on the zone-plane which is that of the figure; it results from the principle of the harmonic division of an angle, which is only a particular case of the anharmonic division, that if Σ be a plane in the zone perpendicular to S,

$$(1)\quad \frac{\sin PP'}{\sin P\Sigma}\cdot\frac{\sin S\Sigma}{\sin SP'} = 2;$$

$$(2)\quad \frac{\sin P\Sigma}{\sin PS}\cdot\frac{\sin P'S}{\sin P'\Sigma} = -1;$$

$$(3)\quad \frac{\sin PS}{\sin PP'}\cdot\frac{\sin \Sigma P'}{\sin \Sigma S} = \frac{1}{2};$$

results easily verified by substituting the values for the angles in the expressions; thus expression (3) is

$$\frac{\sin\phi}{2\sin\phi\,.\,\cos\phi}\cdot\frac{-\cos\phi}{-\sin 90^{\circ}} = \frac{1}{2}.$$

Since the expressions are independent of the value of ϕ and have rational values, we may assert:

1. That where two planes in a zone are equally inclined on a third, a fourth plane perpendicular to the last plane is a possible plane of the zone; and

2. That, in a crystalloid zone containing two perpendicular planes, for every possible plane of the zone another plane may be found equally inclined with it on the two perpendicular planes, which fulfils the condition necessary for its being a plane of the system, namely that of its having rational indices.

In fact if S be (efg) and P and P' be (hkl) and $(h'k'l')$; then,

$$\frac{\sin PP'}{\sin P\Sigma}\cdot\frac{\sin S\Sigma}{\sin SP'} \qquad \frac{PP'}{SP'} \qquad \frac{P\Sigma}{S\Sigma} \qquad \frac{\begin{vmatrix} k & l \\ k' & l' \end{vmatrix} \begin{vmatrix} f & g \\ q & r \end{vmatrix}}{\begin{vmatrix} k & l \\ q & r \end{vmatrix} \begin{vmatrix} f & g \\ k' & l' \end{vmatrix}}$$

and thus $\frac{q}{r}$ is rational; p is determined from the relation

$$p\mathbf{u}+q\mathbf{v}+r\mathbf{w}=0,$$

where $\mathbf{u}\,\mathbf{v}\,\mathbf{w}$ are the indices of the zone $\left[\begin{smallmatrix} h & k & l \\ h' & k' & l' \end{smallmatrix}\right]$. Hence the indices $p\ q\ r$ of Σ are necessarily rational—that is to say, fulfil the *necessary* condition for a possible plane of the system. This however, as will be hereafter seen, is not always a sufficient condition in order that a plane may exist in a symmetrical system of crystalloid planes.

It is obvious that any relations that are established regarding the anharmonic ratios of the planes in a zone apply equally to those of their normals and to the sines of the angles of the arcs joining their poles on the zone-circle.

It will further be evident that what particular values of ϕ may be possible, or whether any two angles between three consecutive planes in a zone can have the same value ϕ, will depend on the nature of the zone, and ultimately on the axial elements to which it is possible for the particular system of planes to be referred.

64. We may next discuss the more general problem regarding the repetition of the same angle of inclination between several consecutive planes in a zone: where however this angle is limited, by the condition that the number of planes in a crystalloid zone cannot be infinite, to such angles as are commensurable with π; so that the recurring angle

$$\phi = \frac{p}{q} 2\pi,$$

where p and q are integers.

And it is obvious that if the anharmonic ratios of four consecutive tautozonal planes inclined at the same angle ϕ are rational, those of three of them and a fifth plane also inclined on one of them at the angle ϕ are rational, and that in fact the whole zone-circle may be divided by n of its origin-planes into $2n$ sectors, the angles of which are each ϕ, i.e. $\frac{p}{q} 2\pi$.

DEF. A zone thus divided by n equally inclined consecutive origin-planes will be termed an *isogonal* zone; and the angular values which ϕ may be able to assume will be termed *crystallometric* arcs or angles.

We shall proceed to prove that the number of possible values for ϕ is extremely limited, and that in fact the only crystallometric angles are 90°, 60°, 45°, and 30°.

It will be only necessary to establish this for the case of four consecutive poles in a zone.

65. The only possible isogonal zones in which the angle (ϕ) between consecutive origin-planes, and therefore between the normals of consecutive faces, is a submultiple of π are those in which ϕ is $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{6}$, and 0°.

Let P, Q, P', Q' be four poles in such a zone-circle, so that

$$PQ = QP' = P'Q' = \phi = \frac{p}{q} 2\pi.$$

Then the anharmonic ratio $\frac{\sin PP' \sin QQ'}{\sin PQ \sin P'Q'}$ is rational, or

$$\frac{\sin 2\phi \,.\, \sin 2\phi}{\sin \phi \,.\, \sin \phi} = \text{a rational quantity} = \lambda;$$

$$\therefore \quad 2\cos 2\phi = \lambda - 2 \text{ and is a rational quantity.}$$

By trigonometry* we have

$$2\cos q2\phi = (2\cos 2\phi)^q + A_2(2\cos 2\phi)^{q-2} + \dots$$
$$+ A_{2r}(2\cos 2\phi)^{q-2r} + \dots,$$

where $q-2r$ is never negative and A_2, A_4, A_6, A_{2r} are all integers.

But $$2\cos q2\phi = 2\cos 4p\pi = 2;$$

$$\therefore\quad (2\cos 2\phi)^q + A_2(2\cos 2\phi)^{q-2} + \dots = 2.$$

Thus $2\cos 2\phi$ must be a rational root of the Equation

$$x^q + A_2 x^{q-2} + A_4 x^{q-4} + \dots = 2.$$

† But if the coefficient of the highest power in a rational algebraical equation be unity and the other coefficients integers, the rational roots must be zero or integers.

Therefore $$2\cos 2\phi = 0, \text{ or } = \pm 1, \text{ or } = \pm 2$$

for any other integer would give an impossible value for $\cos 2\phi$;

$$\therefore\quad 2\phi = \frac{\pi}{2}, \frac{\pi}{3}, \frac{2\pi}{3}, \pi \text{ or } 0^\circ,$$

and $$\phi = \frac{\pi}{4}, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2} \text{ or } 0^\circ.$$

66. If three tautozonal planes in a crystalloid system be inclined on each other at a common crystallometric angle ϕ and any fourth face belonging to the zone be taken; then any planes in the zone inclined on this fourth face at the same crystallometric angle ϕ, or at any angle an integral multiple of ϕ, will fulfil the condition necessary for being possible planes of the system; i.e. their indices will be rational.

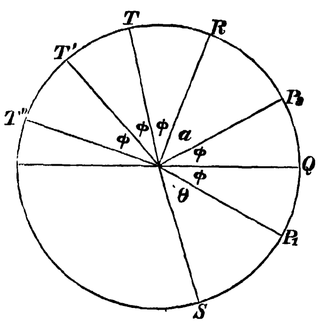


Fig. 29.

Let P_1, Q, P_2, R be four tautozonal planes, the arcs between the poles of which are $P_1Q = QP_2 = \phi$, $P_2R = a$; then, if T be a plane lying in the zone such that $RT = \phi$, T is a possible plane of the system.

* Todhunter, *Trigonometry*, Article 286.

† Todhunter, *Theory of Equations*, 113.

For by the principle of rationality of anharmonic ratios,

$$\frac{\sin P_1 P_2 \sin QR}{\sin P_1 R \sin QP_2} = \frac{\sin 2\phi . \sin(\phi + a)}{\sin(2\phi + a) . \sin\phi} = \lambda,$$

$$= 2 \cos\phi \frac{\sin(\phi + a)}{\sin(2\phi + a)}. \quad \ldots\ldots \quad \text{(i)}$$

Also $$\frac{\sin P_1 Q \sin TP_2}{\sin P_1 P_2 \sin TQ} = \frac{\sin\phi}{\sin 2\phi} \frac{\sin -(\phi + a)}{\sin -(2\phi + a)} = \Lambda,$$

$$= \frac{1}{2\cos\phi} \frac{\sin(\phi + a)}{\sin(2\phi + a)}; \quad \ldots\ldots \quad \text{(ii)}$$

let $2\cos\phi = m$, $\therefore \Lambda = \frac{\lambda}{m^2}$,

which is rational, since λ is by hypothesis rational and m^2 is rational for all crystallometric values of ϕ.

Hence the symbol of T fulfils the condition of rationality in its indices. And similarly another plane T' inclined also at the same angle ϕ on T is a possible plane of the zone, and so on for any more planes.

67. But in the same zone crystallometric angles of 30° and 45° cannot concur; for,

1. If $P_1 Q = QP_2 = \phi = 30$, and $P_1 S = \theta = 45°$, then S is not a plane of the system.

For the anharmonic ratio

$$\frac{\sin P_2 Q}{\sin P_2 P_1} \frac{\sin SP_1}{\sin SQ} = \frac{\sin\phi}{\sin 2\phi} . \frac{\sin\theta}{\sin(\phi + \theta)} = \frac{}{3 + \sqrt{3}},$$

which is irrational. And,

2. If R is, as in the last paragraph, a plane belonging to the zone with P_1, Q and P_2, then T and T' are possible planes of the zone forming with R three planes successively inclined at the same angle ϕ at which the planes P_1 and P_2 are inclined on Q; and therefore a plane inclined on R at 45°, if ϕ be 30°, or at 30° if ϕ be 45°, is not a plane of the system.

It is evident that in a zone containing three or more planes inclined on each other successively at a crystallometric angle, each of these planes will, together with planes perpendicular to them, harmonically divide the zone; and the latter planes therefore fulfil the necessary condition for being possible planes of the zone.

CHAPTER IV.

THEOREMS RELATING TO THE AXES AND PARAMETERS OF A CRYSTALLOID SYSTEM.

SECTION I.—On changing the Axial System to which a Crystalloid Plane-System is referred.

68. IN crystallographic operations it occasionally becomes necessary to change the parametral plane or to refer the crystal to a new set of axes or to effect both operations simultaneously. The expressions necessary for performing these transformations may either be obtained directly by the methods of algebraic geometry or may be deduced from the expression previously obtained for the anharmonic ratios involved in four tautohedral zones.

The latter, the more brief and elegant of these methods, is due to Professor Miller (Tract on Crystallography, § 21).

Both processes will however be given here as affording different points of view of the operations performed.

(i) *To change the parametral plane only.*

69. a, b, c being the original parameters and a', b', c' the new ones as determined by the intercepts of a plane (efg), i.e. by the ratios

$$\frac{a}{e}, \quad \frac{b}{f}, \quad \frac{c}{g};$$

then $$a = a'e, \quad b = b'f, \quad c = c'g,$$

and the intercepts of a plane (hkl) as expressed by the new parameters are

$$a'\frac{e}{h},\quad b'\frac{f}{k},\quad c'\frac{g}{l},$$

and the new indices of the plane are

$$h',\ k',\ l' \quad \text{or} \quad \frac{h}{e},\quad \frac{k}{f},\quad \frac{l}{g},$$

i.e. the symbol is $(hfg,\ \ kge,\ \ lef)$.

If it should happen that any of the indices e, f, g be unity, the corresponding parameter will remain unaltered.

(ii) *To transform a crystalloid system of planes referred to one set of axes to another set of axes formed by edges of the system.*

70. Let the system of planes be referred to axes OX, OY, OZ with a parametral plane ABC, the equation to which is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$

Let OX', OY', OZ' be zone-lines of the system which are to form the new axes; and let the equation to OX' be

$$\frac{1}{u_1}\cdot\frac{x}{a} = \frac{1}{v_1}\cdot\frac{y}{b} = \frac{1}{w_1}\cdot\frac{z}{c} = \frac{1}{\theta_1}.$$

Then the plane ABC will cut the new axis OX' in a point A' in which the coordinates of the plane and line are identical; whence the coordinates of A' are

$$x_1 = a\frac{u_1}{\theta_1},\quad y_1 = b\frac{v_1}{\theta_1},\quad z_1 = c\frac{w_1}{\theta_1},$$

$$\frac{u_1}{\theta_1} + \frac{v_1}{\theta_1} + \frac{w_1}{\theta_1} = 1 \quad \text{or} \quad \frac{1}{u_1+v_1+w_1} = \frac{1}{\theta_1};$$

and
$$x_1 = \frac{a u_1}{u_1+v_1+w_1}. \qquad \text{(1)}$$

The new parameter for the axis X' is OA', which is the diagonal of the parallelepiped of which the sides are the old axial planes, and the edges are the coordinates of A' on the original axes.

If now HKL be a plane of the system the equation to which referred to the original axes is

$$h \cdot \frac{x}{a} + k \cdot \frac{y}{b} + l \cdot \frac{z}{c} = 1;$$

the coordinates of the point H' in which it cuts the new axis of X, that is, cuts

$$\frac{1}{\mathrm{u}_1} \cdot \frac{x}{a} = \frac{1}{\mathrm{v}_1} \cdot \frac{y}{b} = \frac{1}{\mathrm{w}_1} \cdot \frac{z}{c} = \frac{1}{\theta_2},$$

are $$x_2 = a \cdot \frac{\mathrm{u}_1}{\theta_2}, \quad y_2 = b \cdot \frac{\mathrm{v}_1}{\theta_2}, \quad z_2 = c \cdot \frac{\mathrm{w}_1}{\theta_2}.$$

Also $$\frac{hx_2}{a} + \frac{ky_2}{b} + \frac{lz_2}{c} = 1, \text{ or } h\frac{\mathrm{u}_1}{\theta_2} + k\frac{\mathrm{v}_1}{\theta_2} + l\frac{\mathrm{w}_1}{\theta_2} = 1,$$

and $$\frac{1}{\theta_2} = \frac{1}{h \cdot \mathrm{u}_1 + k \cdot \mathrm{v}_1 + l \cdot \mathrm{w}_1}. \quad \ldots \quad (2)$$

But, clearly

$$\frac{OH'}{OA'} = \frac{x_2}{x_1} = \frac{\theta_1}{\theta_2} = \frac{\mathrm{u}_1 + \mathrm{v}_1 + \mathrm{w}_1}{h \cdot \mathrm{u}_1 + k \cdot \mathrm{v}_1 + l \cdot \mathrm{w}_1};$$

similarly, $$\frac{OK'}{OB'} = \frac{\theta_1'}{\theta_2'} = \frac{\mathrm{u}_2 + \mathrm{v}_2 + \mathrm{w}_2}{h \cdot \mathrm{u}_2 + k \cdot \mathrm{v}_2 + l \cdot \mathrm{w}_2},$$

$$\frac{OL'}{OC'} = \frac{\theta_1''}{\theta_2''} = \frac{\mathrm{u}_3 + \mathrm{v}_3 + \mathrm{w}_3}{h \cdot \mathrm{u}_3 + k \cdot \mathrm{v}_3 + l \cdot \mathrm{w}_3},$$

where OK' and OL' are the intercepts of the plane HKL on two zone-lines $\mathrm{u}_2\mathrm{v}_2\mathrm{w}_2$ and $\mathrm{u}_3\mathrm{v}_3\mathrm{w}_3$, taken for the new axes of Y and Z, and cut by the parametral plane ABC in B' and C'. In fact the equation to the plane HKL referred to the new axes

$$\mathrm{u}_1\mathrm{v}_1\mathrm{w}_1, \quad \mathrm{u}_2\mathrm{v}_2\mathrm{w}_2, \quad \mathrm{u}_3\mathrm{v}_3\mathrm{w}_3,$$

is $$\frac{x}{OH'} + \frac{y}{OK'} + \frac{z}{OL'} = 1,$$

that is to say, is

$$\frac{x}{\frac{a'}{h'}} + \frac{y}{\frac{b'}{k'}} + \frac{z}{\frac{c'}{l'}} = 1,$$

in which the denominators are the intercepts, and the indices h', k', l' of the plane HKL as referred to the new axes are the reciprocals of the coefficients in these denominators of the parameters a', b', c', that is to say of OA', OB', OC'.

But $$OH' = \frac{a'}{h'} = \frac{OA'}{h'},$$

wherefore $$h' = \frac{OA'}{OH'} = \frac{h \cdot \mathbf{u}_1 + k \cdot \mathbf{v}_1 + l \cdot \mathbf{w}_1}{\mathbf{u}_1 + \mathbf{v}_1 + \mathbf{w}_1}.$$

Similarly $$k' = \frac{h \cdot \mathbf{u}_2 + k \cdot \mathbf{v}_2 + l \cdot \mathbf{w}_2}{\mathbf{u}_2 + \mathbf{v}_2 + \mathbf{w}_2},$$

$$l' = \frac{h \cdot \mathbf{u}_3 + k \cdot \mathbf{v}_3 + l \cdot \mathbf{w}_3}{\mathbf{u}_3 + \mathbf{v}_3 + \mathbf{w}_3};$$

the zone-axes $\mathbf{u}_1\mathbf{v}_1\mathbf{w}_1$, $\mathbf{u}_2\mathbf{v}_2\mathbf{w}_2$, $\mathbf{u}_3\mathbf{v}_3\mathbf{w}_3$ are the edges of three origin planes $p_1q_1r_1$, $p_2q_2r_2$, $p_3q_3r_3$, so that

$$[\mathbf{u}_1\mathbf{v}_1\mathbf{w}_1] \text{ is } [p_2q_2r_2,\ p_3q_3r_3],$$
$$[\mathbf{u}_2\mathbf{v}_2\mathbf{w}_2] \text{ is } [p_3q_3r_3,\ p_1q_1r_1],$$
$$[\mathbf{u}_3\mathbf{v}_3\mathbf{w}_3] \text{ is } [p_1q_1r_1,\ p_2q_2r_2];$$

$$\mathbf{u}_1 = q_2r_3 - r_2q_3,\quad \mathbf{v}_1 = r_2p_3 - p_2r_3,\quad \mathbf{w}_1 = p_2q_3 - q_2p_3,\ \&c.;$$

values which may be substituted, if required, in the above expressions for h', k', l'.

If only one axis, say the axis of Z be changed, the plane $p_3q_3r_3$, common to the other two unchanged axial zone-lines, remains unchanged and is 001.

Whence

$$\mathbf{u}_1 = q_2,\quad \mathbf{v}_1 = -p_2,\quad \mathbf{w}_1 = 0,$$
$$\mathbf{u}_2 = -q_1,\quad \mathbf{v}_2 = +p_1,\quad \mathbf{w}_2 = 0;$$

and $$h' = \frac{hq_2 - kp_2}{q_2 - p_2},\qquad k' = \frac{hq_1 - kp_1}{q_1 - p_1},$$

$$l' = \frac{h(q_1r_2 - r_1q_2) + k(r_1p_2 - p_1r_2) + l(p_1q_2 - q_1p_2)}{(q_1r_2 - r_1q_2) + (r_1p_2 - p_1r_2) + (p_1q_2 - q_1p_2)}.$$

These expressions give the values for the indices of a plane in a crystalloid system when transformed to a new set of crystalloid lines as axes, but referred to the same parametral plane.

(iii) *If the parameters are to be changed as well as the axes.*

71. Let EFG be the new parametral plane and let efg be its symbol as referred to the original axes. Then the coordinates of

E, the point in which the new axis of X intersects with the new parametral planes, are

$$\frac{a\,u_1}{\theta_3},\quad \frac{b\,v_1}{\theta_3},\quad \frac{c\,w_1}{\theta_3},$$

and
$$\frac{1}{\theta_3} = \frac{1}{e\,u_1 + f\,v_1 + g\,w_1}.$$

Hence

$$\frac{OH_1}{OE_1} = \frac{\theta_3}{\theta_2} = \frac{e\,u_1 + f\,v_1 + g\,w_1}{h\,u_1 + k\,v_1 + l\,w_1};$$

and the indices of the plane HKL as transformed to the new axes, and also to the new parameters, become

$$h' = \frac{h\,u_1 + k\,v_1 + l\,w_1}{e\,u_1 + f\,v_1 + g\,w_1},$$

$$k' = \frac{h\,u_2 + k\,v_2 + l\,w_2}{e\,u_2 + f\,v_2 + g\,w_2};$$

$$l' = \frac{h\,u_3 + k\,v_3 + l\,w_3}{e\,u_3 + f\,v_3 + g\,w_3}.$$

Derivation of the expressions for transformation from those for the anharmonic ratios.

72. Let the plane-system be referred to three new zone-lines as axes, the symbols for which are deduced from any of the faces belonging to their several zones.

Let [**VW**], [**WU**], [**UV**] be zones of which the symbols are

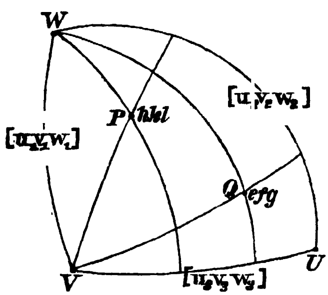


Fig. 30.

$[u_1 v_1 w_1]$, $[u_2 v_2 w_2]$, $[u_3 v_3 w_3]$ as referred to the original axes, and let the new axis of

X be the zone-line $[\mathbf{u}_1\mathbf{v}_1\mathbf{w}_1]$ which accordingly becomes [100],
Y „ $[\mathbf{u}_2\mathbf{v}_2\mathbf{w}_2]$ „ „ [010],
Z „ $[\mathbf{u}_3\mathbf{v}_3\mathbf{w}_3]$ „ „ [001];
$\mathbf{UVW}$ being poles in which the zone-circles intersect.

Let now P be the pole of a plane with the symbol (hkl) under the old axial system, and Q be similarly the pole of a plane (efg); then by the expression in article **51**, which represents the form adopted by Professor Miller for the anharmonic ratio, we have

$$\frac{\sin \mathbf{VW}P}{\sin \mathbf{VW}Q} : \frac{\sin \mathbf{UW}P}{\sin \mathbf{UW}Q} = \frac{\mathbf{u}_1 h + \mathbf{v}_1 k + \mathbf{w}_1 l}{\mathbf{u}_1 e + \mathbf{v}_1 f + \mathbf{w}_1 g} : \frac{\mathbf{u}_2 h + \mathbf{v}_2 k + \mathbf{w}_2 l}{\mathbf{u}_2 e + \mathbf{v}_2 f + \mathbf{w}_2 g}, \dots (1)$$

$$\frac{\sin \mathbf{WV}P}{\sin \mathbf{WV}Q} : \frac{\sin \mathbf{UV}P}{\sin \mathbf{UV}Q} = \frac{\mathbf{u}_1 h + \mathbf{v}_1 k + \mathbf{w}_1 l}{\mathbf{u}_1 e + \mathbf{v}_1 f + \mathbf{w}_1 g} : \frac{\mathbf{u}_3 h + \mathbf{v}_3 k + \mathbf{w}_3 l}{\mathbf{u}_3 e + \mathbf{v}_3 f + \mathbf{w}_3 g}; \dots (2)$$

and if the symbols of P and Q when referred to the new axial system become $(h'k'l')$ and $(e'f'g')$, while those of $[\mathbf{u}_1\mathbf{v}_1\mathbf{w}_1]$, $[\mathbf{u}_2\mathbf{v}_2\mathbf{w}_2]$, $[\mathbf{u}_3\mathbf{v}_3\mathbf{w}_3]$ become [100], [010], [001],

$$\frac{\sin \mathbf{VW}P}{\sin \mathbf{VW}Q} : \frac{\sin \mathbf{UW}P}{\sin \mathbf{UW}Q} = \frac{h'}{e'} : \frac{k'}{f'}, \quad \dots\dots\dots (3)$$

and

$$\frac{\sin \mathbf{WV}P}{\sin \mathbf{WV}Q} : \frac{\sin \mathbf{UV}P}{\sin \mathbf{UV}Q} = \frac{h'}{e'} : \frac{l'}{g'}; \quad \dots\dots\dots (4)$$

where by substitution and division, from (1) and (2),

$$\frac{h'}{k'} \frac{\mathbf{u}_1 e + \mathbf{v}_1 f + \mathbf{w}_1 g}{\mathbf{u}_2 e + \mathbf{v}_2 f + \mathbf{w}_2 g} = \frac{e'}{f'} \frac{\mathbf{u}_1 h + \mathbf{v}_1 k + \mathbf{w}_1 l}{\mathbf{u}_2 h + \mathbf{v}_2 k + \mathbf{w}_2 l}, \quad \dots\dots (5)$$

$$\frac{k'}{l'} \frac{\mathbf{u}_2 e + \mathbf{v}_2 f + \mathbf{w}_2 g}{\mathbf{u}_3 e + \mathbf{v}_3 f + \mathbf{w}_3 g} = \frac{f'}{g'} \frac{\mathbf{u}_2 h + \mathbf{v}_2 k + \mathbf{w}_2 l}{\mathbf{u}_3 h + \mathbf{v}_3 k + \mathbf{w}_3 l}; \quad \dots\dots (6)$$

two equations which are satisfied if

$$\left.\begin{array}{ll} h' = \mathbf{u}_1 h + \mathbf{v}_1 k + \mathbf{w}_1 l, & e' = \mathbf{u}_1 e + \mathbf{v}_1 f + \mathbf{w}_1 g, \\ k' = \mathbf{u}_2 h + \mathbf{v}_2 k + \mathbf{w}_2 l, & f' = \mathbf{u}_2 e + \mathbf{v}_2 f + \mathbf{w}_2 g, \\ l' = \mathbf{u}_3 h + \mathbf{v}_3 k + \mathbf{w}_3 l, & g' = \mathbf{u}_3 e + \mathbf{v}_3 f + \mathbf{w}_3 g; \end{array}\right\} \dots\dots \mathbf{L}$$

which two sets of equations are those given by Professor Miller, and thus comprehend all the processes of transformation. Besides the symbols of the new axial planes in the two axial systems, it is obvious that those of some fourth plane in each axial system must be known. Such a fourth plane serves to give the parametral ratios of the new axial system.

The equations **L** then afford the means of determining either the symbols of planes in the new system when their symbols as referred to the old system are known; or conversely.

Thus, if $(e'f'g')$ and (efg) be the known symbols under the new and the old systems respectively of any plane Q of the system, and $(h'k'l')$ be the symbol of the plane P as referred to the new axial system, (hkl) being its symbol as referred to the old system, then, from the equations **L**, we have

$$\left.\begin{aligned} h' &= e'\frac{\mathbf{u}_1 h+\mathbf{v}_1 k+\mathbf{w}_1 l}{\mathbf{u}_1 e+\mathbf{v}_1 f+\mathbf{w}_1 g}, \\ k' &= f'\frac{\mathbf{u}_2 h+\mathbf{v}_2 k+\mathbf{w}_2 l}{\mathbf{u}_2 e+\mathbf{v}_2 f+\mathbf{w}_2 g}, \\ l' &= g'\frac{\mathbf{u}_3 h+\mathbf{v}_3 k+\mathbf{w}_3 l}{\mathbf{u}_3 e+\mathbf{v}_3 f+\mathbf{w}_3 g}; \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad \mathbf{M}$$

which will be equivalent to the equations previously obtained in the last article in the case where $e'f'g'$ is the face (111).

If then the symbols (efg), $(e'f'g')$ be given as well as $[\mathbf{u}_1\mathbf{v}_1\mathbf{w}_1]$, $[\mathbf{u}_2\mathbf{v}_2\mathbf{w}_2]$, $[\mathbf{u}_3\mathbf{v}_3\mathbf{w}_3]$, the values of $h'k'l'$ can be found from those of hkl, and conversely.

And where Q is the parametral plane (111) under the new system, $e'=f'=g'=1$, and where the parametral plane remains the same for the two systems, only the axes being changed,

$$e=f=g=1=e'=f'=g'.$$

73. Let **U**, **V**, **W**, Fig. 27, be the poles $(p_1 q_1 r_1)$, $(p_2 q_2 r_2)$,

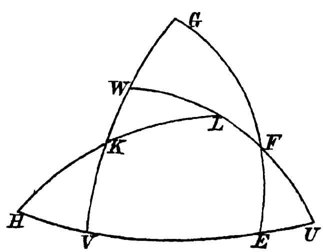


Fig. 31.

$(p_3 q_3 r_3)$; and let two zone-circles $[\mathbf{hkl}]$, $[\mathbf{efg}]$ intersect with the zone-circles $[\mathbf{UV}]$, $[\mathbf{VW}]$, $[\mathbf{WV}]$, viz.

[hkl] with [UV] in H, with [VW] in K, with [WU] in L,
[efg] with [UV] in E, with [VW] in G, with [WU] in F;
then, by D′, article **46**,

$$\frac{\sin UH}{\sin UE} : \frac{\sin VH}{\sin VE} = \frac{p_1 h + q_1 k + r_1 l}{p_1 e + q_1 f + r_1 g} : \frac{p_2 h + q_2 k + r_2 l}{p_2 e + q_2 f + r_2 g}, \quad \ldots \ldots (1)$$

$$\frac{\sin WL}{\sin WF} : \frac{\sin UL}{\sin UF} = \frac{p_3 h + q_3 k + r_3 l}{p_3 e + q_3 f + r_3 g} : \frac{p_1 h + q_1 k + r_1 l}{p_1 e + q_1 f + r_1 g}. \quad \ldots \ldots (2)$$

On referring the plane-system to new axes, let the poles U, V, W become (100), (010), (001) under the new axial system, and let the new symbols of [hkl], [efg] at the same time become [h′k′l′], [e′f′g′].

Then, by substituting and transposing in (1) and (2), we have

$$\frac{h'}{k'} \cdot \frac{p_1 e + q_1 f + r_1 g}{p_2 e + q_2 f + r_2 g} = \frac{e'}{f'} \frac{p_1 h + q_1 k + r_1 l}{p_2 h + q_2 k + r_2 l},$$

$$\frac{h'}{l'} \cdot \frac{p_1 e + q_1 f + r_1 g}{p_3 e + q_3 f + r_3 g} = \frac{e'}{g'} \frac{p_1 h + q_1 k + r_1 l}{p_3 h + q_3 k + r_3 l}.$$

Equations which are satisfied if

$$\left.\begin{array}{ll} h' = h p_1 + k q_1 + l r_1, & e' = e p_1 + f q_1 + g r_1, \\ k' = h p_2 + k q_2 + l r_2, & f' = e p_2 + f q_2 + g r_2, \\ l' = h p_3 + k q_3 + l r_3, & g' = e p_3 + f q_3 + g r_3; \end{array}\right\} \quad \ldots \ldots \ldots \mathbf{N}$$

from which expressions the symbols of any zones in the system may be determined if the symbols of the three poles referred to the old system are known.

The symbol of a zone when the parameters but not the axes are changed.

74. [uvw] being the symbol of the zone [*hkl*, *efg*] as determined by the original parameters, let [u′v′w′] or [*h′k′l′*, *e′f′g′*] be its symbol as determined by the new parameters, and let the new parameters be $a'b'c'$;

$$\mathbf{u\,v\,w} \text{ is } kg - lf, \quad le - hg, \quad hf - ke,$$

$$\mathbf{u'v'w'} \text{ is } k'g' - l'f', \quad l'e' - h'g', \quad h'f' - k'e'$$

and from article **69**,

$$e' = eh'kl, \quad f' = fhk'l, \quad g' = ghkl',$$

$$h' = he'fg, \quad k' = kef'g, \quad l' = lefg';$$

whence $\mathbf{u}' = k'g' - l'f' = (kg - lf)\, e f' g' = \mathbf{u}\, e f' g',$

$$\mathbf{v}' = l'e' - h'g' = (le - hg)\, e' f g' = \mathbf{v}\, e' f g',$$

$$\mathbf{w}' = h'f' - k'e' = (hf - ke)\, e' f' g = \mathbf{w}\, e' f' g.$$

Hence the new symbol of the zone is

$$[\mathbf{u} e f' g', \quad \mathbf{v} f g' e', \quad \mathbf{w} g e' f'].$$

Examples.

I. To transform a crystal of Quartz from the customary axes to a new set of axes such that the pole $(41\bar{2})$ becomes the pole (100) of the axial plane YZ, $(\bar{2}41)$ the pole (010) of the axial plane ZX, and $(1\bar{2}4)$ the pole (001) of the axial plane XY.

Then $[\mathbf{u}_1 \mathbf{v}_1 \mathbf{w}_1]$ is $[\bar{2}41, 1\bar{2}4]$, i.e. $[210]$,

$[\mathbf{u}_2 \mathbf{v}_2 \mathbf{w}_2]$ is $[1\bar{2}4, 41\bar{2}]$, i.e. $[021]$,

$[\mathbf{u}_3 \mathbf{v}_3 \mathbf{w}_3]$ is $[41\bar{2}, \bar{2}41]$, i.e. $[102]$.

The symmetry of quartz is of a type which renders it possible and convenient to refer the crystal to an axial system in which the angles at which the axes are inclined to each other are equal while the parameters also are equal. It results from this that in transforming a crystal of quartz from one axial system of this kind to another of the same kind, the same face serves as the parametral plane for both systems. Consequently,

$$e' = f' = g' = 1 = e = f = g;$$

and the equations **M** become

$$h' = \frac{\mathbf{u}_1 h + \mathbf{v}_1 k + \mathbf{w}_1 l}{\mathbf{u}_1 + \mathbf{v}_1 + \mathbf{w}_1} = \frac{2h + k}{3},$$

$$k' = \frac{\mathbf{u}_2 h + \mathbf{v}_2 k + \mathbf{w}_2 l}{\mathbf{u}_2 + \mathbf{v}_2 + \mathbf{w}_2} = \frac{2k + l}{3},$$

$$l' = \frac{\mathbf{u}_3 h + \mathbf{v}_3 k + \mathbf{w}_3 l}{\mathbf{u}_3 + \mathbf{v}_3 + \mathbf{w}_3} = \frac{2l + h}{3};$$

or $(h'k'l')$ is $(2h+k, \quad 2k+l, \quad 2l+h)$.

Therefore, for the face 100 or r in the lettering of Miller (Brooke and Miller's Mineralogy, p. 246),

$$h' = 2, \quad k' = 0, \quad l' = 1,$$

and the symbol under the new axial system becomes 201; for z, or $\bar{1}22$, the transformed symbol is 063, i.e. 021.

And the new symbols for the following faces,

a, $0 1 \bar{1}$; b', $2 \bar{1} \bar{1}$; b, $\bar{1} 2 \bar{1}$; k, $11\ \bar{4}\ \bar{7}$; $\overline{11}\ 4\ 7$,
become $1 1 \bar{2}$; $1 \bar{1} 0$; $0 1 \bar{1}$; $6 \bar{5} \bar{1}$; $\bar{6} 5 1$;

those for x, $4\ \bar{1}\ \bar{2}$; S', $4 \bar{2} 1$; y, $10\ \bar{2}\ \bar{5}$; u, $8\ \bar{1}\ \bar{4}$,
become $7 \bar{4} 0$; $2 \bar{1} 2$; $2 \bar{1} 0$; $5 \bar{2} 0$;

and those for w, $\overline{14}\ 16\ 7$; $\overline{14}\ 7\ 16$; q, $\overline{16}\ 17\ 8$; $\overline{16}\ 8\ 17$,
become $\bar{4}\ 13\ 0$; $\bar{7}\ 10\ 6$; $\bar{5}\ 14\ 0$; $\bar{8}\ 11\ 6$;
and $17\ 8\ \overline{16}$; p, $\overline{10}\ 14\ 5$, become $14\ 0\ \bar{5}$; and $\bar{2}\ 11\ 0$.

II. A crystal of the (oblique) mineral Sphene as referred to the axial system employed by M. Des Cloizeaux may present the faces

010, 001, $\bar{1}00$, $\bar{1}11$, 110, $\bar{1}02$, 102, 112, 021.

It is required to transform these symbols into accordance with the axial system employed by Professor Miller for this mineral; in which the zone-axis

$\begin{vmatrix} \bar{1}00 \\ 010 \end{vmatrix}$, i.e. $[00\bar{1}]$ is $[u_1 v_1 w_1]$ and becomes $[100]$;
$[010]$ is $[u_2 v_2 w_2]$ and remains $[010]$;
$\begin{Vmatrix} 102 \\ 010 \end{Vmatrix}$, i.e. $[\bar{2}01]$ is $[u_3 v_3 w_3]$ and becomes $[001]$.

The equations **M** here become

$$h' = e'\frac{-l}{-g}, \quad k' = f'\frac{k}{f}, \quad l' = g'\frac{-2h+l}{-2e+g}.$$

And any plane of the system, the symbols for which according to both axial systems are known, will serve to give the ratios of the indices in the new symbol $(h'k'l')$ to which the original symbol of the plane (hkl) has been transformed; or, conversely, will give those of (hkl) the original symbol from the indices of $(h'k'l')$.

The face $\bar{1}11$ on the old system has for its symbol 123 on the new: let (efg) then be $\bar{1}11$, $(e'f'g')$ be 123.

Then $h' = l, \quad k' = 2k, \quad l' = l - 2h.$

Let now (hkl) be 110; then for the new symbol

$$h'k'l' = 02\bar{2}, \text{ i.e. } = 01\bar{1}.$$

So if (hkl) be $\bar{1}02$, $h' = 2$, $k' = 0$, $l' = 4$, and $(h'k'l')$ is 102; i.e. the old symbol $\bar{1}02$ becomes the new symbol 102.

Similarly, 102 becomes 100, 001 becomes 101, and 112 becomes 110.

Conversely, for the face $(h'k'l')$ or 112 of Miller the original symbol would be $(h\,k\,l)$, and we have for this

$$h' = 1 = l, \quad k' = 1 = 2k, \quad l' = 2 = l - 2h;$$

$$\therefore \quad 2h = -1, \text{ and } (h\,k\,l) \text{ is } \bar{1}12.$$

Similarly, 163 on the new system corresponds to $\bar{1}31$ on the old.

III. A crystal of the (anorthic) mineral Axinite is referred under the axial system adopted by Professor vom Rath * to axes which would be the zone-lines of the zones [010, $10\bar{1}$], [$10\bar{1}$, 101], and [101, 010] under the axial system to which Professor Miller refers this mineral. Professor Miller's axes on the other hand are the zone-lines which in vom Rath's system would have the symbols

$\begin{matrix}100\\011\end{matrix}\Big|\Big|$ or [011], i.e. $[u_1 v_1 w_1]$ for the axis of X,

$\begin{matrix}0\bar{1}1\\011\end{matrix}$ or [100], i.e. $[u_2 v_2 w_2]$ „ „ Y,

$\begin{matrix}011\\100\end{matrix}$ or [$01\bar{1}$], i.e. $[u_3 v_3 w_3]$ „ „ Z.

Whence by the equations **M**

$$h' = e'\frac{k+l}{f+g}, \quad k' = f'\frac{h}{e}, \quad l' = g'\frac{k-l}{f-g}.$$

The face 102 under vom Rath's system becomes the parametral plane 111 under Miller's; and therefore efg is 102, and $e'f'g'$ is 111.

Hence, $\quad h' = \dfrac{k+l}{2}, \quad k' = h, \quad l' = \dfrac{l-k}{2}.$

For the new symbols therefore of the face 101 $(h\,k\,l)$ we have

$$h' = \tfrac{1}{2}, \quad k' = 1, \quad l' = \tfrac{1}{2},$$

or 121 is the symbol required; and the symbols on the system of vom Rath,

101, 001, $\bar{1}02$, $\bar{1}20$, 120, $11\bar{1}$, $1\bar{1}1$, 211, 111, $\bar{1}11$,

* Pogg. Annal. cxxviii. 20 and 227.

become

$1 2 1,\ 1 0 1,\ 1 \bar{1} 1,\ 1 \bar{1} \bar{1},\ 1 1 \bar{1},\ 0 1 \bar{1},\ 0 1 1,\ 1 2 0,\ 1 1 0,\ 1 \bar{1} 0$

severally, on the system of Professor Miller.

So also for the symbol $1 \bar{1} 2$ of Miller; since

$$h' = 1 = \frac{k+l}{2}, \quad k' = -1 = h, \text{ and } l' = 2 = \frac{l-k}{2},$$

the symbol under vom Rath's system is $\bar{1} \bar{1} 3$, and Miller's face $0 1 2$ is $1 \bar{2} 2$ with vom Rath's axial system.

Section II.—The axes of a crystalloid system are necessarily origin-edges or face-normals.

75. In establishing the expressions in article **70** for the transformation of an axial system the ratios of the values $u_1 v_1 w_1$ &c. were assumed as rational; these formulae however are entirely general and would be equally true were this restriction not introduced. In that case however the component symbols resulting in the symbol $[u_1 v_1 w_1]$ would not be necessarily rational. It will however be seen that in every case in which the symbols of the planes as referred to the new axes are rational, these axes must necessarily be zone-lines of the system and as such present rational symbols.

In the article referred to it has been proved that if the equation to the new axis X be

$$\frac{1}{u_1} \frac{x}{a} = \frac{1}{v_1} \frac{y}{b} = \frac{1}{w_1} \frac{z}{c},$$

and $h_1 k_1 l_1$ be the transformed symbols of the plane hkl as referred to the new axes, we have

$$h_1 = \frac{h u_1 + k v_1 + l w_1}{u_1 + v_1 + w_1}.$$

or $$(1) \quad \frac{u_1}{w_1}(h_1 - h) + \frac{v_1}{w_1}(h_1 - k) = -h_1 + l;$$

where h_1 and hkl are by hypothesis rational.

In a similar way, if $p_1 q_1 r_1$ be the new rational symbols for a plane pqr, we have

$$(2) \quad \frac{u_1}{w_1}(p_1 - p) + \frac{v_1}{w_1}(p_1 - q) = -p_1 + r;$$

and, solving these two simple equations, we get values for the ratios $\frac{u_1}{w_1}$ and $\frac{v_1}{w_1}$ which must necessarily be rational. Hence any axes for which the law of a crystalloid system of planes holds good are themselves possible zone-lines of the system.

On the reciprocity of a zone-system and a plane-system.

76. It has been previously established that while the intersection of two origin-planes is a zone-axis, that of two zone-planes is a normal to a face of the system; or, which is the same thing, that while two zone-axes lie in an origin-plane, two normals must lie in a zone-plane.

It may further be proved, that whereas in a crystalloid system of planes any set of *origin-edges* may be taken, together with a *face* intersecting with them, for the axial system to which the system of planes and of zone-axes parallel to the edges of these is referred; so, on the other hand, any set of *normals* of faces belonging to the system may be taken together with a parametral *zone-plane*, for the axial system to which may be referred the system of *zone-planes* and of '*rays*' (i.e. of radii of the sphere coincident with normals to the planes of the system) in which these zone-planes intersect.

And the expressions for a zone-axis as referred to three zone-axes with a parametral plane belonging to the system are identical in form with those for a 'ray' as referred to the axial system formed by three normals and a zone-plane. In the one case, for instance, the expression for the zone-axis is that already obtained in art. **53**, namely

$$\frac{x}{\mathrm{u}a} = \frac{y}{\mathrm{v}b} = \frac{z}{\mathrm{w}c};$$

in the other case, that for the ray belonging to the pole hkl is

$$\frac{x}{h\alpha} = \frac{y}{k\beta} = \frac{z}{l\gamma},$$

where α, β, γ are the parameters on a *normal* system of axes, in which the axes are the normals to the axial planes of the former system.

This may be thus proved. If OA, OB, OC be normals to the axial planes YZ, ZX, XY to which a plane-system has been referred, and $(h\,k\,l)$ be the symbol of P a pole referred to these normals as axes, see Fig. 32; then a zone-circle through A and P will intersect the zone-circle BC in a pole D which will have for its symbol $(o\,k\,l)$.

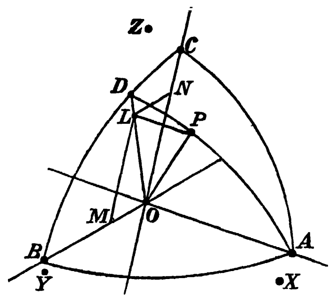


Fig. 32.

The position of a point P on the normal of the plane $(h\,k\,l)$ is determined by the lines LM, LN, LP parallel to the axes (and proportional to the intercepts of the plane); and clearly the points O, L, D are in a right line.

$$\frac{y}{z} = \frac{LN}{NO} = \frac{\sin LON}{\sin OLN} = \frac{\sin DC}{\sin DB};$$

by the fundamental equation (A)

$$\frac{b}{k}\cos DY = \frac{c}{l}\cos DZ,$$

or
$$\frac{b}{k}\sin DC \sin BCA = \frac{c}{l}\sin DB \sin CBA,$$

$$\frac{\sin DC}{\sin DB} = \frac{\frac{kc}{lb}\sin CA}{\sin AB}.$$

Thus
$$\frac{y}{\frac{k}{b}\sin CA} = \frac{z}{\frac{l}{c}\sin AB}, \text{ and by symmetry}$$

$$\frac{x}{\frac{h}{a}\sin BC}:$$

and
$$\frac{x}{h\alpha} = \frac{y}{k\beta} = \frac{z}{l\gamma}, \text{ where}$$

$$\frac{\alpha}{\sin BC} = \frac{\beta}{\sin CA} = \frac{\gamma}{\sin AB}.$$

Hence equation (1) represents the normal or 'ray' of the face P as referred to normal axes; and again, if

$$\frac{x}{h_1 a} = \frac{y}{k_1 \beta} = \frac{z}{l_1 \gamma} \text{ and}$$

$$\frac{x}{h_2 a} = \frac{y}{k_2 \beta} = \frac{z}{l_2 \gamma}$$

be the equations to two such rays, these rays will lie in a plane of which the equation is

$$\mathrm{u}\frac{x}{a} + \mathrm{v}\frac{y}{\beta} + \mathrm{w}\frac{z}{\gamma} = 0;$$

where $\quad \mathrm{u} = kl' - lk', \;\; \mathrm{v} = lh' - hl', \;\; \mathrm{w} = hk' - kh',$

and are rational.

Thus, if OU, OV, OW be the intercepts of a plane parallel to [u v w], then

$$\frac{OU}{\frac{a}{h}} = \frac{OV}{\frac{\beta}{k}} = \frac{OW}{\frac{\gamma}{l}},$$

and a, β, γ are the intercepts of the parametral zone-plane [1 1 1].

The reciprocity in the expressions for a face-normal and for a zone-plane as referred to an axial system of the kind adopted in this treatise, and for a zone-plane and a face-normal referred to such an axial system as has been discussed in this article, is complete. And it will further be seen that the indices are the same for the same face or for the same zone-plane as referred to the two axial systems, though the symbols in which they are embodied have a different significance in the different systems.

Some of the results of this reciprocal relation between the zone-system and the plane-system of a crystalloid system of planes have been examined by Professor Miller in his elegant tract 'On the Crystallographic Method of Grassmann' (Part V of the Proceedings of the Cambridge Philosophical Society, Cambridge, 1868), wherein he has shewn that all the problems of crystallography may be approached from the side of a system of rays referred to normals as axes, and that this method yields expressions identical in form with those which are obtained by the other method.

Case of a zone-axis being coincident with a plane-normal.

77. An important question however arises as to the possibility of a zone-axis being not only a normal belonging to a system of zone-planes reciprocal to the system of actual planes, but being also a normal to one of the faces of this actual system.

It is clear that in general it is not so; i.e. that a zone-axis and a plane-normal, or that a zone-plane and an actual plane, can only in particular cases be coincident.

Those cases will form a subject of enquiry in the Fourth Chapter, in connection with the subject of Symmetry.

It will be well however to investigate here the general conditions to which a plane-system is subject when presenting a plane or planes the normals of which are zone-axes.

Taking the most general case, that namely in which the plane-system is referred to oblique coordinates, let the origin-line lmn perpendicular to the plane

$$Ax+By+Cz=0$$

be

$$\frac{x}{\mathbf{u}a}=\frac{y}{\mathbf{v}b}=\frac{z}{\mathbf{w}c},$$

then the condition for this line to be perpendicular to the plane is that it shall be so to every origin-line $l'm'n'$ lying in the plane.

But for the angle θ between the lines lmn and $l'm'n'$,

$$\cos\theta = ll'+mm'+nn'+(mn'+m'n)\cos\xi + (nl'+n'l)\cos\eta+(lm'+l'm)\cos\zeta,$$

ξ, η, ζ being the axial angles YZ, ZX, and XY; and this expression equals zero, when the lines are perpendicular.

The condition for the line $l'm'n'$ to lie in the plane

$$Ax+By+Cz=0$$

is

$$Al'+Bm'+Cn'=0.$$

We have therefore

$$l+n\cos\eta+m\cos\zeta=kA,$$
$$m+l\cos\zeta+n\cos\xi=kB,$$
$$n+m\cos\xi+l\cos\eta=kC;$$

where

$$k=\frac{1}{Al+Bm+Cn}.$$

By substitution, the equation to the plane to which the line

$$\frac{x}{\mathbf{u}a} = \frac{y}{\mathbf{v}b} = \frac{z}{\mathbf{w}c}$$

is perpendicular is

$$(l + n\cos\eta + m\cos\zeta)\,x + (m + l\cos\zeta + n\cos\xi)\,y + (n + m\cos\xi + l\cos\eta)\,z = 0;$$

that is to say, is

$$(\mathbf{u}a + \mathbf{w}c\cos\eta + \mathbf{v}b\cos\zeta)\,x + (\mathbf{v}b + \mathbf{u}a\cos\zeta + \mathbf{w}c\cos\xi)\,y + (\mathbf{w}c + \mathbf{v}b\cos\xi + \mathbf{u}a\cos\eta)\,z = 0.$$

Comparing this expression for the plane in question with the equation to an origin-plane parallel to the plane (efg), i.e. with

$$\frac{e}{a}x + \frac{f}{b}y + \frac{g}{c}z = 0,$$

where $\frac{e}{a}, \frac{f}{b}, \frac{g}{c}$

are the inverse ratios of the intercepts on the axes of the plane (efg); we find for the case where the zone-axis [uvw] is normal to the plane (efg),

$$\left.\begin{aligned}\frac{\mathbf{u}a + \mathbf{w}c\cos\eta + \mathbf{v}b\cos\zeta}{\frac{e}{a}} &= \frac{\mathbf{v}b + \mathbf{u}a\cos\zeta + \mathbf{w}c\cos\xi}{\frac{f}{b}} \\ &= \frac{\mathbf{w}c + \mathbf{v}b\cos\xi + \mathbf{u}a\cos\eta}{\frac{g}{c}}\end{aligned}\right\} \dots\dots \mathbf{P}$$

Hence if we assume any rational values for efg, by these equations we may determine the symbol of a zone-plane [uvw] parallel to the face (efg), but only in the case of **u v w** being also rational will [u v w] be a zone-plane of the system.

CHAPTER V.

ON THE VARIETIES OF SYMMETRY POSSIBLE IN A CRYSTALLOID SYSTEM OF PLANES.

SECTION I.—**Application of the principles of Geometrical Symmetry to crystals and crystalloid plane-systems.**

78. *Symmetry* in Nature consists in the rhythmical recurrence of a morphological element, that is to say, the repetition of the element in accordance with a law of arrangement.

In animal and vegetable organisms the conditions of growth appear never, or rarely, to be compatible with that geometrical exactitude in the distribution of morphological features which is requisite for their treatment by a geometrical science.

Crystallography, which treats of the morphology of inorganic nature, offers, on the other hand, in the persistence of the angular inclinations of the corresponding faces of crystals the means of bringing the subject of their symmetry within the domain of Geometry.

79. *Geometrical symmetry of plane figures.* In Geometry a plane figure is said to be symmetrically divided by a straight line (as a 'line of symmetry') when a perpendicular falling on this line from each point in the figure meets at an equal distance beyond the line a point corresponding to the first point.

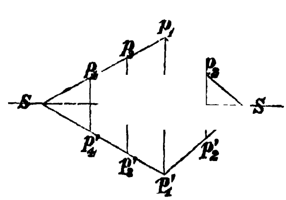


Fig. 33.

If either of the halves into which the line of symmetry S divides the figure be reverted round that line of symmetry, as an axis of revolution, through an angle of 180°, it falls into congruence with the other half.

A plane figure may also be symmetrical to a point as a 'centre of symmetry.' This point will then bisect all the lines traversing it that join the several points of the figure with points corresponding to them. Any one such line divides the figure into two halves which become directly congruent by the revolution of the figure in its own plane through an angle of 180° round the centre of symmetry.

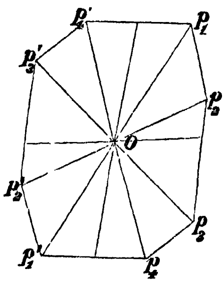


Fig. 34.

A plane figure may furthermore be symmetrical with regard to a point within it as a *pole of symmetry* when by successive revolutions in its own plane through an angular distance of $\frac{2\pi}{n}$ round that point the figure in each new position is congruent with itself as seen in its original position.

If $n = 2$ the plane figure is symmetrical to a centre, and where $n = 2$, or $= 3$, 4, or 6, the symmetry may be defined as being *diagonal, trigonal, tetragonal,* or *hexagonal.*

And a plane figure may be simultaneously symmetrical to two or more lines of symmetry and to a pole of symmetry.

DEF.—A face of a crystal or any other plane surface or figure symmetrical to one line will be said to be *euthysymmetrically* divided by that line, as by S in Fig. 35; where it is symmetrical

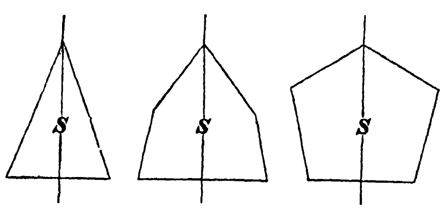


Fig. 35.

to two lines perpendicular to each other it will be said to be *orthosymmetrically* divided by these lines.

An isosceles triangle, a deltoid, a symmetrical (as distinguished from a regular) pentagon, Fig. 35, are euthysymmetrical figures;

and a rhomb, Fig. 36, is orthosymmetrical to its diagonals, as a rectangle is to diameters parallel to its sides; and so the polygon in Fig. 36 is also orthosymmetrical to the lines S and Σ.

80. *Symmetry of solid figures.* In an analogous manner a solid figure may be symmetrical to one or to several *planes of symmetry*, to an *axis of symmetry*, or to a *centre of symmetry;* or simultaneously to several of these.

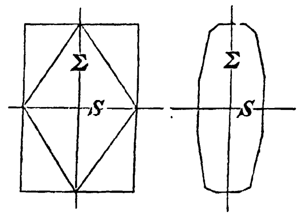


Fig. 36.

It is so to a plane of symmetry when corresponding points equidistant from the plane would lie on any line drawn perpendicularly to the plane. Where the solid figure presents symmetry to only a single plane (and not to a centre also) the corresponding portions of its surface cannot be brought by reversion into congruence. They are to each other as either would be to its own image if seen reflected by the plane of symmetry as by a mirror.

Def.—Such a correspondence of form will be termed *antistrophe*, and such figures will be said to be *antistrophic* to each other.

A solid figure is symmetrical to an axis when every *radius vector* moving in a plane perpendicular to the axis and meeting a point of the figure would also meet corresponding points at the same distances from the axis at each revolution through an arc-angle of $\frac{2\pi}{n}$.

The aspect of such a solid figure will not therefore be changed by a revolution of the solid round this axis through the angle $\frac{2\pi}{n}$, and any portion of its surface so revolving will move into a position in which it will be congruent with another portion of the surface entirely corresponding to it.

Def.—Congruence of this kind will be termed *metastrophe*, and such corresponding parts will be said to be *metastrophic* to each other.

Def.—Where $n = 2$, or 3, or 4, or 6, the axis is one of *diagonal*, of *trigonal*, of *tetragonal*, or *hexagonal* symmetry.

A solid figure may likewise be symmetrical to a point, or centre of symmetry. Corresponding points will lie equidistant from this centre on straight lines traversing it.

Def.—Such a figure will be termed *centrosymmetrical.*

Other varieties of symmetry may be imagined: such, for instance, as a spiral symmetry resulting from symmetry round an axis along as well as around which the radius vector would be supposed to move at a given relative rate of motion.

Application of principles of symmetry to crystals.

81. *Equipoised polyhedra.* It would obviously be futile to attempt to apply these geometrical definitions directly to the interpretation of such symmetry as crystals may exhibit. Crystals, in fact, only in exceptional cases present any such complete geometrical symmetry, since the magnitudes and the distances of their faces from any point or planes within the crystal follow no law.

But in the distribution on the Sphere of Projection of the Poles (which represent the relative directions in space) of the faces of a crystal, as also in the relative directions of origin-planes drawn parallel to the faces, we have the means of establishing that a crystal is, in a sense which is not the less real because somewhat more elastic than the strictly geometrical sense, a symmetrical polyhedron. The poles of its faces will in fact be found to be symmetrically distributed on the sphere, and inasmuch as any planes parallel to the actual faces of the crystal would equally represent those faces in a crystallographic sense, we may conceive of a polyhedron so constructed that while its faces were all parallel to those of the crystal, such of these as correspond symmetrically, that is to say which represent the same repeated face of the crystal, should be taken at equal distances from the point within the crystal which is the centre of the Sphere of Projection. An imaginary polyhedron of such a nature is said to be *in equipoise.*

The outline figures in which crystals are commonly represented by the projection of their edges are drawn in equipoise; though faces of different kinds are generally projected with differently proportioned magnitudes.

Before discussing the results established by observations made upon actual crystals it will be well to determine what different sorts of symmetry it is possible that a crystalloid polyhedron may present, and the character of the limitations that the law of Rationality of Indices must impose on the number of such varieties of symmetry; the expression 'symmetry' being of course understood in the sense just defined.

82. *Crystalloid symmetry. Two planes*, whether referred as origin-planes to the centre or taken as faces of a polyhedron, will be *symmetrical in respect to or 'on' a third* (which it will be preferable to consider as an origin-plane) when they are tautozonal with the third plane and the dihedral angle which forms their edge is bisected by it. Their poles lying on the same side of the plane of symmetry will be equidistant from either one of its poles, those on opposite sides of it will be so from opposite poles of that plane.

Evidently the two faces are antistrophically symmetrical: and their edge lies in the plane of symmetry.

DEF.—Two poles or planes thus symmetrically disposed in regard to an origin-plane will be termed *homologous to each other in respect to that plane of symmetry;* and this term will be extended to embrace all the planes or poles or other features which in a system of planes correspond to each other *as being symmetrically repeated*, whether in respect to one or more planes, or to an axis or axes, or to a centre of symmetry.

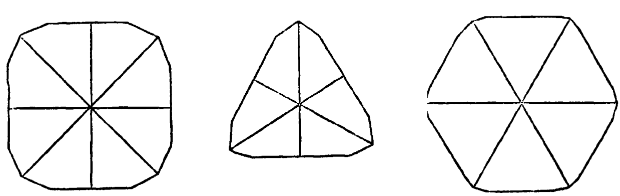
DEF.—A plane or its pole will be termed an *independent plane or pole* when the pole is not the pole of a plane of symmetry, and does not lie on the great circle in which a plane of symmetry intersects the sphere of projection.

DEF.—In a crystalloid system a group of homologous planes will be comprised under the term '*a form.*' The general symbol of a form will be represented by the indices of one of the planes of the form enclosed in brackets, e.g. $\{hkl\}$, $\{110\}$, &c.

DEF.—The great circle in which a plane of symmetry intersects the Sphere of Projection will be occasionally termed *a circle of* *symmetry*, and the axial point of an Axis of Symmetry *a pole of symmetry;* such an axis will be frequently designated by the symbol of or by a letter indicating a face-pole in which it will be found to meet the Sphere of Projection.

The adjective form *symmetral* will further be employed as a convenient expression as applied to a plane, an axis, &c. of symmetry. And we may speak of an axis or pole of symmetry being diagonally symmetral, or diasymmetral, or orthosymmetral, tri-, tetra-, or hexa-symmetral, according as the face or zone of which it is the pole or the axis presents diagonal, orthosymmetrical, trigonal, tetragonal, or hexagonal symmetry.

83. *Di-n-gonal symmetry.* But the most general and frequent kind of symmetry that we have to deal with is that in which an axis of a zone is an axis of symmetry by reason of the symmetral character of a certain number of the planes belonging to the zone. In such cases the recurrences of each feature take place in pairs from repetition over each plane of symmetry; but so that adjacent features are antistrophically, alternate features metastrophically, repeated. It thus will happen that the zone-axis of n tautozonal planes of symmetry will only be an axis of n-gonal symmetry, notwithstanding that any single feature occurs $2 \times n$ times. For instance, to take illustrations from plane-figures, those known as the symmetrical hexagon, symmetrical octagon, or symmetrical dodecagon (as distinguished from the regular figures), and of which indeed all the sides but only the alternate angles are similar, present as do the polygons in Fig. 37 only trigonal, tetra-



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gonal, and hexagonal symmetry when considered solely in the aspect of symmetry round the centre of the figure as a pole of symmetry, while the sides are repeated in three, four, or six pairs.

It will be in fact observed that the same distribution of features in n *pairs* round an n symmetral axis, resulting from the influence of n planes of symmetry, is virtually extant, even in such figures as the equilateral triangle, regular hexagon, &c.; since the adjacent halves of their edges or angles as well as of the faces themselves may be considered as due to repetition over three, six, &c. planes of symmetry.

Where homologous elements of form are thus coupled or repeated in pairs, the symmetry may be designated as *di-trigonal*, *di-tetragonal*, and *di-hexagonal* in character.

It is obvious that a polyhedron will equally, with a plane figure, present such $2n$-gonal symmetry in the distribution of its morphological features around an axis where that axis is the zone-line in which n planes of symmetry intersect.

It is to be observed that, for reasons which will be developed when the principles of crystalloid symmetry are further discussed, n cannot exceed 6 and can never be 5, pentagonal symmetry being precluded by those principles. The orthosymmetry which is the result of two supplemental planes of symmetry perpendicular to each other, to the exclusion of any other plane of symmetry in the zone, is distinguished from merely diagonal symmetry by this duplication of the recurring features.

84. *Morphological features of crystals.* The morphological features of a crystalloid polyhedron are recognised in its *edges*, its *faces*, and its solid angles or (its coigns or) *quoins*. Two edges may be accounted in a *geometrical sense* as *similar* when their dihedral angles are the same, and when further, for any plane in the zone of or otherwise inclined on the one edge, a plane is also possible in the zone of or inclined on the other edge in such manner that the angular relations of this second plane to the planes forming the latter edge are the same as those of corresponding planes in the case of the former edge.

Since a face of a polyhedron is bounded by edges, two crystalloid faces are geometrically similar when they are bounded by the same number of actual or possible edges severally similar and inclined at the same plane-angles, each to each. Such similar planes distributed as they are on the surface of an equipoised polyhedron, may be considered as plane figures irrespective of their relative position in space, capable of being brought into congruence either by direct superposition or by superposition after retroversion: i.e. they are either metastrophic by the imagined congruence of the inner surface of the one with the outer surface of the other; or they are mutually antistrophically congruent by the supposed contact of their outer or inner surfaces.

The faces of a form will, by the fact of their being mutually homologous, fulfil these conditions of similarity.

In a crystalloid plane, edges that are symmetrically repeated will be thus far similar, and such directions on the plane will be similar as are equally inclined on similar edges.

85. *Similar faces.* In order that two faces *on a crystal* may be *crystallographically* similar, their *physical properties* as *well as their geometrical characters* will have to be identical in respect to similar directions on the two faces. These must also be identical along similar directions on the same face. And this will also be true in respect of any sectional planes carried in similar directions through a crystal; the normals of such arbitrary planes being similarly inclined on similar edges.

N.B. We shall have hereafter to discriminate between this geometrical similarity of features in the case of a crystalloid polyhedron and the crystallographic similarity involving identity of physical characters also, in that of a *crystal.*

86. *Quoins: their symmetry. Similar quoins.* Quoins are of different kinds. Thus the faces which meet in a quoin may have all their poles symmetrically distributed on the same small circle of the sphere. In such a case the diameter of the sphere passing through the pole of the small circle will, if continued, meet the vertex of the quoin, and will be an axis of symmetry to it.

And such a quoin may be formed of groups of faces of which the poles are symmetrically disposed on different small circles having a common pole.

Such quoins may evidently present the various kinds of crystalloid symmetry that are possible round an axis of symmetry.

And on the other hand there may be quoins that are symmetrical to one plane passing through their vertex, or again that are devoid of any symmetry at all.

Two quoins of a crystalloid polyhedron will be *similar* if composed of the same number of actual or possible edges severally similar and inclined at the same angles, each on each.

87. *Octants as quoins.* *An octant* formed by three axial planes is a trihedral quoin transferred to the origin. Any two of its edges, i.e. of the axes of the system, are therefore similar when they are equally inclined on the third axis, and when the intercepts of a plane, actual or possible, upon these axes are the same.

Of the octants into which the axial planes divide space any two will be similar when their respective elements are the same; that is to say when the corresponding axes are inclined at the same angle in both octants.

88. *Faces in parallel pairs.* Since every origin-plane has two poles lying at the opposite extremities of its normal on the sphere of projection, while the indices in the symbol of such a plane differ only in having opposite signs; we must consider the two faces on either side of and parallel to the origin-plane as equally representing that plane in the polyhedral system in the absence of any special geometrical condition to the contrary.

Def.—A polyhedron or a 'form' of a polyhedron presenting both the planes thus parallel to but on opposite sides of each of its origin-planes will be termed a *diplohedral* form or polyhedron. Where for each of its origin-planes the system or a form belonging to it has only one plane extant parallel to the origin-plane, the system or form will be termed *haplohedral.*

It is evident that a centrosymmetrical system of planes must be *diplohedral*, and *vice versâ.*

89. *Planes of symmetry in a zone presumably supplementary.*. The consequence of the anharmonic ratios of four tautozonal planes in a crystalloid zone becoming harmonic in the case where two of the planes are symmetrical to a third, has been alluded to in article **63**.

It was there shown that a plane of symmetry in a zone implies a second or supplementary plane perpendicular to the first, which is also a possible plane of the system, and like the first plane invests with rationality the symbol of any plane symmetrical in respect to it with some actual plane belonging to the zone. We have at present only to deal with this general statement regarding the symmetry of a zone to one of its planes, or to two perpendicular planes in it.

If in a zone the planes are *diplohedral*, the plane perpendicular to a plane of symmetry will clearly be *actually* symmetral: in a *haplohedral* zone one of these planes is actually, while the plane supplementary to it is only *potentially* symmetral, the zone being as it were incomplete and its symmetry so far in abeyance.

SECTION II.—**Conditions for a crystalloid system of planes to be symmetrical to one of its planes.**

90. *A plane of symmetry is parallel to a possible face.* (i) If, in a crystalloid system, an origin-plane S be at once a zone-plane and a plane of the system, it is a possible plane of symmetry to the system.

Let s, Fig. 38, be the pole of the plane S, p be the pole of any plane P of the system. Then the zone-circles $[ps]$ and $[S]$ will intersect in m a pole of a possible plane of the system.

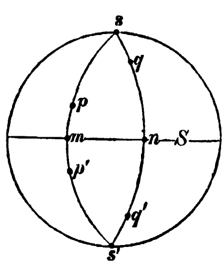


Fig. 38.

If now $mp' = mp$, the poles s and m harmonically divide the zone-circle $[pp']$, and p' is a possible plane of the system. Thus, for any pole existing in the system, another pole symmetrical with it in respect to the plane S is a possible pole of the system. The entire plane-system must therefore be potentially symmetrical to S.

(ii) Conversely, any symmetry plane of the system must be at once a possible zone-plane and possible plane of the system.

Let pp' and qq' be two pairs of poles symmetrical to a plane S, Fig. 38. Then the zone-circles $[pp']$, $[qq']$ will intersect in a possible pole of the system. But they intersect in s the pole of the great circle in which S intersects the sphere. S is therefore a possible plane of the system. And if the arcs pp', qq' are bisected in m and n, the zone-circles $[pp']$ and $[qq']$ are harmonically divided by s and m and s and n respectively: and therefore m and n are possible poles of the system. Therefore $[mn]$ is a possible zone of the system. But m and n lie in the plane S which must therefore be at once, potentially, a zone-plane and a plane of the system.

The necessary conditions therefore for a crystalloid system of planes to be symmetrical to a plane are, that this plane shall be at once a plane of the system and a zone-plane; or, they may be expressed in the form, that two planes of the system are perpendicular to a third plane of it.

91. *Restrictions imposed on a crystalloid system by its being symmetrical to one only of its planes.* In the general case considered in the last article, if the arc $mn = \frac{\pi}{2}$, the origin-planes M and N, of which m and n are the poles, will be perpendicular to each other and will become potentially planes of symmetry orthosymmetrically dividing the zone $[S]$; but also each becomes a zone-plane, M of a zone $[PP']$, N of $[QQ']$. They are therefore potentially planes of symmetry to the entire polyhedral system: a condition inconsistent with the uniqueness of S as a plane of symmetry to that system.

Evidently therefore, if S is to be the only plane of symmetry of the system, no two planes belonging to the zone $[S]$ can be perpendicular to each other.

Nor, it may be added here, can any two of the planes belonging to the zone, the zone-plane of which is the primarily assumed plane of symmetry, be inclined on a third plane at any other crystallometric angle: for it will hereafter become apparent that if such were the case, the zone lying in the plane of symmetry would itself present trigonal, tetragonal, or hexagonal symmetry, according to the value of the particular crystallometric angle between the planes. And the symmetry of the whole plane-system will be found to follow that of the zone in question, so that we should pass from the symmetry of the plane-system to a single plane on to a kind of symmetry of a much higher order.

92. *Axial-system where there is one plane of symmetry. Character of a form.* The two conditions that a crystalloid system symmetrical to one of its planes must satisfy are most simply embodied in an axial system wherein the plane of symmetry *S*, and any pair of planes such as *M* and *N*, Fig. 39, perpendicular to it but inclined on each other at an angle generally greater than 60° and less than 90°, are taken as the axial planes, so that the axes

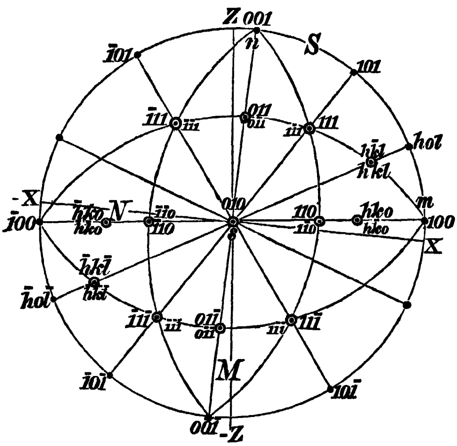


Fig. 39.

are the normal of *S* taken for the axis *Y* and the intersections of the planes *M* and *N* with the plane *S* which are taken for the axes *Z* and *X*. And, whereas these two planes are necessarily oblique in their inclination on each other, the obtuse angle β which represents their inclination (and cannot be of crystallometric value) is taken for the positive angle; that namely of XOZ and $\overline{X}O\overline{Z}$; the supplemental angle $\beta'(=\pi-\eta)$ being taken for that of $\overline{X}OZ$ and $XO\overline{Z}$. The symbol of the plane of symmetry is then 010, and its normal, the axis *Y*, will coincide with the axis of the zone containing the poles 100 and 001—that is to say, with the zone-axis [010].

The planes M and N will have for their symbols ± 100 and ± 001, and their poles will lie on the great circle of symmetry S, distant from each other by an arc β'.

The parametral ratios may be provided either by a single plane intersecting with all three axes, or by a plane in each of two out of the three zones [MS], [NS], or [MN].

The axial-system is thus represented by the expression

$$\xi = 90^\circ = \zeta, \quad \eta > 90^\circ < 120^\circ, \quad a : b : c,$$

wherein two out of the five elements are fixed and the remaining three are unfettered by conditions.

The first, fourth, fifth, and eighth octants, viz.

$$XYZ, \ X\bar{Y}Z, \ \bar{X}\bar{Y}\bar{Z}, \ \bar{X}Y\bar{Z}$$

are similar: so are the remaining four octants adjacent to them.

The poles of an independent form $\{hkl\}$ will lie on a great circle passing through the pole 010 and the diplohedral form will have four faces. If the pole in which this great circle containing the poles of the form intersects with the zone-circle [010] lie between 001 and 100, the symbol is $\{hkl\}$, and the four faces are $h\,k\,l$, $h\,\bar{k}\,l$, $\bar{h}\,k\,\bar{l}$, $\bar{h}\,\bar{k}\,\bar{l}$; if it intersects with the zone-circle [010] in a pole lying between 001 and $\bar{1}00$, the symbol of the form is $\{\bar{h}\,k\,l\}$ and its faces are $\bar{h}\,k\,l$, $\bar{h}\,\bar{k}\,l$, $h\,k\,\bar{l}$, $h\,\bar{k}\,\bar{l}$. A pole lying on the zone-circle of symmetry [S] will belong to a form $\{h\,0\,l\}$ or to a form $\{\bar{h}\,0\,l\}$ which will comprise only two parallel faces.

A form $\{h\,k\,0\}$ will have four faces the poles of which lie on the great circle [100, 010]. The four poles of a form $\{0\,k\,l\}$ will lie in the great circle [010, 001]. The normal to the plane of symmetry 010 is an axis of diagonal symmetry when the system of planes is diplohedral.

SECTION III.—Conditions involved in a crystalloid polyhedron being symmetrical to more than one of its planes.

93. Where a plane-system is symmetrical to more than one plane it is obvious that not only must each pole or plane or other actual morphological feature, but that also each zone-circle and each plane of symmetry, must be virtually as such repeated over each and every plane of symmetry of the system. Where one plane of symmetry S_1 is so repeated over a second plane of symmetry Σ the third plane of symmetry S_2 is originated; and the distribution of all the features of the system, as illustrated for instance by the poles belonging to a form, will necessarily be identical when viewed as circumjacent to the two planes of symmetry S.

But the distribution of these features of the system in respect to either of the groups of planes will not be the same, nor will it ever be so, in respect to two adjacent planes of symmetry.

DEF.—Where the situation of a pole or where the distribution of the poles of a form is different when considered as circumjacent to one or to another of two planes of symmetry, these two planes are said to present *unconformable symmetry;* where the situations of the poles fall into congruence by such a revolution of the system round the zone-axis of the two planes as brings one of the planes of symmetry into coincidence with the position previously occupied by the other, the two planes of symmetry will be termed *similar*, or of *conformable*, or also of *congruent symmetry*.

Further, it may happen in certain cases that the distribution of the features of a crystal may be unconformable in respect to the two halves into which each of three planes of symmetry is divided by the common zone-axis; so that for instance three similar zone-circles, as in Fig. 45, article **115**, may present conformability in the symmetry due to alternate hemizones. Such zone-circles or planes of symmetry will be termed *hemicyclically conformable* in the symmetry they govern.

94. *Nomenclature for planes of symmetry.* DEF.—The plane or planes to which a plane-system is symmetrical will hereafter be called its *systematic* planes; where it is symmetrical to different planes or groups of planes not conformable in their symmetry, these several planes or groups of planes are designated as *proto-systematic*, *deutero-systematic*, and *trito-systematic* planes or groups of planes: and in this treatise these designations will correspond to the letters S, Σ, and C by which the different planes and groups of planes of symmetry are denoted.

95. *Planes of symmetry are inclined at crystallometric angles.*

It follows from what has been said regarding the mutually repetitive character of planes of symmetry, that two planes S_1 and Σ_1, reflected each by the other in planes S_2 and Σ_2, &c., form a zone of planes of symmetry in which the planes S alternate with the planes Σ, and the inclinations of the planes on each other are equal.

This equality in the angles at which successive planes of symmetry are inclined to each other, leads directly to the necessity of these angles having only crystallometric values; and indeed having only a single crystallometric value in the case of planes of symmetry lying in any particular zone. And it is further evident that a zone cannot be symmetrical simultaneously to two independent sets of planes.

Were the angles in question not commensurate with π, these angles would continue to recur in the zone through each successive revolution round the zone-axis, while also in each such revolution new series of planes of symmetry inclined on each other at new angles of inclination would present themselves until the number of such planes would become indefinitely great, and the symmetrical character of the zone would entirely disappear.

The necessity will thus become apparent for the limitation which was imposed in article **65**, on the character of crystallometric angles, whereby they were confined to such as were commensurate with π and not greater than $\frac{\pi}{2}$.

96. And this necessity for the angles between planes of symmetry being crystallometric is no less imperative in the case of a plane system than it is in the case of the symmetry of a zone; so that we have to recognise that the condition, necessary and sufficient for a single plane to divide the plane system symmetrically—namely, that it shall be simultaneously a zone-plane and parallel to a face—is no longer sufficient in the case where a second plane is, simultaneously with the former plane, a plane of symmetry; but that this has to be supplemented by the condition that when a certain plane is established as a plane of symmetry to a zone or plane-system, any other plane or planes of symmetry can only be inclined on it at crystallometric angles.

97. *Relations between the degree of symmetry of a crystalloid plane-system and the axial systems to which it may be referred.* The equations *P*, obtained in article **77**, implicitly contain the conditions under which a crystalloid plane-system may present symmetry to one or more of its planes.

It has been proved that a plane of symmetry must be parallel to a face of the system, and that when there are more planes of symmetry than one these must be inclined on each other at crystallometric angles. The equations *P* give the conditions necessary for a *zone-axis* to be perpendicular to a *face*, and it is obvious that these equations will become greatly simplified in cases where the plane-system can be referred to axes that are rectangular and where two or all three of the parameters are equal.

On the other hand it is evident that where there is a plane of symmetry the plane system is capable of being referred, as in art. **92**, to axes whereof one is perpendicular to the other two.

We might proceed to enquire what would be the conditions under the different varieties of axial systems to which a crystalloid polyhedron might be referred, in order that the normal of a face and a zone-axis may be coincident in direction.

But hereafter, when the changes of volume accompanying changes of temperature in *a crystal* are discussed, it will be shewn that, in a crystal as distinguished from a crystalloid polyhedron, the parametral ratios can never be permanently rational except where one or both of them is unity; and that the cosines of the axial angles are equally only capable of being momentarily rational where the axes containing them are not coincident with the normals of actual or possible planes of the system, or, which is an equivalent statement, do not lie in an isogonal zone.

98. It will be sufficient, then, for our purpose, to consider here the cases arising under each kind of axial system in which zone-planes will be parallel to faces of the system; i.e. the conditions under which the substitution of the designated values for the axial elements in the equation **P**, article **77**, gives rational indices for the coincident normals and zone-axes. And in fact it will be seen that with the selected axial elements this resolves itself into determining all the cases in which the equations **P** will reduce to the form

$$\mathbf{u} \quad \mathbf{v} \quad \mathbf{w}$$

i.e. to the cases in which the zone and the face to which it is parallel have the same indices in their symbols.

I. Clearly one series of values by which this condition is satisfied will be that in which the plane-system is capable of being referred to an axial-system in which the axial-angles are

$$\xi = \eta = \zeta = 90^\circ,$$

and the parameters are equal, i.e.

$$a = b = c.$$

Then
$$\frac{\mathbf{u}}{e} = \frac{\mathbf{v}}{f} = \frac{\mathbf{w}}{g}$$

is true *for the symbol of every plane and of a possible zone-line normal to it,* in the system.

II. If, the axes being as before rectangular, two only of the parameters are equal, for instance

$$a = b \lesseqgtr c,$$

the equations P become

$$\frac{\mathbf{u}}{e} = \frac{\mathbf{v}}{f} = \frac{c^2}{a^2}\frac{\mathbf{w}}{g};$$

where the third expression must be taken as irrational (since $\frac{c^2}{a^2}$ can only be temporarily rational), unless,

(*a*) $\mathbf{w} = 0$ and $g = 0$, and therefore the condition holds good for *all the normals in the zone* [001] for which $ef0$ is the symbol so long as e and f are integers or one only of them 0. It holds therefore for the normals of the faces 100 and 010: and—

(*b*) it is true also for the normal of the face 001, since it is true for the case where

$$\mathbf{u} = \mathbf{v} = 0 = e = f,$$

simultaneously with $\mathbf{w}$ and g being finite; as then the equations become

$$\frac{a^2}{c^2}\frac{\mathbf{u}}{e} = \frac{a^2}{c^2}\frac{\mathbf{v}}{f} = \frac{\mathbf{w}}{g}$$

and are satisfied if $\mathbf{w} = g = 0$.

III. If again, with rectangular axes, the parameters be all unequal, the equations P would become

$$a^2\frac{\mathbf{u}}{e} = b^2\frac{\mathbf{v}}{f} = c^2\frac{\mathbf{w}}{g};$$

writing the equations as

$$\frac{\mathbf{u}}{e} = \frac{b^2}{a^2}\frac{\mathbf{v}}{f} = \frac{c^2}{a^2}\frac{\mathbf{w}}{g},$$

(*a*) the second and third expressions are irrational (since $\frac{b^2}{a^2}$ and $\frac{c^2}{a^2}$ must be treated as if they were permanently so), if $\mathbf{v}, f, \mathbf{w}$ and g are not all zero, while $\mathbf{u}$ and e remain finite. The condition holds, then, for the zone-axis [100], which is also the axis X, and coincides with the normal of the axial plane.

And similarly (*b*) for the zone-axis [010] or axis Y,

and (*c*) for the zone-axis 001 which is the axis Z.

Hence in this case the *intersections of the axial planes* are the only cases of coincidence in zone-line and normal, and in the symbols of these.

IV. Assuming the parameters still to be unequal but only two of the axial-angles to be right-angles, i.e.

$$a \gtrless b \gtrless c \text{ and } \xi = \zeta = 90,$$

η being an angle greater than 90°, the cosine of which may be treated as irrational, the equations become

$$\frac{a\mathbf{u} - c\mathbf{w}\cos\eta}{\frac{e}{a}} = \frac{b\mathbf{v}}{\frac{f}{b}} = \frac{c\mathbf{w} - a}{\frac{g}{c}}$$

and can only be satisfied by the first and third ratios which contain the irrational cosine becoming indeterminate, in which case $\mathbf{u} = \mathbf{w} = 0$ simultaneously with $e = g = 0$ and $\mathbf{v}$ and f remain finite.

Under the assumed conditions therefore the only zone-axis that is also a face-normal is that in which the indices for the X and Z axes are zero; namely, *where the symbol is* [010].

99. V. In the case where only a single axial angle can be a right angle and the parameters are unequal it will be found that no zone-axis can be a face-normal of the system. It remains then to consider the cases in which all the axial angles are oblique. And here we have first that in which these angles are equal as well as the parameters; i.e.

$$a = b = c;\ \cos\xi = \cos\eta = \cos\zeta = \frac{1}{\delta}.$$

Here, as in case IV, the angle η not being a permanently persistent angle the value of $\frac{1}{\delta}$ may be treated as irrational. Then the equations become

$$\frac{\delta\mathbf{u}+\mathbf{v}+\mathbf{w}}{e} = \frac{\mathbf{u}+\delta\mathbf{v}+\mathbf{w}}{f} = \frac{\mathbf{u}+\mathbf{v}+\delta\mathbf{w}}{g}$$

$$= \frac{(\delta+2)(\mathbf{u}+\mathbf{v}+\mathbf{w})}{e+f+g} = \lambda;$$

and $\quad \delta\mathbf{u}+\mathbf{v}+\mathbf{w} = \lambda e,\ \mathbf{u}+\delta\mathbf{v}+\mathbf{w} = \lambda f,\ \mathbf{u}+\mathbf{v}+\delta\mathbf{w} = \lambda g;$

and $$\mathbf{u}+\mathbf{v}+\mathbf{w} = \frac{\lambda}{\delta+2}(e+f+g).$$

Subtracting the last from each of the three previous equations in turn,

$$(\delta-1)\mathbf{u} = \lambda\left(e - \frac{e+f+g}{\delta+2}\right) = \lambda\frac{e(\delta+2)-S}{\delta+2},$$

$$(\delta-1)\mathbf{v} = \lambda\left(f - \frac{e+f+g}{\delta+2}\right) = \lambda\frac{f(\delta+2)-S}{\delta+2},$$

$$(\delta-1)\mathbf{w} = \lambda\left(g - \frac{e+f+g}{\delta+2}\right) = \lambda\frac{g(\delta+2)-S}{\delta+2},$$

where $$S = e+f+g.$$

Hence $$\frac{\mathbf{u}}{e(\delta+2)-S} - \frac{\mathbf{v}}{f(\delta+2)-S} - \frac{\mathbf{w}}{g(\delta+2)-S}.$$

Since $\frac{\mathbf{u}}{\mathbf{v}}$ is rational, let $\frac{\mathbf{u}}{\mathbf{v}} = n$, and $\frac{\mathbf{v}}{\mathbf{w}} = n'$,

$$n(f(\delta+2)-S) = e(\delta+2)-S,$$

$$(\delta+2)(nf-e) = (n-1)S.$$

Similarly, $$(\delta+2)(n'g-f) = (n'-1)S.$$

So long as δ is irrational this can only be true provided either that

$$nf-e = 0,\ n'g-f = 0,\ \text{and}\ S = 0 = e+f+g,$$

that is, $\frac{u}{e}=\frac{v}{f}=\frac{w}{g}$, where $e+f+g=0$;

or that $nf-e=0$, $n'g-f=0$, and $n-1=0=n'-1$;

that is, $u=v=w$ and $e=f=g$.

There are therefore two cases that here arise in which the coincidence of face-normals and zone-axes is possible; the one is the case of all normals of faces the indices of which fulfil the condition $e+f+g=0$, i.e. for *the normal of every face in the zone* [111]; and the other case is that of this zone itself, the axis of which is *the normal of the face* (111).

100. VI. It will have been noticed that the cases in which the coincidences under consideration are the more numerous are those in which the greater number of the axial elements are fixed in their values, and that in case IV, where only a single zone-plane was parallel to a face of the system, two of the axial elements were fixed.

And all remaining cases that may be conceived will be found, where they present any zone-line coincident with a plane-normal, to resolve themselves into one or other of those already discussed.

Thus, for example, if we take

$$\xi=\zeta=90^\circ \gtrless \eta, \quad a=c\gtrless b,$$

we have to eliminate the cosines of ζ and ξ from the equations P, and we get

$$\frac{u-w\cos\eta}{e}=\frac{b^2}{a^2}\cdot\frac{v}{f}=\frac{w-u\cos\eta}{g}$$
$$=\frac{(u+w)(1-\cos\eta)}{e+g},$$

or $\mu\frac{u+w}{v}=\frac{e+g}{f}$, where $\mu=\frac{a^2}{b^2}(1-\frac{1}{\delta})$

and is irrational; and the equation is satisfied if

$$v=0,\ f=0,$$

and if $u+w=0,\ e+g=0,$

i.e. if $w=-u,\ g=-e.$

In the case supposed then there will occur at least one case of a coincident symbol for a zone-line and a normal, that, namely, of the normal to the plane $(e\,0\,\bar{e})$ or $10\bar{1}$. But this normal must of necessity bisect the arc $\bar{1}00,001$, since the parameters *a* and *c* are equal; and the normal 101 will bisect the supplementary arc 100,001. And the zone [010] is thus symmetrical to the two perpendicular planes 101 and $10\bar{1}$: and, since the axis of that zone is also perpendicular to the normals 100 and 001, and therefore to 101 and $10\bar{1}$, the case in question resolves itself into one presenting identical conditions with those discussed under case III of this article.

The assumption of irrationality of the parameters and of the cosines of the axial angles, on which the reasoning in articles **97**, **98** and **99** is founded, involves statements regarding physical characters which, while true of a crystal, have no place in a plane-system that is simply crystalloid.

For the discussion of the principles involved in the parallelism of zone-axes and normals in a system of the latter kind, see a memoir by H. J. S. Smith, Savilian Professor of Geometry, in the *Proceedings* of the London Mathematical Society, vol. viii. Nos. 109 and 110, an abstract of which is given in the *Proceedings* of the Crystallogical Society, *Phil. Mag.*, Ser. V. vol. iv. p. 18.

101. *Conditions for more than one plane of symmetry.* It has been seen that the conditions necessary for a single plane to be a plane of symmetry to a crystalloid system, namely, that it be at once a zone-plane and parallel to a face, are not sufficient to impart to a second plane the character of a plane of symmetry. Such a second plane, in fact, must furthermore *be inclined* on the first plane, and therefore also *on every other plane of symmetry, at one of the crystallometric angles.*

The method of reasoning adopted in article **90** suffices to prove that *every* plane of symmetry to a crystalloid plane-system must be also, by that fact, the plane of a zone: and it is shown in article **96** that two planes of symmetry must always belong to an isogonal group in the zone that contains them.

In fact, if S_1 be a plane of symmetry and Σ_1 be a plane of the system, Σ_1 will be symmetrically repeated over S_1; and where the two planes are inclined at a crystallometric angle, the zone $[S\Sigma]$ will be isogonal as regards the repeated planes and symmetrical to each of them. And by article **63** the zone will further be symmetrical to a series of planes perpendicular to the planes S and Σ, and therefore also included with them in an isogonal group harmonically dividing the zone: indeed in all cases except where $\phi = \frac{\pi}{3}$ these supplementary planes will each fall into coincidence with one or other of the planes S or Σ.

It is evident that in this case each of the planes of the isogonal group is at once parallel to a face and to a zone-plane. And each will, in accordance with the reasoning in article **90**, be potentially a plane of symmetry to the entire plane-system.

That a plane inclined at other than a crystallometric angle on a plane of symmetry cannot be a true plane of symmetry to a plane-system may be seen by considering the class of cases that have been already alluded to in articles **98**. I and II *a*, and **99**. V; in which a series of tautozonal planes fulfil the *primâ facie* conditions for being planes of symmetry to the zone they lie in, while, from the plane of that zone being parallel to a possible face, it might be assumed that all or some of them would be planes of symmetry to the entire plane-system. And yet they are not true planes of symmetry either to the system or even to the zone; for the reason that they are not inclined at crystallometric angles on certain planes belonging to the zone, which, being at once (like the planes in question) zone-planes and parallel to faces, are on the other hand unlike those planes in belonging also to an isogonal group, and so being established in the position of true planes of symmetry alike to the zone and to the entire plane-system.

102. *Planes of abortive symmetry.* The cases just alluded to, in which all the planes lying in a zone may in unison with other planes perpendicular to them (in accordance with article **63**) orthosymmetrically divide the zone, need some further consideration. It is clear that for any plane belonging to such a zone the symbol of another plane can be calculated that shall be rational, provided this second plane is equally inclined with the first on some third plane belonging to the zone. In fact, the symbols of the three planes will, with the symbols of a plane perpendicular to the last, form a harmonic ratio. Every plane of the zone would in short fulfil the first condition of a possible plane of symmetry to the zone; and by virtue of a plane perpendicular to it, and of the plane of the zone itself being both planes of the system, each plane of the zone would seem presumably to be a plane of symmetry to the entire plane-system.

But such presumptive planes of symmetry are evidently precluded from really possessing symmetral characteristics by the principle that if they were actual planes of symmetry they would of necessity be *recurrent;* whereas if they were so they would mutually repeat each other in numbers without limit, since their angles of mutual inclination cannot be commensurate with π. For it is the essence of a plane of symmetry, actual or potential, that all the morphological features of a zone or of any form be not capriciously or partially, but systematically, or not at all, repeated in respect to it. Where a plane may hereafter be spoken of as only potentially a plane of symmetry, it will be implied that its influence on a form is as it were suspended or *in abeyance* in respect to the *entire form.* In the cases under consideration however a zone would contain an indefinite number of planes; and the faces of a form would be repeated at an ever-increasing variety of angles: from a plane-system we should pass to a solid with a curved surface. Symmetral characters in planes of the kind under discussion cannot therefore be spoken of as being merely in abeyance: they are impossible, and such planes are consequently *abortive* as planes of symmetry, and that for the reason that they are not inclined at crystallometric angles on certain particular planes which in a crystal are naturally selected from among the possible planes of the zone as true planes of morphological symmetry.

103. *The several cases of abortive symmetry.* In articles **98** and **99** the whole of the cases have been recounted in which planes of *abortive* symmetry occur. Case I in that article presents an entire plane-system the faces of which are all such that they are parallel to possible zone-planes; and it will be seen that these faces, when a system of this kind falls under consideration, will divide themselves into such as are parallel to true systematic planes (of symmetry) and such as cannot be parallel to such planes nor be inclined on them at crystallometric angles.

The other cases are those of case II, article **98**, and case V, article **99**, which however are each confined to the faces of a single zone. In either case we shall find that the true planes of symmetry in the zone have a crystallometric angle less than $\frac{\pi}{2}$: such, in fact, are the only cases not included under case I in which we shall meet with zones of this kind.

The point under discussion is sufficiently important to receive illustration from an example, which may be taken from the second case in article **98.** In that case the plane 001 is parallel to the zone-plane containing the axes X and Y and also the poles 100 and 010 which lie on the normals of the planes YZ and ZX; the parameters on these axes being equal. Consequently, two planes Σ_1, Σ_2, Fig. 40, perpendicular to each other bisect the right angles between the planes ZX and YZ which may be designated S_1 and S_2, and these four planes are successively inclined at 45°, and will be recognised in a future article as the planes of symmetry for the zone [001] and also for the whole plane-system. A pole p_1 on the zone-circle [001] will be repeated over the plane Σ_1 in a pole p_1', and this over the plane S_2 in a pole p_2; and the angle 100 on p_1 equals the angle 010 on p_2, and the arc $p_1\,p_2$ is a quadrant. So p_1 fulfils the primary condition for being a plane of symmetry to the zone. In fact, if we assign a rational symbol, say 320, to the pole p_1, then a pole q_1 equidistant with 100 from p_1 must have a rational symbol. In fact, the symbol of p_1' will be 230, and of p_2 will be $\bar{2}30$, and the ratio

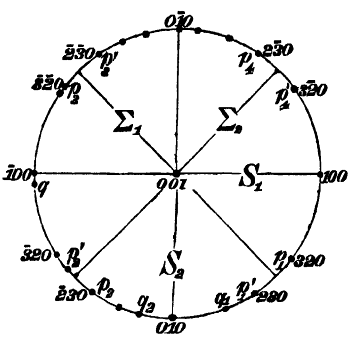


Fig. 40.

$$\frac{\sin p_2 q_1}{\sin p_2\, 100} \cdot \frac{\sin p_1 q_1}{}$$

is harmonic, and

$$= -1 = \left\| \frac{\left|\begin{matrix}\bar{2}30\\hk0\end{matrix}\right|}{\left|\begin{matrix}\bar{2}30\\100\end{matrix}\right|} \right\| : \left\| \frac{\left|\begin{matrix}320\\hk0\end{matrix}\right|}{\left|\begin{matrix}320\\100\end{matrix}\right|} \right\| = \frac{2k+3h}{3} : \frac{3k-2h}{-2};$$

whence $5k = 12h$, and the symbol of q_1 is $5\ 12\ 0$.

Now taking the poles $100, p_1, 110, 010$, by the problem of four planes, the arc $100, p_1 = 33°41' = p_1 q_1$, and the arc $q_1, 010 = 22°37'$. So that if p_1 were the pole of P a plane of symmetry, this plane P and the plane S_1 would be mutually repeated at angles of $33°41'$, while with Σ_1 and S_2, P would form a series of planes inclined at $11°19'$ and $56°19'$, while Q the plane of which q is the pole would furnish repetitions with these planes at again new angles.

It would be futile to pursue such repetitions into their results. In a word, the zone can exhibit no symmetry other than to the planes S and Σ.

104. It has been established in the preceding articles that in order for a plane-system to be symmetrical to more than one of its planes, each such plane must be parallel to a possible zone-plane and inclined on each other plane of symmetry at a crystallometric angle. It is however to be observed that where this condition is fulfilled by two planes of an isogonal group in a zone that are not supplementary, it is necessarily true of the remaining planes belonging to that group.

We may now proceed to discuss the various kinds or types of symmetry which result, in the first place from assuming different crystallometric angles between planes in a zone fulfilling the above conditions, and in the next place from the discussion of the problem of the possible modes in which such isogonal zones are capable of intersecting with each other.

Case I.—The type of symmetry where $\phi = \frac{\pi}{2}$.

105. The first case to be considered will be the simplest, that namely in which two planes of symmetry are perpendicular to each other.

If there be two and only two planes of symmetry S and Σ,

Fig. 41, in a zone, these must be perpendicular to each other and divide the zone-circle orthosymmetrically: and the zone-plane C containing the normals of the two planes will be parallel to the face in which their two zone-circles intersect and be itself potentially a plane of symmetry. And the three zone-lines ss', $\sigma\sigma'$, cc', which are the normals as well as origin-edges of the three planes S, Σ, C, will, where the plane C is an actual plane of symmetry, be axes of orthosymmetry to the system. And it will further be seen that the system will, *as a consequence of its symmetry* to the plane C as well as to S and Σ, be *centrosymmetrical.*

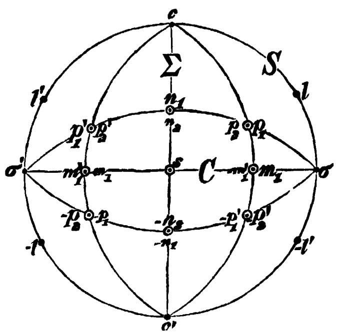


Fig. 41.

The conditions of symmetry assumed in the particular type under discussion preclude the possibility of other than two perpendicular symmetral planes in either of the zones $[S\Sigma]$, $[\Sigma C]$, or $[CS]$.

And they therefore preclude the possibility of a pole of any form lying on a great circle that should bisect either of the right angles formed by the intersection of the planes (or zone-circles) S, Σ, C, and therefore also of a pole bisecting any of the quadrantal arcs connecting the poles of the planes S, Σ, C. For if such a pole existed, e.g. if a pole m_1 were to bisect one of the quadrants on the great circle $[C]$ between the poles of the planes S and Σ, a great circle $[m_1 c]$ would bisect the angle $S\Sigma$ and would be perpendicular to a second zone-circle $[cm_1']$ bisecting the supplementary right angle of $S\Sigma$, and these two zone-circles thus intermediate to, and equally inclined at $45°$ on S and Σ would each be potentially a plane of symmetry to the system; and this would superpose a fresh condition to those assumed for the type of symmetry under discussion; so that the plane-system would in fact belong to another and a more complex type of symmetry than that to only three perpendicular planes.

In the type under consideration, therefore, any independent pole p must be situated asymmetrically, that is to say, at a point on the sphere of projection the distances of which from the great circles $S\Sigma$ and C must all be different; wherefore, every plane of an independent form (to which such a point p would be the pole) must be inclined at different angles on the three planes of symmetry and will meet their zone-axes ss', $\sigma\sigma'$, cc' at different distances from the origin.

106. *The systematic triangle.* The necessity for this unsymmetrical or eccentric position of the pole in a case where three adjacent great circles of symmetry so intersect to form a spherical triangle as that other great circles of symmetry are precluded from invading the triangle, is not confined to the case in which the triangle is quadrantal. It in fact is true for every spherical triangle formed by the intersections of adjacent great circles of symmetry. And it will be well therefore to make a brief digression, in order to give form to this principle.

Evidently a spherical triangle of the kind in question will have for each of its angles an axial-point in which a zone-axis of a zone of symmetry-planes and is consequently an axis of symmetry meets the sphere of projection; while, further, each such point is the pole of a possible face.

Def.—A spherical triangle formed by the intersection of adjacent planes of symmetry with the sphere of projection, and which therefore may not be intersected by any other circle of symmetry, will be termed the *systematic triangle* for the particular type of symmetry to which it belongs and which moreover it characterises; since it is not conceivable that the surface of the sphere should be symmetrically subdivided into systematic triangles of more than one kind.

107. *The general independent form is a scalenohedron.* It is obvious that one and only one pole of a form can occur in each systematic triangle, and that the position of this pole in every such triangle will be the same relatively to corresponding sides and angles. It is also evident that the *angles* of a systematic triangle can have only crystallometric values.

Furthermore, since from the nature of a systematic triangle no two of its sides can be in a crystallographic sense homologous, and since the edges of adjacent faces of an independent form drawn in equipoise will lie in the several planes to which those faces are symmetrical, the edges of each face will necessarily be three in number and will all be dissimilar; so that every face of such a general independent form will have the character of a scalene triangle. Such a general form therefore, in every case where the type of symmetry admits of a systematic triangle, may be termed a general scalenohedron of the system.

108. *Systematic triangle and axial system where* $\phi = \frac{\pi}{2}$. The systematic triangle in the case last under consideration, that namely of three perpendicular planes of symmetry, will be formed by three quadrantal arcs of the great circles S, Σ, and C, and be represented by the expressions

$$S = \Sigma = C = \frac{\pi}{2},$$

$$s = \sigma = c = 90^\circ;$$

S, Σ, C being the arcs that form the sides; s, σ, c the angles opposite to them, of this quadrantal triangle.

Taking the three planes of symmetry for the axial planes, their normals become the axes for the polyhedral system and the axial octants coincide with the eight systematic triangles. The octants and the systematic triangles adjacent to each other, and the faces of any form of which the poles lie in them, will evidently be antistrophic in the character of their symmetry, those belonging to attingent octants will be metastrophic.

The axes being the zone-lines or origin-edges [SC], [$C\Sigma$], [ΣS] are dissimilar, since from § **105** it is seen that no plane is possible in the system which intersects any pair of these axes with equal intercepts; so that equal, and therefore commensurate, parameters are equally precluded.

For the purpose of uniformity in the representation of crystals that accord with the same type of symmetry, it will be desirable to assign an '*orientation*' of a definite kind to the different axes; and for this purpose in the type under consideration the order of the magnitudes of the parameters will be adopted. Thus the longest parameter will be that assigned to the X axis, the mean to the Y axis, and the least to the vertical or Z axis.

The expression representing the axial system will therefore be

$$\xi = \eta = \zeta = 90, \quad a > b > c;$$

in which the parametral ratios $a : b : c$ are unfettered by conditions and may be different for each distinct system of planes conforming to this type of symmetry. The remaining three permanently fixed elements of the axial system are therefore engaged in satisfying those conditions which all plane-systems must fulfil in order to belong to this type of symmetry; the conditions namely that a plane of the system is the plane of a zone symmetrical to one of its planes, or, as a consequence, that the plane-system is symmetrical to three perpendicular planes which are also zone-planes.

109. *The character of a form.* As each octant is conterminous with a systematic triangle, eight faces will be comprised under the symbol of an independent form $\{hkl\}$; see Fig. 42. And as the intercepts on any one of the several axes will have the same value for every face of the form, the position of the indices in their symbols will not admit of permutation; the signs of these will however change, following those that designate the octant in which a particular pole may lie; and therefore undergoing every possible interchange. The symbols of the eight planes are therefore

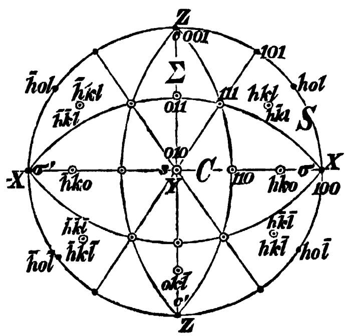


Fig. 42.

$$hkl,\ \bar{h}kl,\ h\bar{k}l,\ hk\bar{l},\ \bar{h}\bar{k}l,\ \bar{h}k\bar{l},\ h\bar{k}\bar{l},\ \bar{h}\bar{k}\bar{l}.$$

The position of the axes in this system being so taken that the zone-lines $[SC]$, $[\Sigma C]$, and $[S\Sigma]$ are the axes X, Y, Z respectively, the plane S has for its symbol (010), Σ is (100), and C is (001).

The prismatic forms $\{hko\}$, $\{hol\}$, and $\{okl\}$ are constituted each of four planes, the first form being technically termed a *prism;* the other two, *domes* (from the Greek δῶμα). Thus the dome-form $\{101\}$ comprises the planes (101), ($\bar{1}$01), (10$\bar{1}$), ($\bar{1}$0$\bar{1}$).

The parametral form $\{111\}$ is a scalene octahedron.

Case II.—The type of symmetry in which $\phi = \frac{\pi}{4}$.

110. The case next to be considered involves the condition precluded by the conditions of the last case, which required two *adjacent* planes of symmetry to be perpendicular to each other. If a pole be recognised as possible on a great circle that would bisect an angle and the quadrantal side opposite to it of the systematic triangle in the case last considered, we admit the condition that a face of the system may be equally inclined on and intersect with equal intercepts two of the axes, for instance, the axes X and Y; for this would evidently be true of every plane the pole of which lies on such a great circle.

Then the pole in question will with the pole C (or 001) serve to designate a zone-circle intersecting with the great circle C in another pole equidistant, that is to say, separated by an arc of $\frac{\pi}{4}$ from the poles of the axial-planes ZX and YZ; see Fig. 42.

The plane corresponding to this last pole would thus be inclined at the crystallometric angle of $\frac{\pi}{4}$ on each of the two perpendicular planes of symmetry S and Σ characteristic of the last considered type. It, and obviously also a plane perpendicular to it and also lying in the zone $[S\Sigma]$, would therefore each fulfil the necessary conditions for a plane potentially symmetral.

Adjusting the letters to indicate conformable planes of symmetry, it will be seen that we have in the case under consideration two proto-systematic perpendicular planes S_1, S_2 alternating with two perpendicular deutero-systematic planes Σ_1, Σ_2, tautozonal with the planes S and forming with them an isogonal zone with the crystallometric angle of 45°.

Each pair of planes S and of planes Σ will be conformable *inter se*, but unconformable each with the other. And the plane C of the zone-circle $[S\Sigma]$ will also be potentially a trito-systematic plane of symmetry to the system unconformable with both the pairs of planes S and Σ. Moreover, it will be evident that the poles of a form on either of the hemispheres divided by the equatorial plane C will be repeated in the same manner of distribution on the opposite hemisphere, whether the repetition be due to the system being centrosymmetrical or to the influence of the plane C as an actual plane of symmetry.

The sphere of projection will be divided by the five planes S_1, Σ_1, S_2, Σ_2, and C into sixteen systematic triangles, each having two of its sides S and Σ quadrants and the third side $C = \frac{\pi}{4}$. The systematic triangle will therefore be represented by the expression

$$S = \Sigma = \frac{\pi}{2}, \quad C = \frac{\pi}{4},$$

$$s = \sigma = 90^\circ, \quad c = 45^\circ.$$

111. *The distribution of the poles of the tetragonal scalenohedron.* If under any type of symmetry the letters designating the systematic triangles be read in one order of rotation (e.g. that of the hand of a clock) it will be seen that these triangles fall into two groups indicated by letters taking the order, in the one group $c\sigma s$, in the other group $cs\sigma$.

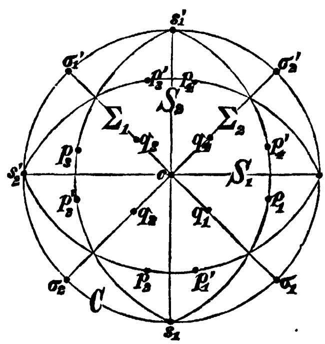


Fig. 43.

The triangles of the one group and the faces whose poles lie within them are mutually metastrophic, but are antistrophic to those of the group designated by letters in inverse order to theirs. Thus if in the type under consideration the poles in the group of triangles $cs\sigma$ be designated as p_1, p_2, &c., and those in the triangles $c\sigma s$ as p_1', p_2', &c., the scalenohedron $\{p\}$ will consist in the assemblage of the faces

$$p_1 \, p_2 \, p_3 \, p_4, \quad p'_1 \; p'_2 \; p'_3 \; p'_4,$$

$$\bar{p}_1 \, \bar{p}_2 \, \bar{p}_3 \, \bar{p}_4, \quad \bar{p}'_1 \; \bar{p}'_2 \; \bar{p}'_3 \; \bar{p}'_4;$$

wherein the minus sign is used to indicate poles on the lower of the hemispheres divided by the plane C, and the faces belonging to the poles p are antistrophic to those belonging to the poles p'.

112. *The axial system for this type.* The necessary conditions for the type of symmetry under discussion are four in number, and may be embodied in the statement that a plane of the system is a zone-plane, and that a second plane of the system also parallel to a zone-plane is inclined on the former plane at 45°. Of the five elements constituting the axial system four will be constants occupied in satisfying these conditions, a single element only remaining to vary with the particular system of planes. As in the previous case, three of the perpendicular planes of symmetry, namely the two planes S and the equatorial plane C, are taken as axial-planes, the zone-axis $[S\Sigma]$ becoming the Z axis, and the zone-axes $[S_1C]$, $[S_2C]$ the axes X and Y, respectively.

While any plane having its pole situate on one of the zone-circles $[\Sigma]$ will meet the X and Y axes with equal intercepts, such a plane will, under the conditions assumed for this type of symmetry, meet the Z axes at a distance incommensurable with the intercepts on the axes X and Y; that is to say, the parameters a and b are equal but are unequal to and generally incommensurate with the parameter c. Hence also there can be no pole on a great circle bisecting the quadrantal arcs S or Σ of the systematic triangle or the right angles which they subtend.

Taking then for a parametral plane a face belonging to the zone Σ_1, the axial system is represented by the expression

$$\xi = \eta = \zeta = 90,$$
$$a = b \gtrless c.$$

The symbols of the axial-planes will be for S_1, S_2, the poles of which lie on the Y and X axes respectively, 010 and 100; the symbol for the pole of the plane C which will lie on the Z axis being 001.

Def.—Where two or more planes of symmetry lying in a zone are conformable, their zone-axis will be termed a *morphological axis* or *axis of form* for the system; and where there is only one such axis to the plane-system, the plane (potentially symmetral) to which this axis is the normal will conveniently be termed the *equatorial* plane.

In the type of symmetry under discussion the morphological axis is an axis of tetragonal symmetry for all forms of the system, since two congruent planes of symmetry perpendicular to each other intersect in this axis; and when the symmetry is complete it is ditetragonal.

Of each zone-circle S and Σ the two hemizones on either side of the morphological axis are similar.

113. *Symbols for forms of tetragonal type.* Each octant of the axial system is composed of two systematic triangles antistrophic to each other, in which the order of the indices, like that of the letters indicating the systematic triangle, will be direct for metastrophic, inverse for antistrophic planes. Hence the indices

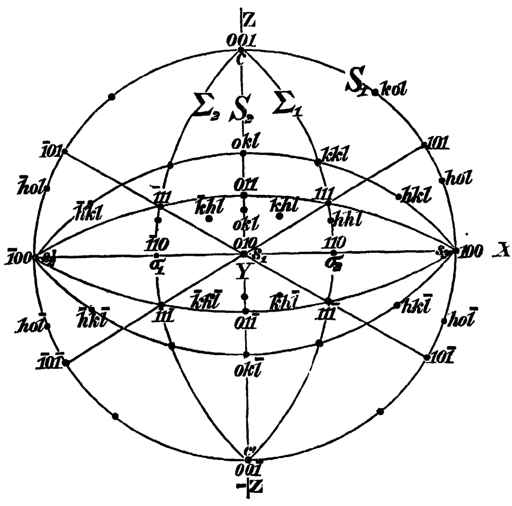


Fig. 44.

h and k relating to the similar axes X and Y will be interchanged in position in the symbols for the two faces of a form $\{h\,k\,l\}$ that lie in the same octant; the position and value of the index l remaining unchanged, except that its sign is + or — according as the face intersects with the axis Z above or below the origin; see Fig. 44.

The symbols of the faces of the general scalenohedral form will therefore be

$$hkl\ khl,\quad \bar{h}kl\ \bar{k}hl,\quad \bar{h}\bar{k}l\ \bar{k}\bar{h}l,\quad h\bar{k}l\ k\bar{h}l,$$
$$\bar{h}\bar{k}\bar{l}\ \bar{k}\bar{h}\bar{l},\quad h\bar{k}\bar{l}\ k\bar{h}\bar{l},\quad hk\bar{l}\ kh\bar{l},\quad \bar{h}k\bar{l}\ \bar{k}h\bar{l}.$$

If the poles lie on the zone-circles Σ, h and k do not differ in value, and the symbol of the form is $\{hhl\}$, which is an isosceles octahedrid form and, for the parametral plane, becomes $\{111\}$; while the zones C and Σ are tautohedral in the faces of a four-faced form $\{110\}$.

If the poles of the form lie on the zone-circles S, the symbol is $\{h0l\}$, and the form has the eight faces—

$$h0l,\ 0hl,\ \bar{h}0l,\ 0\bar{h}l,\ \bar{h}0\bar{l},\ 0\bar{h}\bar{l},\ h0\bar{l},\ 0h\bar{l}.$$

And the symbol for a form consisting of eight poles distributed on the zone-circle C will be $\{hk0\}$, all the faces being parallel to the axis Z. The angles between planes belonging to this zone will be constant for all plane-systems presenting this type of symmetry.

The zone-axes $[SC]$ and $[\Sigma C]$ are, as in the type of symmetry previously discussed, axes of orthosymmetry, since they are the intersections of two perpendicular planes of symmetry.

The abortive character of the symmetry which every plane in the zone-circle C simulates has already been exposed in article **103**. It is only in the case of the systematic planes that this symmetry is real.

Case III.—The type of symmetry in which $\phi = \frac{\pi}{3}$.

114. We have next to enter on the consideration of the case in which a plane of symmetry is inclined on another plane of the system at the angle $\phi = \frac{\pi}{3}$. Here therefore there will be two planes of symmetry S_1 and S_2 inclined to each other at 60°, which will be mutually repeated in a third plane S_3 symmetrical to each in respect of the other and inclined on both at 60°. Three simultaneous tautozonal planes S_1, S_2, S_3 result; their common zone-line being a morphological axis for the system of planes. This axis divides the zone-circles into hemicyclically conformable hemizones. And the great circles [S] divide the sphere of projection into six lunes, the alternate lunes becoming congruent by a revolution through 120° round their zone-axis as an axis of trigonal symmetry.

115. *Distribution of the poles of a form of trigonal type.* A form of a system symmetrical to three such planes will present, if p be an independent pole, six poles p ditrigonally grouped round the axis of form: viz. $p_1, p'_1, p_2, p'_2, p_3, p'_3$, as in Fig. 45, in which the poles on the nether hemisphere are indicated by eyelets, those on the hitherward hemisphere by dots.

If the pole lie in a zone-circle S, there will be but three such poles, viz. r_1, r_2, r_3, or t_1, t_2, t_3.

Since however the form is to be generally assumed to be centro-symmetrical, and therefore diplohedral, there will be six additional poles p in the former case, or three new planes r or t where they lie on the zone-circles S. There are, thus, the poles

$$p_1, p'_1, p_2, p'_2, p_3, p'_3; \qquad \bar{p}_1, \bar{p}'_1, \bar{p}_2, \bar{p}'_2, \bar{p}_3, \bar{p}'_3;$$

and $\quad r_1, r_2, r_3, \bar{r}_1, \bar{r}_2, \bar{r}_3;$ or $t_1, t_2, t_3, \bar{t}_1, \bar{t}_2, \bar{t}_3$.

Since the zone to which the planes S belong is symmetrically divided by these three planes, it may therefore, by article **101**, also be symmetrical to three new planes Σ, each perpendicular to one and inclined at 30° on the other two planes S; and these planes Σ will thus be also *potentially* planes of symmetry for the entire plane-system: as is also the equatorial zone-plane C.

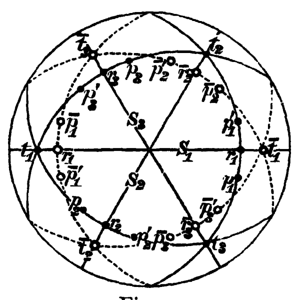


Fig. 45.

Hence, that these planes Σ or the plane C may become actual planes of symmetry entails no condition that is not involved in those originally assumed in this case; a point in which the type of symmetry now being considered differs from the otherwise somewhat analogous case where two intermediate planes of symmetry are equally inclined on two perpendicular symmetral planes.

116. *The quasi-independent nature of this trigonal type of symmetry.* Another and very characteristic point of dissimilarity between the rectangular systems already discussed and the system now under discussion is that diplohedral and therefore centro-symmetrical symmetry is independent in the present case of the influence of the equatorial plane as a plane of symmetry. In the former systems centro-symmetry involved symmetry to the plane C and *vice versâ.* Here these are independent of one another, and, according as we may suppose one or the other principle of symmetry to be in abeyance, the system will assume distinct characters. To assume a form to be haplohedral is however inconsistent with the view hitherto taken of the developement of a plane-system as a system of faces parallel in pairs to a system of origin-planes, though hereafter we shall have often to deal with such forms. But the supposition that the influence of a group or groups of planes of symmetry may be in abeyance puts no such strain upon the conceptions of symmetry that we have formed. And in fact, though it involves no new geometrical condition for a system symmetrical to three planes in a zone inclined at $\frac{\pi}{3}$ to pass into a system symmetrical at once to two triads of tautozonal planes inclined at the crystallometric angle of $\frac{\pi}{6}$, and to a plane equatorial to these, yet the distinct and quasi-independent characters of the symmetry to a trigonal axis as in the former case, and of symmetry to a hexagonal axis as in the latter case, would alone justify a separate treatment of the two resulting varieties of symmetry. But this treatment will be further borne out when we find ourselves hereafter dealing with a veritable trigonal system of symmetry, such that the circumstances of the particular system (the tesseral system) in which it occurs are incompatible either with the three planes intermediate to the original three planes or with their equatorial zone-plane being recognised as actual planes of symmetry.

But where the zone-circles Σ are only potentially symmetral, i.e. where they are not actual planes of symmetry, a pole may lie on one and be repeated on each of these great circles simply as the result of the symmetral character of the S-triad of planes: since the existence of such poles does not invalidate the conditions assumed for this type of symmetry, as it did in the case where $\phi = \frac{\pi}{2}$; inasmuch as in the present case no new condition has to be assumed as necessary to be fulfilled by the system of planes in order to render the presence of the faces corresponding to these poles possible.

Such poles, which we may designate by the letter u, will be repeated according to the same law as the poles p, namely in six poles which, through hexasymmetrically grouped round the morphological axis, are in the case in question to be viewed (see Fig. 46) as grouped in a ditrigonal manner round that axis. And if the poles u fall on the zone-circle containing the poles of the planes S they coincide with the poles s of the planes of symmetry.

In the same way, if the three poles r on the great circles S fall on the equatorial zone-circle C they coincide with three alternate poles σ of the great circles Σ; and if this form be diplohedral, the poles opposite to r falling on the zone-circle C would coincide with the remaining poles σ: and in this case also the result is a form with six poles grouped round the axis $[S]$, that may be viewed as an axis of ditrigonal symmetry.

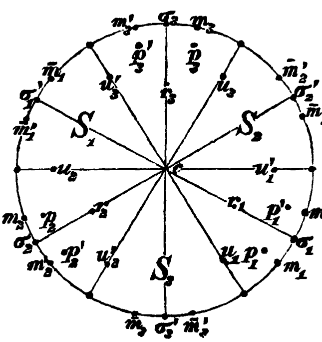


Fig. 46.

It will be seen then that some forms presenting even the aspect of regular hexagonal symmetry may be conceived as resulting from a law of trigonal symmetry in which, as it were, there lurks though concealed the potentiality of a hexagonal type of symmetry. But such a system is evidently incapable of being represented by a systematic triangle, since the surface of the sphere is partitioned into six similar lunes alternately congruent.

The only real axis of symmetry is the zone-axis of the planes S, ditrigonal in its character.

Case IV.—**The type of symmetry where** $\phi = \frac{\pi}{6}$.

117. The discussion of the last case shows that no new fundamental conditions will have to be assumed in order to pass to the consideration of a plane system in which a plane of symmetry is inclined at the crystallometric angle of $\frac{\pi}{6}$ on another plane of the system which is at the same time a zone-plane. It is in fact the case in which the symmetry resulting from a plane parallel to a zone-plane and to a face being inclined at the angle $\frac{\pi}{3}$ on a plane of symmetry has received its complete developement: it is the case, namely, in which the latter kind of plane system is symmetrical simultaneously to its centre and to its equatorial plane; and in which as a consequence the planes Σ influence the system as a triad of planes of actual symmetry (of deutero-symmetry). The polyhedron thus becomes symmetrical to seven planes, namely, to one triad of proto-systematic planes S, a triad of deutero-systematic planes Σ unconformable with the planes S, and to the equatorial plane C as a trito-systematic plane. These planes Σ_1, Σ_2, Σ_3 are then perpendicular each respectively to one of the planes S_1, S_2, S_3 and inclined at 30° to the other two, their mutual inclination being 60°: the systematic triangle where the symmetry is thus complete being represented by the symbol

$$S = \Sigma = \frac{\pi}{2}, \quad C = \frac{\pi}{6},$$
$$s = \sigma = 90°, \quad c = 30°.$$

It will be seen that the sphere of projection is partitioned into twenty-four such triangles. The number of faces presented by an independent form is therefore twenty-four. And the morphological axis $[S\Sigma]$ becomes an axis of hexagonal or generally of dihexagonal symmetry.

The conditions which the crystalloid axial system—and therefore also the particular polyhedral system that is to accord with such an axial system—has to satisfy are embodied in the statements that a plane is to be at once a plane of the system and a zone-plane, and that it is to be inclined either at $\frac{\pi}{3}$ or at $\frac{\pi}{6}$ on a second plane which must be parallel to a face of the system. In satisfying these conditions four out of the five axial elements will be employed, one only being left, as in the case where the morphological axis was tetragonal, to vary with the different varieties of the plane systems.

118. *Axial Systems.* Various axial systems may serve for the geometrical treatment of such a polyhedral system. Thus :—

(1) Three of the planes of symmetry, perpendicular to each other, e.g. S_1, Σ_1, and C, may be taken as the axial-planes; and if C be taken as parallel to the plane 001 and therefore the morphological axis be the axis Z, a plane the pole of which lies on the zone-circle S_3 may be taken for the parametral plane; or the plane S_2 may be taken to determine the parametral ratio for the axes X and Y; this ratio having therefore the constant value $\frac{a}{b} = \sqrt{3}$; while another plane the pole of which lies in a great circle S_1 or Σ_1 will serve to determine the parametral ratios $\frac{a}{c}$ or $\frac{b}{c}$ for the axis Z.

The axial elements would thus be represented by the expression

$$\xi = \eta = \zeta = \frac{\pi}{2}, \quad a : b : c, \quad \sqrt{3} : 1 : c;$$

the paramental ratio for the axis Z being the single varying element.

Such an axial system, the details of which have been elaborated by Schrauf (Sitzb. d. k. Acad. Wien, 1863), presents the insuperable disadvantage, that while the three planes of symmetry S divide the sphere symmetrically with respect to a trigonal axis, the axial-planes so divide it that congruent lunes are not similarly situated in regard to the axes, so that even the simplest trigonal forms have to be represented by double symbols.

(2) This difficulty of representing by a single symbolical ex-

pression even the simplest and most frequent forms of a crystal presenting trigonal or hexagonal symmetry, when its forms are referred to rectangular axes, led to the adoption by some crystallographers of an axial system itself according with this symmetry, the three proto-symmetral planes being taken with the equatorial plane C as axial-planes. Thus, their intersections form four axes to which the system of planes is referred, three of these presenting an axial-angle of 60° with each other and being perpendicular to the fourth which is the morphological axis. Since however three points are sufficient to determine a plane, an axial system by which a plane has in fact or virtually to be represented by four points involves an element in excess of what is needed. The further discussion of such an axial system will be entered on in a future chapter.

119. *The axial system of Hauy and Miller.* On the other hand, an axial system more in accord with geometrical method is provided by the selection for axial-planes of three origin-planes parallel to faces of the system, symmetrical in regard to and therefore equally inclined on the morphological axis; the poles of which planes lie on the great circles of proto-symmetry S.

Thus, $\mathbf{r}_1$ being the pole of a plane R_1 and lying on the zone-circle $[S_1]$, two other poles $\mathbf{r}_2$ and $\mathbf{r}_3$, poles of planes R_2 and R_3, will lie on the alternate hemizones of the zone-circles $[S_2]$ and $[S_3]$, equidistant with $\mathbf{r}_1$ from c the pole of the equatorial plane C.

Accordingly the zone-axes $[\mathbf{r}_2\mathbf{r}_3]$, $[\mathbf{r}_3\mathbf{r}_1]$, $[\mathbf{r}_1\mathbf{r}_2]$ which are the edges of the planes R_2R_3, R_3R_1, and R_1R_2 become the axes X, Y, Z, and the axial-points λ_1, λ_2, λ_3 in which they meet the sphere will also be equidistant from c, on the same side with r_1, r_2, r_3 respectively on the circles $[S_1]$, $[S_2]$, $[S_3]$; and they will only coincide with the poles $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$ in the case in which the planes R and therefore their edges also are perpendicular to each other.

This however involves a condition that would remove the system of planes into another type of symmetry, in which not only would the parameters be equal, but the axial-angles would, besides being equal, become right angles.

In fact it will be seen hereafter that

$$\tan \text{arc}\, r_1 c = 2 \,\text{cotan arc}\, \lambda_1 c,$$

so that where r_1 and λ_1 coincide

$$r_1 c = \lambda_1 c = 54^\circ\, 44.14' \text{ and } r_1 r_2 = \lambda_1 \lambda_2 = 90^\circ.$$

120. That in the axial system under consideration the parameters will be equal is involved in the statement that the pole c is equidistant alike from the poles **r** and the axial-points λ.

For then, evidently, a face parallel to C, the equatorial plane, meets the axes with equal intercepts and serves as the parametral plane 111 or $\bar{1}\bar{1}\bar{1}$; the former symbol being taken for that lying in the upper hemisphere on C.

Such an axial system as that here defined will be represented by the expression

$$\xi = \eta = \zeta, \quad a = b = c;$$

where the required four conditions are satisfied by an equal number of the axial elements, and only the particular angle η at which the axes are inclined remains a variable element characteristic of the particular system of planes.

Great circles passing through each pair of the axial-points XYZ, $\bar{X}\bar{Y}\bar{Z}$ will divide the sphere into octants, two of which, XYZ and $\bar{X}\bar{Y}\bar{Z}$, are similar, while the remaining six, viz.

$$Z\bar{X}\bar{Y}, \quad Y\bar{X}\bar{Z}, \quad X\bar{Y}\bar{Z},$$

are also similar to each other but in 'zig-zag,' that is, alternately inverted in their position on the sphere.

It will thus be seen that the systematic triangles are not coterminous with the octants formed by the axial system: and, as a consequence, the symbols of a complete hexagonal form will in certain cases have a double character. Thus of the general scalenohedron, one pole will lie in each of the twenty-four systematic triangles: but the poles lying in those adjacent pairs of triangles that have in common for one of their sides an arc S containing one of the axial-points X, Y, Z or of the poles 100, 010, 001, will have different intercepts on these axes from those of the planes the poles of which lie in the triangles which in pairs alternate with them. Hence the symbol for the general form will be of a double kind; the poles corresponding to those of a simple trigonal form retaining in their symbol indices which will be distinct from those in the symbols for the poles of the other trigonal form correlative to the former, and which united with the former complete the form in its hexagonal type.

Retaining the letter p for the general trigonal form and designating the trigonal form correlative to it by the letter q, the general symbol for the completed form will be $\{pq\}$. The distribution of the poles, including certain of those on the negative hemisphere, is seen in Fig. 47.

They will be $p_1 p_2 p_3$, $p'_1 p'_2 p'_3$,

the faces in the two left-hand blocks being, as in article **109**, antistrophic to those in the right-hand blocks, while the faces the symbols of which lie in either block are mutually metastrophic.

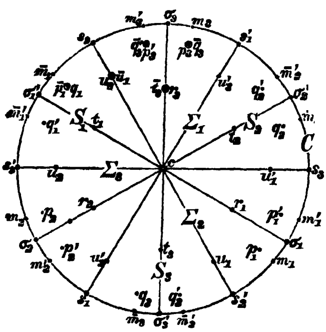


Fig. 47.

The six poles r and six poles t in Fig. 47 unite to form a double form $\{rt\}$, the poles of which lie on the proto-systematic great circles S; while the poles u, alike twelve in number in the trigonal (article **115**) and in the hexagonal type, constitute a form of which the poles lie on the great circles Σ. The poles of a form m on the great circle C will also be twelve in number in the hexagonal as in the trigonal type of symmetry.

The poles of the proto-systematic plane S are six, lying in the zone-axes $[\Sigma C]$ on the great circle C, namely,

$$s_1,\ s'_1,\ s_2,\ s'_2,\ s_3,\ s'_3;$$

those of the planes Σ being

$$\sigma_1,\ \sigma'_1,\ \sigma_2,\ \sigma'_2,\ \sigma_3,\ \sigma'_3$$

on the axes $[SC]$; while the zone-circle C has two poles c and c'.

121. *Symbols of the different forms.* The poles r_1, r_2, r_3 of the axial-planes R_1, R_2, R_3 must have for their symbols 100, 010, 001; and since (in accordance with the principle established in article **99**) the planes of symmetry [*S*] which pass through these poles and the pole *c* or 111 have the same indices in their symbols as the zone-axes have which are their normals, their symbols are, severally,

for S_1, $\begin{Vmatrix} 111 \\ 100 \end{Vmatrix}$, i.e. $01\bar{1}$ or $0\bar{1}1$;

for S_2, $10\bar{1}$ or $\bar{1}01$; and for S_3, $1\bar{1}0$ or $\bar{1}10$.

Of these symbols, by article **45** the former of the two will in each case be that of a pole lying on the same side with 100 of the great circle passing through the poles 111 and 010, the latter will be the poles lying on the same side of that zone-circle with the pole 001.

The symbols of the planes Σ_1, Σ_2, Σ_3, the poles of which lie in the intersection of the zone-circles *S* and *C*, are, of Σ_1, $\bar{2}11$ or $2\bar{1}\bar{1}$;

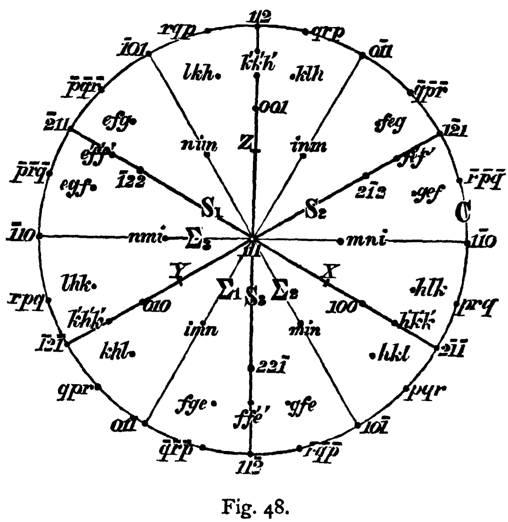


Fig. 48.

of Σ_2, $1\bar{2}1$ or $\bar{1}2\bar{1}$; and of Σ_3, $\bar{1}\bar{1}2$ or $11\bar{2}$. And of these the former will be the symbol of the poles lying on the same side with 001 of the zone-circle passing through 111 and $1\bar{1}0$. The symbols will then be distributed as in Figs. 48 and 49.

The symbols of all the axial-points forming the angles of the systematic triangles having been determined, we may proceed to consider those of a pole lying on one or other of the sides of the systematic triangle.

With respect to the symbol of any form of which the poles lie on the zone-circle [111], it is evident that the sum of the three indices is zero; i.e. the symbol has to fulfil the condition

$$p+q+r=0 \quad \text{or} \quad p=-q-r.$$

And this is precisely the zone which corresponds to one (namely to Case V) of the two cases in which it was shown in articles **98** and **99**

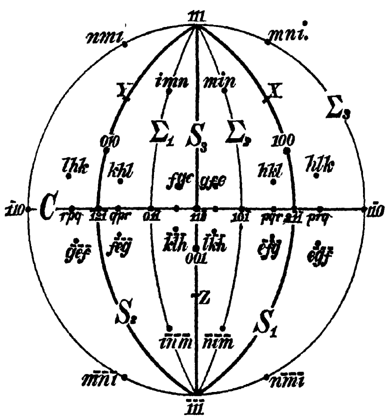


Fig. 49.

that a plane-system might present a zone which could be conceived as being symmetrical to each of its planes, all these planes being parallel to possible zone-planes. In respect to these planes however it was shewn that this symmetrical character can be only abortive where the planes in question are not inclined at crystallometric angles upon any of the systematic planes S or Σ, that is to say, must be so for all planes other than these.

In considering the arcs S forming the sides of the systematic triangle we shall have to distinguish between those which contain the poles of the form {100} and those which do not; since the poles of this form on either of the hemispheres that stand on C lie only on the quadrantal arcs S of alternate systematic triangles. Now whereas the symbol of any pole lying on the great circle $[S_1]$ i.e. $[01\bar{1}]$ which traverses the pole r_1, must fulfil the condition $k-l=0$, it is clear that the symbol must be of the form mnn; and if it lie on the positive side of the great circle $[111]$, $m+2n$ must be greater than zero; if, again, it lie on the same side of the great circle $[2\bar{1}\bar{1}]$ with the pole 100,

$$2m-2n>0 \quad \text{and} \quad m>n;$$

and if it lie on the other side of $[2\bar{1}\bar{1}]$, $m<n$. Forms fulfilling the former condition will be termed *direct*, those fulfilling the latter will be termed *inverse*.

If the pole r_1 for which $m>n$ have for its symbol (hkk), and the so-called '*transverse*' pole t equidistant with r from 111 on the other side of 111, and therefore also homologous with r, be (eff), then t_1, t_2, t_3 will be three poles homologous with and severally correlative to r_1, r_2, r_3, that is, to (hkk), (khk), and (kkh). Now t_1 will be symmetrical to r_2 in respect to the plane Σ_3 and to r_3 in respect to the plane Σ_2, and a great circle passing through t_1 and r_2 will also pass through $11\bar{2}$; t_1 therefore is the pole in which the two great circles $[11\bar{2}, 010]$ and $[01\bar{1}]$ are tautohedral and the indices in the symbol (eff) are

$$4k-h, \quad k+2h, \quad k+2h.$$

Thus, for example, the poles correlative to 100, 010, 001 are $\bar{1}22$, $2\bar{1}2$, and $22\bar{1}$.

Hence the six homologous poles r and t lying on the great circles S on one side of the zone-plane C will have their two correlative triads of symbols connected by the relation just established between the symbols hkk, khk, kkh and the symbols eff, fef, ffe; and on the opposite hemisphere the symbols of planes parallel to these will only differ from them in having opposite signs.

The symbol for a pole (min) lying on the side Σ of a systematic triangle is characterised by one of its indices being the arithmetic mean of the other two. Thus, a pole hkl lying on the great circle $[\Sigma_1]$, i.e. on $[2\bar{1}\bar{1}]$ or $[\bar{2}11]$, must fulfil in its indices the condition

$$2h-k-l=0 \quad \text{or}$$

where $h = \frac{k+l}{2}$. For a pole situated on the positive side of the zone-plane C, i.e. of [111],

$$h+k+l > 0 \text{ or } k+l > 0;$$

and if it be also situated on the same side with 010 of the zone [01$\bar{1}$], $k-l > 0$, that is $k > l$. Whence if min be the symbol of a pole u where $m > n$ and $i = \frac{m+n}{2}$, the poles on the upper or positive hemizones of the great circle

$$\Sigma_2 \text{ are } min, \; nim,$$
$$\Sigma_1 \text{ ,, } imn, \; inm,$$
and on $$\Sigma_3 \text{ ,, } mni, \; nmi;$$

the poles on the nether or negative hemisphere of faces parallel to these having their signs reversed.

122. *Composite symbol of the general di-scalenohedron.* Of the general independent form hkl a face lies eccentrically in each of the twenty-four systematic triangles of the hexagonal system. If the pole hkl be that lying in the triangle $c\sigma_1 s'_2$, Fig. 47, the pole symmetrical with it in respect to the plane S_1 will lie in the triangle $c\sigma_1 s_3$ on a great circle passing through the poles 01$\bar{1}$ and 0$\bar{1}$1: and since the positive sides of the great circles [2$\bar{1}\bar{1}$] and [111] are those containing the pole 100, the symbols for the poles in the two systematic triangles in question must satisfy the conditions

$$2h-k-l > 0 \text{ and } h+k+l > 0,$$

or $$2h > (k+l) \text{ and } h > -(k+l).$$

So that the first index must be greater than the other two; and as $h > k > l$ is the assumed order of magnitudes of the indices, the first index is h in the symbols for both the poles. So again the pole hkl is on the same side with 010 of the great circle S_1 or [01$\bar{1}$], and therefore the second index is greater than the third, while for the pole lying in adjoining systematic triangles the third index is greater than the second.

Hence the symbols of the poles in the triangles

$$c\sigma_1 s'_2 \text{ and } c\sigma_1 s_3 \text{ are } hkl \text{ and } hlk,$$
and in $$c\sigma_2 s_1 \text{ and } c\sigma_2 s'_3 \text{ are } khl \text{ and } lhk,$$
$$c\sigma_3 s'_1 \text{ and } c\sigma_3 s_2 \text{ are } klh \text{ and } lkh.$$

The symbol of the pole q transverse to and correlative with a pole p or (hkl) may be readily found by the problem of four planes. It lies on the zone $[pc]$, i.e. $\left\|\begin{matrix} h\,k\,l \\ 1\,1\,1 \end{matrix}\right\|$ or $[k-l, l-h, h-k]$, at a distance $qc=pc=\theta$ from c. And as this zone will intersect with the great circle $[111]$ in a pole d, or (pqr), the symbol of which will be

$$2h-k-l,\quad 2k-l-h,\quad 2l-h-k,$$

$$\frac{m}{n}\quad \frac{cd|}{|cp|}\quad \frac{\sin\theta}{\sin 2\theta}\cdot\frac{\cos\theta}{1}\ \text{(article 49)},$$

$$=\frac{3(k-l)}{k-l}\,\frac{3(l-h)}{l-h}\,\frac{3(h-k)}{h-k}\cdot\frac{\sin\theta\cos\theta}{2\sin\theta\cos\theta}=\frac{3}{2};$$

and the indices in the symbol efg of the pole q are obtained by equations **F**, article **49**. They are, substituting the values obtained for $p\,q\,r$,

$$e=np-mh=2(k+l)-h,$$
$$f=nq-mk=2(l+h)-k,$$
$$g=nr-ml=2(h+k)-l;$$

where $e<f<g$ for $e-f=3k-3h$, and, h being greater than k, $f>e$; also $f-g=3l-3k$, where, k being greater than l, $f<g$.

The same ratios for efg are obtained directly from the symbols of the zones $[hkl, 111]$ and $[1\bar{2}1, lkh]$ which are tautohedral in $[efg]$, since $[efg]$ and $[lkh]$ are symmetrical on the plane Σ_2. They may also be written

$$2(h+k+l)-3h=e,$$
$$2(h+k+l)-3k=f,$$
$$2(h+k+l)-3l=g.$$

123. *Symbols of the prisms.* Reverting to the poles lying on the equatorial zone-circle, we have from the last article

$$p=2h-k-l,\quad q=2k-l-h,\quad r=2l-h-k,$$

whence $q+r=k+l$, $q-r=3(k-l)$; and, since $k>l$, also $q>r$. So that the absolute relative magnitudes of the indices are $p>q>r$ for all poles not lying at the intersections with C of the great circles S or Σ; and these for the axial points at the intersections of the zone-circles S give $q=r=1$, while those at the intersections of the zone-circles Σ gives $p=r=1$, the values of $p\,q\,r$ being taken absolutely.

Planes of congruent symmetry not in one zone.

124. The four cases have now been considered in which it is possible for two or more tautozonal planes of symmetry to be inclined on each other at one of the crystallometric angles; and they have been seen to involve three distinct types of symmetry. In fact the symmetry of the whole system of planes is controlled in these cases by the symmetry of the zone to which the proto- and deutero-systematic planes belong.

The question however remains as to what other types of symmetry may be possible that are not included under these three and that of symmetry to a single plane.

Thus of the polyhedral systems hitherto considered some were symmetrical to planes at once tautozonal and conformable in their symmetry; but it remains to be determined whether it may not be possible for three or more *heterozonal* planes of symmetry to present conformability; and after these have been considered, there will remain the question whether there may not be yet other types of crystalloid symmetry.

Since it follows that an axis of symmetry potentially tetragonal will result from the existence of an independent pole lying on a circle that bisects one of the right-angles formed by three perpendicular symmetry-zones and bisects therefore also the quadrant of the systematic triangle which subtends that right-angle; we may enquire what will result from the further condition that any pole or poles may lie on a great circle bisecting the right-angle formed by another pair of the three perpendicular planes. Or, which is the same thing, it may be asked what will be the nature of the symmetry resulting from a third plane of symmetry being conformable with as well as perpendicular to either and therefore to both of two perpendicular and conformable planes of symmetry.

Case V.—Three heterozonal planes of congruent symmetry.

125. *Case of three conformable planes of symmetry that are heterozonal.* In the case suggested in the last article each of the three right-angles s, σ, and c of the systematic triangle, Fig. 40, will be bisected by a zone-circle lying in what will be potentially a plane of symmetry but not conformable with the two planes with which it intersects. Hence also each of the three zone-lines in which the proto-symmetral planes intersect becomes an axis of tetragonal symmetry, since in these same zone-lines two perpendicular deutero-symmetral planes will intersect with the two former planes at 45°.

Assigning similar letters to conformable planes of symmetry, namely, S to the three planes of proto-symmetry and Σ to the planes of deutero-symmetry, we have three perpendicular planes S intersected by six planes Σ (see Fig. 50) that are in pairs tautozonal with two and perpendicular to the third of the planes S.

It will be seen that one zone-circle Σ of each of the three deutero-symmetral pairs must intersect with other two belonging to

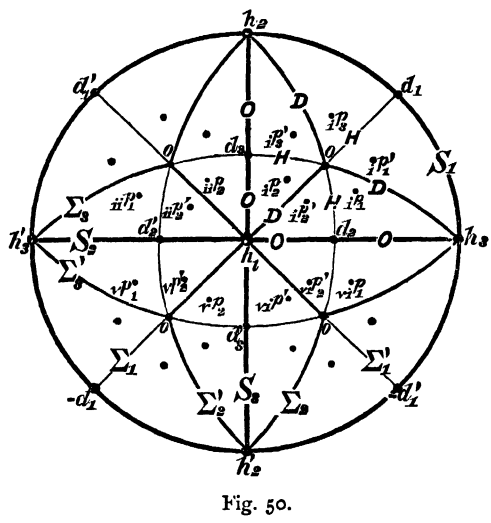


Fig. 50.

the other pairs, in a point o that will be equidistant from the poles of the planes of proto-symmetry, and this will be the axial point of an axis of trigonal symmetry.

For the three zone-planes Σ_1, Σ_2, Σ_3 (Fig. 50) will obviously intersect with each other at the crystallometric angle of 60°; since in the triangles hdo (Fig. 50) we have the angles $h = 45°$, $d = 90°$, and the side hd or $O = \frac{\pi}{4}$; whence also ho or $D = 54° 44{\cdot}14'$, and do or $H = 35° 15{\cdot}86'$, and $D + H = 90°$, and of the eight points o the mutual distance of any two adjacent poles as measured on that arc of the zone-circle Σ which traverses a pole d is $70° 31{\cdot}7'$, while the distance of two poles o measured on the arc traversing a pole h is $109° 28{\cdot}3'$.

On comparing the relative distances of the points o and the points h with the angles between faces of the polyhedron known as the cubo-octahedron, it will be seen that they are supplementary to each other, and that in fact the points o and h correspond to the poles of the regular octahedron and cube respectively which combine to build up that figure. The conditions supposed will thus give rise to a type of crystalloid symmetry different from those which have been hitherto considered.

126. *The systematic triangle, and axial system.* In it the sphere of projection will be divided by the nine intersecting circles S and Σ into forty-eight systematic triangles, of which the sides O, D, H opposite respectively to the angles o, d, h have the values

$$O = \frac{\pi}{4}, \quad D = 54° 44{\cdot}14', \quad H = 35° 15{\cdot}86',$$

while the angles are $\quad o = 60°, \quad d = 90°, \quad h = 45°$;

and there will be

three pairs of tetra-symmetral poles h,
four pairs of tri-symmetral poles o,
six pairs of ortho-symmetral poles d.

The general scalenohedron under such a type of symmetry will therefore have forty-eight faces.

Taking the proto-symmetral planes S for the axial planes and consequently their perpendicular intersections for the axes, and taking a face of the octahedron for the parametral plane, it is clear that since the latter truncates a quoin of the cube it will cut the axes with equal intercepts, so that all the parameters are equal: and the conditions for this type of symmetry as embodied in the axial system thus chosen are expressed by the symbol

$$\xi = \eta = \zeta = 90°, \quad a = b = c.$$

It will be seen that all the five axial elements are required to have special values in order to fulfil the general conditions of the system; no element remaining variable to characterise any different systems of planes that may conform to this type of symmetry.

The six poles *h* are those of three symmetral planes *S*, parallel to the faces of the cube; and they are also the axial points in which the axes of the quoins of the octahedron meet the sphere; the eight poles *o* are those of the faces of the octahedron and are the axial points of the quoins of the cube; and the twelve poles *d* are those of the six symmetral planes Σ, parallel to twelve faces which, truncating the edges alike of cube and octahedron, constitute the form {110}, termed the rhombic dodecahedron (the dodecahedral rhombohedron), a figure the quoins of which have axial points in common with those of the cube and of the octahedron.

127. *Symbols for the forms in such a system.* Since the axes and the edges of the proto-symmetral planes *S* are coincident, it is evident that the faces of the cube will have the six symbols arising from the various permutations of the indices 100 and the interchanges of the sign of the unit index; and the different faces of the octahedron will be represented by the eight interchanges of + and − sign of which the general symbol {111} for this form is susceptible. And since the axial octants are formed by the intersections of the proto-systematic planes *S*, each of them will be conterminous with six of the systematic triangles; so that six poles of the general scalenohedron will lie in every octant.

In fact, since the edges $S_2 S_3$, $S_3 S_1$, $S_1 S_2$, and therefore also the axes, are necessarily similar, a face presenting three different indices in its symbol will be so often repeated in the octant as is necessary to interchange each pair of the indices for every two axes; so that the different symbols for the faces in an octant will be six in number, corresponding to the six interchanges of position in their indices of which three different numbers are susceptible, and falling each into one of the six systematic triangles in the octant. Further, if we consider the symbols thus indicating the six poles of the form $\{hkl\}$ in an octant, it will be seen (see Fig. 49) that those in which the indices follow the same order as

$hklhk$ or as $lkhlk$ are symbols of poles lying in alternate systematic triangles, which, as in article **125**, may be designated as hdo or hod, and their faces are metastrophic; while those in which the order of the indices is reversed belong to antistrophic faces lying in adjacent triangles.

In passing from one octant to another, it is obvious that the six faces belonging to any octant will have their signs in the same position in their symbols, the position namely of the signs of the XYZ which designate the octant.

The symbols of the faces of the general independent scalenohedral form in this system will consequently represent all the permutations of three numbers taken positively and negatively. The variety and distribution is given in the following table:

TABLE A.

	i.	iii.	v.	vii.	ii.	iv.	vi.	viii.	
	hkl,	$\bar{h}\bar{k}l$,	$\bar{h}k\bar{l}$,	$h\bar{k}\bar{l}$,	$\bar{h}kl$,	$h\bar{k}l$,	$hk\bar{l}$,	$\bar{h}\bar{k}\bar{l}$,	
I μ.	lhk,	$\bar{l}\bar{h}k$,	$\bar{l}h\bar{k}$,	$l\bar{h}\bar{k}$,	$\bar{l}hk$,	$l\bar{h}k$,	$lh\bar{k}$,	$\bar{l}\bar{h}\bar{k}$,	a II.
	klh,	$\bar{k}\bar{l}h$,	$\bar{k}l\bar{h}$,	$k\bar{l}\bar{h}$;	$\bar{k}lh$,	$k\bar{l}h$,	$kl\bar{h}$,	$\bar{k}\bar{l}\bar{h}$;	
	hlk,	$\bar{h}\bar{l}k$,	$\bar{h}l\bar{k}$,	$h\bar{l}\bar{k}$,	$\bar{h}lk$,	$h\bar{l}k$,	$hl\bar{k}$,	$\bar{h}\bar{l}\bar{k}$,	
III a.	khl,	$\bar{k}\bar{h}l$,	$\bar{k}h\bar{l}$,	$k\bar{h}\bar{l}$,	$\bar{k}hl$,	$k\bar{h}l$,	$kh\bar{l}$,	$\bar{k}\bar{h}\bar{l}$,	μ IV.
	lkh,	$\bar{l}\bar{k}h$,	$\bar{l}k\bar{h}$,	$l\bar{k}\bar{h}$;	$\bar{l}kh$,	$l\bar{k}h$,	$lk\bar{h}$,	$\bar{l}\bar{k}\bar{h}$;	

where the *columns* represent each an octant; the first four giving the symbols of the poles lying in the octants XYZ and those attingent to it and adjacent to $\overline{X}\overline{Y}\overline{Z}$; the remaining four columns giving the poles in the octant $\overline{X}\overline{Y}\overline{Z}$ and those attingent to it.

The blocks I μ. and IV μ. represent the systematic triangles metastrophic to each other and to the triangle containing the pole hkl, the blocks II a. and III a. represent the triangles antistrophic to the former, but mutually metastrophic.

The symbols of a form the poles of which lie on a great circle S will have a zero in the place of the index corresponding to that axis which is normal to the great circle; and since all the eight arcs O on each of the great circles S are similar and will each carry a pole of the form, the form will have twenty-four poles, in the symbols for which two of the indices are different and one is zero. These then will correspond with the twenty-four various interchanges of position in the indices and of character in their signs of which the general symbol $\{hko\}$ is susceptible.

The poles of a form lying on a great circle Σ will be the poles of planes intersecting with identical intercepts on two of the axes. Two of the indices must therefore be identical; hence the general symbol for a form the poles of which lie on the great circles Σ will be either $\{hkk\}$ or $\{hhl\}$; and according as its position lies on one or the other side of the pole of the octahedral form $\{111\}$, it will present the one or the other of these types of symbol. In fact, when the pole lies on an arc H its indices for two axes are greater than that for the third and the form is $\{hhl\}$, when it lies on an arc D its symbol is $\{hkk\}$. The form in either case has twenty-four faces symmetrical to the great circles Σ; their symbols interchanging the positions of the indices in each octant and the character and position of their signs from octant to octant.

General discussion of the systematic triangle.

128. We have so far considered certain special cases in which a crystalloid polyhedral system may be supposed to be symmetrical to one or simultaneously to several planes. These cases have included all the possible conditions under which such planes of symmetry may lie in the same zone, and certain conditions under which other planes heterozonal to these may also be planes of symmetry for the system.

It remains however to determine whether these are the only possible cases of crystalloid symmetry, or whether some polyhedral systems may not exist the law of whose symmetry may have to be represented by a different systematic triangle from any yet considered. And for this we may discuss the general characters of such a triangle and the limits these impose to its variation; and the most general form of stating this problem will be that of determining under what conditions three great circles may intersect at crystallometric angles.

129. *The six systems of crystallography deduced from the crystallometric law.* Let S, C, Σ be the sides and s, c, σ be the angles respectively opposite to them of such a spherical triangle. Then as s, c, and σ can neither of them be greater than 90°, while their sum must be greater than 180°, only the values for these angles in the three first columns of the following table are possible, the corresponding values for the arcs forming the sides of the triangle being those in the three last columns:

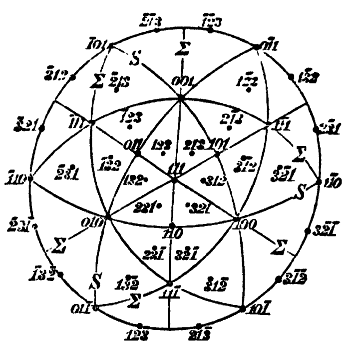
	s	c	σ	S	C	Σ
I.	90°,	60°,	60°;	70° 31·7′,	54° 44·14′,	54° 44·14′;
	90°,	60°,	45°;	54° 44·14′,	45°,	35° 15·86′;
II.	90°,	90°,	60°;	$\frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{\pi}{3}$
	90°,	90°,	30°;	$\frac{\pi}{2}$,	$\frac{\pi}{2}$,	$\frac{\pi}{6}$;
III.	90°,	90°,	45°;	$\frac{\pi}{2}$,	$\frac{\pi}{2}$,	$\frac{\pi}{4}$,
IV.	90°,	90°,	90°;	$\frac{\pi}{2}$,	$\frac{\pi}{2}$,	[illegible]

If to these we add (V) the case of a system symmetrical to a single plane, and (VI) that of a system merely symmetrical to a centre, we shall have represented every possible case in which a crystalloid polyhedral system can be said to be symmetrical at all. It is evident that these are precisely the cases that have been investigated in this chapter. These varieties of symmetry will hereafter be distinguished by the following designations:

I.	3 planes S,	6 planes Σ,	....	Cubic System.
II.	3 ,, S,	3 ,, Σ,	1 plane C,	Hexagonal System.
III.	2 ,, S,	2 ,, Σ,	1 ,, C,	Tetragonal System.
IV.	1 ,, S,	1 ,, Σ,	1 ,, C,	Orthosymmetric System.
V.	1 ,, S,	....	....	Monosymmetric System.
VI.	a centre of symmetry,		....	Anorthic.

And the planes of symmetry which characterise these several systems, and by their intersections with the sphere of projection in arcs of great circles determine the systematic triangle in each system, will henceforward be designated only as *systematic planes.*

130. *Relations of the cubic system to the trigonal and hexagonal system.* It results from this that the trigonal axes of the first of these systems—those namely which represent the normals of the regular octahedron—can in no case become axes of hexagonal symmetry, and therefore the planes of the octahedron cannot be planes of symmetry for that system. Of course planes belonging to a form analogous to the form $\{min\}$, the u of Fig. 47 of the trigonal system, and lying in great circles that bisect the angles



at which three planes Σ meet each other, may exist on crystals of this type. But they are essentially trigonal and not hexagonal in the character of their symmetry round the axes *O*. It will be furthermore apparent in the comparison of the cubic system with the trigonal type of symmetry, Figs. 50 and 51, that the case alluded to in article **119**, where $\tan rc = 2 \cot \lambda c$, is that in which a form in the trigonal system is directly comparable to one belonging to the cubic system; since then

$$rc = ho = 54^\circ 44{\cdot}14' = H$$

and $$r_1 r_2 = h_3 h_1 = 90^\circ$$

in the cubic system, and the three axes are perpendicular.

131. *Symmetry of faces.* The features that undergo symmetrical repetition in the faces of a polyhedron are the edges that form its sides and the plane angles in which any two of these edges meet so as to form one of the sides of a quoin.

The character of the boundary line of a face, as being an edge, is determined, not by the length of the side, but by the angle at which the face is inclined on the other face meeting it in the edge; and generally edges of which the angles are the same are geometrically similar and homologous, except in the ambiguous case of such angles being right-angles.

It is however necessary, especially where they are right-angles, to enquire as to the character of the faces that lying in the zone can intersect each pair of edge-forming faces, and to determine whether two different edges can have faces so replacing them as to cut off the edge in each case at the same angles to the corresponding faces.

And the symmetry of a face will be known when we know the law of repetition of its edges and angles; and this will obviously depend on the number and nature of the planes of symmetry that may intersect with it perpendicularly.

When it is not parallel to a zone-plane and is therefore also not parallel to a plane of symmetry it can only be perpendicularly intersected by a single plane of symmetry, and where it is not perpendicular to such a plane of actual or potential symmetry it can have no symmetry at all, except where the system is symmetrical to a single plane, when a face parallel to that plane will be centro-symmetrical. And when we consider any other position that a face may occupy on a crystalloid polyhedron, it is clear that its pole can only lie on a side or at an angle of a systematic triangle, or must be an independent pole; in which last case it is without symmetry.

When its pole lies on a side of a systematic triangle, the face is traversed perpendicularly by one plane of symmetry to the trace of which it is euthy-symmetrical; when the pole lies in an angle of a systematic triangle, and therefore at the intersection of two or of four, of three or of six planes, the face is symmetrical to an axis of symmetry and is ortho-symmetrical or ditetragonal, ditrigonal or dihexagonal in character.

Striations and recognisable physical features in the case of a crystal often throw light on the character of the symmetry that the face obeys which exhibits them; and it is from the aid which such characteristics, as exhibited in important faces, afford for the determination of the type of symmetry, not merely of the face but of the entire crystal, that this study of these features derives its significance: and indeed the information thus attained will be found often to go deeper than a symmetry that is merely morphological, and to involve the symmetry that rules in the distribution of physical properties and underlies the geometrical symmetry.

132. *The symmetry of a quoin* is determinable in a similar manner to that of a face. For the quoin is composed of edges and plane angles, and where these recur symmetrically the quoin will be symmetrical to a line which is an axis of symmetry of a corresponding order.

Thus, where a quoin is symmetrical to a zone-axis, its summit is capable of being truncated by a plane parallel to the corresponding zone-plane, and according with it in symmetry; where it is not symmetrical to a zone-axis, it may be symmetrical to a single plane or it may be altogether without symmetry. And it is evident that the symmetry which a quoin will present will correspond with that of the diameter of the sphere which meets its vertex. If, as in the case of the axis of symmetry in the clinorhombic system, the diameter in question is an axis of diagonal symmetry only, the quoin or a face replacing it is diagonally symmetrical to this axis. If, again, the diameter is a normal to a plane the pole of which lies on a side of a systematic triangle, the quoin is euthy-symmetrical to the systematic plane in which the arc lies that carries the pole. Where the axis of the quoin meets the sphere at the angle of a systematic triangle, the quoin is symmetrical to the zone-axis which at that point meets the sphere.

In every other case the quoin is devoid of symmetry. And the characteristics of the quoins of a crystal, as in the case of the faces forming them, offer one of the most important features by which to recognise the type of the crystal's symmetry.

133. *Symbols of truncating and bevilling planes.* From the principles established regarding crystalloid symmetry it will be seen that the only cases where the edge formed by two faces is truncated by a third face or bevilled by pairs of faces of the system will occur when the faces forming the edge are adjacent faces of a form symmetrical to a systematic plane, or else lie on a great circle traversing an axis of symmetry in regard to which they are symmetrical; so that the pole of a face truncating an edge will lie on an arc and may lie at an angle of a systematic triangle. It further results from the symmetrical character of the axial systems adopted for each crystalloid type that the symbols for two adjacent faces of a form either differ only in the signs of one or of two indices, or else differ by the permutation of certain of their indices. In either case, the ratios of the indices in the symbol of the truncating face is obtained by the addition of corresponding indices in the symbols of the planes whose edge is truncated.

In the Hexagonal system this rule will be found to hold directly only for those faces of, for instance, the general form $\{hkl\ efg\}$ which belong to the trigonal semiform $\{hkl\}$, or else to those belonging to the correlative semiform inverse to it $\{efg\}$; for in attempting to apply the rule to adjacent faces belonging, the one to a direct the other to an inverse form, it appears to fail. In fact, however, the principle involved in the rule is only obscured. The symbol of a form derived from the symbol of an inverse form by the same method as the latter is derived from that of a direct form should evidently be the identical symbol of the original direct form; so that the inverse form to $\{efg\}$ should be $\{hkl\}$. If however it be derived by aid of the formulae given in article **122**, this symbol will be found to be not (hkl) but $(9h\ 9k\ 9l)$; whence it is evident that if the condition above asserted is to be fulfilled we should have to consider the indices of the form inverse to hkl, i.e.

$$2(k+l)-h, \quad 2(l+h)-k, \quad 2(h+k)-l,$$

as equivalent not to efg but to $3e\ 3f\ 3g$, and in order to compare hkl with efg on, so to say, equal terms, we should take efg as ranking with $3h\ 3k\ 3l$. In order then to determine the symbol of a face truncating an edge of adjacent faces of a form $\{hkl\ efg\}$ we have, where the faces belong the one to a direct the other to the inverse semiform, to multiply the indices in the symbol of the face of the direct semiform by a common factor 3, and to add them to the indices belonging to the face of the inverse semiform. The rule will then be found to be general, and it applies equally, of course, to other forms besides those of the general scalenohedron.

Thus, for example, the face truncating the edge $(h\,l\,k)\,(\overline{e\,f\,g})$, that is to say, the edge of the faces $(3h\;3l\;3k)$, and

$(h-2$

is $\quad 4h-2k+l,\;\; k+l-2h,\;\; k+l-2h,\;$ or $\;2\bar{1}\bar{1}.$

CHAPTER VI.

CRYSTALS AS CRYSTALLOID POLYHEDRA.

SECTION I.—**Mero-symmetry.**

134. THE properties of a system of planes mutually related by the law of the 'Rationality of Indices' have been so far investigated as a *crystalloid system* from a purely geometrical point of view; and, by establishing as one of these properties the principle that the varieties of isogonal zones that can be extant in such a system are limited to four, it has been possible to shew that only a limited number of types or systems of symmetry can be illustrated in crystalloid plane-systems. When we turn to the natural polyhedra presented in crystals in order to determine to what extent these actually accord in their geometrical characters with the crystalloid systems hitherto considered, we cannot fail to recognise that whereas the crystallographer, guided heretofore solely by observation and experience, referred every crystal to one or other of six crystallographic systems, those systems furnish precisely the several types of symmetry which coincide in their distinctive features with the six crystalloid types of symmetry resulting from the above principle.

But in order to carry on the enquiry by means of exact observation into the geometrical relations connecting the faces of a crystal, we must have recourse to instrumental methods admitting of the requisite precision. This object is attained by the use of the Goniometer, an instrument constructed for the measurement of the angular inclinations of planes, of which the description and the use will be given in a future chapter.

The results that have been accumulated by means of this instrument form a body of observations on which crystallography as a science rests; but in dealing with these results the crystallographer is often more or less embarrassed by errors incidental to the use of instruments, and still more by difficulties due to peculiarities and imperfections presented by the faces of the crystals themselves.

In proportion however as the errors arising from such sources are diminished, it has been found that the values obtained for the angular inclinations of the faces of a crystal more and more closely accord with those which would result from the crystal being a crystalloid polyhedron; that is to say, from the Law of the Rationality of Indices being the fundamental law presiding in its construction.

And that this is true for all temperatures at which the integrity of the crystal is maintained may be assumed, since it is true within the limited ranges of temperature at which such measurements can be effected; while within these ranges of temperature some crystals are near the highest limit at which they can exist, while others are examined at temperatures far lower than those at which they have been formed.

The analogy of the law—that indices are integral coefficients of the parametral ratios—to the fundamental law of chemical combination by which bodies unite in simple multiples of their weight-equivalents, can hardly escape notice; and the inductive method by which each has been arrived at has consisted in an accumulation of experimental results scarcely less extensive and exact in the case of the crystallographic than in that of the chemical law.

135. A significant illustration of the occurrence on crystals of only such forms as are possible in a crystalloid polyhedron, is furnished in the fact that of the five regular solids, three, namely, the cube, octahedron, and tetrahedron, are frequently crystal forms, whereas the dodecahedron and the icosahedron have never been met with on any crystal. The three first figures are crystalloid in their symmetry; the faces of the two last cannot be expressed by symbols with rational indices, and they furthermore present a pentagonal symmetry which is of an order impossible, as has been seen, in a plane system obeying the law of rational indices. Experience however has, on the one hand, proved that a crystal is not merely thus externally and geometrically, but is also physically and throughout its substance symmetrical and æolotropic; while, on the other hand, it has led to the recognition of a natural principle which in a great number of cases limits the completeness of the symmetry of the crystal. Thus it not unfrequently happens that certain of the faces on a crystal which, in accordance with the geometrical principles of symmetry laid down in the last chapter, should constitute a form, will present in their physical characteristics differences so marked and occasionally so contrasted that it is impossible to view them all as equally repetitions of the same face.

136. This partition of the faces geometrically similar into physically dissimilar groups is, however, found to be itself obedient to principles of symmetrical distribution which concord with those of the crystallographic system to which the crystal belongs: and such an interruption in the complete accord of physical and geometrical symmetry will be seen to be a particular case only under a more general law which deals not only with a division of the faces of the crystal into correlative groups, but in general also with the entire suppression of all the faces not belonging to one of the groups.

137. *Mero-symmetry.* In the last chapter the character of a form under each of the different types of symmetry were considered, and the trigonal type was treated as a partial and incompletely developed variety of hexagonal symmetry.

While observations directed to such crystals shew that this incompletely developed type is represented abundantly in nature, numerous analogous cases of incomplete symmetry are also met with in other crystallographic systems.

Def. A holo-symmetrical form in any system will be the term applied to a form in which all the faces required to complete the symmetry of the system are present, and are physically as well as geometrically similar.

The term mero-symmetrical will be employed in all cases in which the faces requisite to build a geometrically complete form are partially suppressed, or in which these faces fall into physically contrasted groups; the suppression of faces or of the features characteristic of the form taking effect however in a certain symmetrical manner.

Mero-symmetrical forms may be hemi-symmetrical, and will then present one-half of the faces of the complete form; or tetarto-symmetrical, presenting one-quarter only of the faces of the holo-symmetrical form; a form of the hemi-symmetrical kind will be termed a semiform or a hemihedron, one of the latter kind a tetartohedron. And the term merohedral will be reserved for certain cases in which a defalcation is met with in the faces of a crystal out of accord with any fixed law of symmetry; though sometimes such a merohedral crystal simulates the mode of grouping of a crystal belonging to a different type of symmetry from its own.

138. *Hemi-symmetry.* Now in considering in what ways it may be possible, while conserving the essential idea of each type of symmetry, to suppress one-half of the faces of a form, we have to keep in view the principle that in surrendering or modifying the symmetral character of a systematic plane or zone-line, each corresponding similar systematic plane or zone-line must simultaneously undergo the same degree of deprivation of its symmetral character. And again, the kinds of mero-symmetrical forms will be essentially different according as the suppression of their faces is due to the form ceasing to be symmetrical to its centre, or to the symmetral character of a plane or group of planes of symmetry falling into abeyance.

If the crystal be not centro-symmetrical, each origin-plane will be represented by only one of its two poles on the sphere; or, to adopt the corresponding fiction of centro-normals, each normal will be represented by a single *ray*, that is to say, will carry but one face and one pole.

Where the form is centro-symmetrical, on the other hand, the suppression will affect the faces in parallel pairs; so that only one-half the origin-planes corresponding to faces of the crystal—and only one-half the number of normals—will be extant, the remaining half being absent.

And a mero-symmetrical form may further be conceived such that alternate normals only should be extant and should each carry but a single face.

139. *Holo-* and *mero-systematic forms.* Since failure of symmetry in the number of normals or of origin-planes belonging to a form in any system can only result from abeyance of symmetral character in one or more of the planes or groups of planes of symmetry which have already been designated as (*proto-*, *deutero-*, *trito-*) *systematic planes*, we may avoid the ambiguity in which the terms hemihedrism, tetartohedrism, &c. are involved by the more restricted or wider senses in which different authors have employed them, if we adopt a nomenclature consistent with our use of the former term.

Def. A *holo-systematic form*, then, is a form in which all the *origin-planes* or *normals* required by the complete symmetry of the system are extant.

A *hemi-systematic form* is a form in which only half the *origin-planes* or *normals* are extant, the correlative half being absent.

In a *tetarto-systematic form* only one-fourth of the origin-planes or normals can be considered as extant.

A *diplohedral form* will, as before defined, be a form in which every origin-plane is parallel to two faces (or has both its poles extant); or in which each normal is made up of two rays or carries two faces and their poles.

In a *haplohedral* form each origin-plane or each normal is represented by a single face and its pole.

140. *Kinds of mero-symmetry.* Whence there is—

I. Holo-symmetry, where a form is at once holo-systematic and diplohedral.

II. Hemi-symmetry, where a form is

i. holo-systematic and haplohedral, or ii. hemi-systematic and diplohedral. } *Semiforms* or *hemihedra.*

III. Tetarto-symmetry, where the form is

i. hemi-systematic and haplohedral, or ii. tetarto-systematic and diplohedral. } *Tetartohedra.*

IV. **Hemimorphism** is the term for a particular case of haplohedral mero-symmetry. One-half or, it may be, one-fourth of the faces of the original form are present in the hemimorphic form: but these all lie on one side of a systematic plane, the symmetral character of which is in abeyance.

141. *The law of mero-symmetry.* The conceivable modes in which either one-half of the normals or one-half of the faces corresponding to the full complement of normals (each represented by a single face) might be suppressed are evidently numerous and varied. And in the resulting polyhedra we should in many cases look in vain for any characteristic features of the symmetry of the system to which the holo-symmetrical form belonged. But in crystals, as has been before observed, some special quality distinguishing the original type of symmetry is always preserved in all their mero-symmetrical forms; and it is in accordance with this experience that we seek for a geometrical principle that shall embody such a condition.

Now the only geometrical assumption that can be made regarding the mero-symmetrical suppression of the faces of a system so as to satisfy this condition is one which we find in Nature to include all known cases of mero-symmetry, while without extension of its terms it will also be found to embrace the symmetrical conditions presented by holo-symmetrical forms. It may be stated then in the form of a general law of crystallographic symmetry, that on a crystal the extant or absent features of a form must be extant or absent in the same way in respect to equivalent systematic planes. This is the second fundamental law of crystallography.

142. The nature of the forms necessitated by this law in the different systems will be discussed hereafter. But certain of the general results which are involved in its application may be pointed out here.

Thus hemimorphism can only exist in relation to a unique systematic plane, since it could not hold in the case of two conformable systematic planes. It is thus precluded from every form of the Cubic system. It may, on the other hand, occur on a form otherwise holo- or hemi-systematic, and so be either hemi- or tetarto-symmetrical in its character (i. e. presenting only the half or the fourth of the faces of the complete form).

143. So, again, a tetragonal axis of symmetry is, so to say, the creature of the two pairs of alternating planes of symmetry *S* and Σ of which it is the zone-line, and in the holo-symmetrical case the poles of an independent general form $\{h\,k\,l\}$ are grouped ditetragonally round this zone-line. If, now, we suppose one of these pairs of systematic planes—say the planes Σ, Fig. 52 (1)—to fail of being symmetral, the result will be that either the planes *S* continue planes of symmetry and their zone-axis becomes an axis of ortho-symmetry, as in Fig. 52 (2); or the planes *S* also fail as planes of symmetry, while the zone-axis retains its character as an

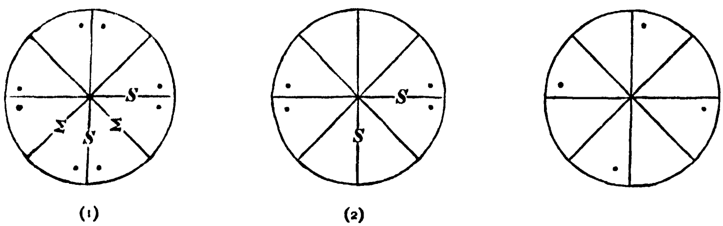


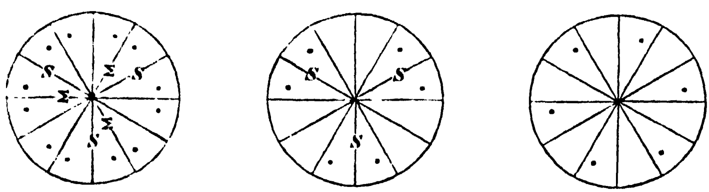
Fig. 52.

axis of tetragonal but not of ditetragonal symmetry, the grouping of the faces round it being in alternate systematic triangles, as in Fig. 52 (3).

144. In the same way, Fig. 53 (1), an axis of hexagonal (in the general case dihexagonal) symmetry is the zone-axis of two triads of alternating planes of symmetry; and the mero-symmetrical suppression of half the poles of a form can be effected by the suppression either of the symmetral character of one of the triads of systematic planes, or of both triads simultaneously. If, for example, the Σ planes are the triad of which the symmetral character is in abeyance, the grouping of the six faces that remain extant out of the twelve of the original scalenohedral form $\{hkl\}$ lying on one side of the equatorial plane will be such that the dihexagonal axis becomes a ditrigonal axis; see Fig. 53 (2).

If, on the other hand, both triads of planes are no longer planes of symmetry, the twelve faces may become reduced to six in such a way that the poles lie in alternate systematic triangles, see Fig. 53 (3); the zone-axis, then, continues to be an axis of hexagonal but not of dihexagonal symmetry.

In a trigonal system, indeed, in which the axis of form is an axis of ditrigonal symmetry and is the zone-line of three conformable systematic planes S_1, S_2, S_3, it is evident that the abeyance of the symmetral character must take effect on all three or on none of these planes; that is to say, the ditrigonal axis becomes a trigonal axis, or else the system, if conceived of as



(2) (3)

Fig. 53.

derived from one originally diplohedral, must become hemimorphous in respect to the zone-plane $[S_1 S_2 S_3]$.

145. In the case of an axis of ortho-symmetry which is the zone-axis of two perpendicular planes of symmetry S and Σ, Fig. 54 (1),

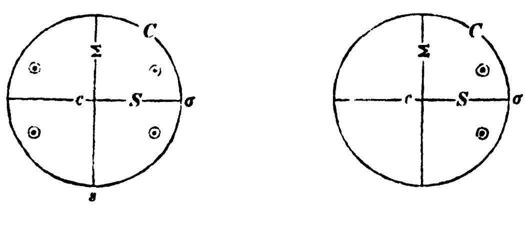
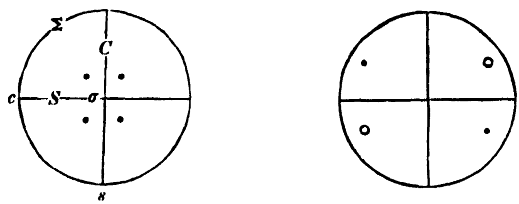


Fig. 54.

if the symmetral character of one and one only of these planes is in abeyance, for instance that of the plane Σ, their zone-axis will not be an axis of symmetry. In this case, however, it will be seen that in a holo-systematic system the zone-axis of the planes *S* and *C* (*C* being in a diplohedral system a third systematic plane perpendicular to *S* and Σ) must be an axis of ortho-symmetry, and that the system will be hemimorphous in regard to the plane Σ, see Fig. 54 (2) and (3). On the other hand, the symmetral character of the two planes *S* and *C* may both be in abeyance; and then there is no symmetral plane and each zone-axis becomes an axis of diagonal symmetry, as in Fig. 54 (4).



(3) (4)

Fig. 54.

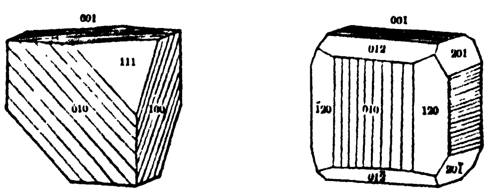
146. In the case of the Mono-symmetric system, obeying only one plane of symmetry of which the normal is an axis of diagonal symmetry, the only available varieties of mero-symmetry are hemimorphism resulting from the abeyance of the symmetral character of this single systematic plane, and the case of a semi-form presenting two poles symmetrical to that plane. For, were the form diplohedral, it would retain but a single normal and its symmetry be undistinguishable from that of an anorthic crystal.

The varieties of mero-symmetrical partition, which faces of a general independent form may undergo in any particular crystallographic system, will have to be considered in detail in chapters specially devoted to the description of the forms of the different systems. But their character is always determined by the necessary conditions that the distribution of the faces must be consistent both with the law of mero-symmetry, and with the rules which have been laid down in the preceding articles regarding the order of the symmetral influence which an axis of symmetry may retain when some or all of the systematic planes of which it is the zone-axis lose their symmetral character.

147. That the different forms of the same crystal cannot be simultaneously obedient to different types of symmetry, that is to say, cannot belong to different systems, is as obvious a necessity as that the different features of the same form cannot be so. There remains however the question whether a crystal hemi-symmetrical in regard to a particular form or forms can exhibit holo-symmetry in regard to other forms; for instance, where geometrically diplohedral forms are found concurrent with haplohedral forms, it may be asked whether the possibility of such a concurrence is not proved.

The general answer to this question is however to be found in the principle that planes of symmetry are such for all features of a crystal, and, where they are in abeyance at all, they are in abeyance for all the forms.

If the effect of any particular mero-symmetrical principle in influencing the different kinds of forms of the same crystal be considered, it will be seen that what in the case of one variety of form will result in the suppression of the half of its faces, may in another produce not a suppression of any of the faces, but a corresponding loss of symmetry in the outline or in the physical characters and molecular structure of each and all the faces of the form. The latter case, that namely in which the suppression takes effect, not in the obliteration of half the faces of the form but in that of half the features of each face—in fact in the partial suppression of the symmetry which these faces would obey in the holo-symmetrical form—occurs wherever the poles of the form lie on any systematic great circles whereof the symmetral character is



(1) (2)

Fig. 55.

in abeyance. A cube with its alternate quoins truncated by the faces of a tetrahedron would illustrate such a suppression. Its six faces are all ortho-symmetrical, not tetra-symmetrical. Thus the striations on the faces of the cube in Fig. 55 (1) will be seen to run parallel only to the edges these faces form with those of the tetrahedron which truncate its alternate quoins; while in Fig. 55 (2) the cube faces are striated parallel to their intersections with alternate pairs of faces of the rhomb-dodecahedron. Fig. 55 (1) represents a crystal of blende symmetrical to the Σ-planes only and haplohedral; Fig. 55 (2) is that of a crystal of pyrites symmetrical to the S-planes only, but diplohedral. In both the symmetry of the cube-faces is seen to be orthogonal, not tetragonal.

148. A uniform notation to represent the kind of mero-symmetry presented by any correlative pair of hemihedra or group of four tetartohedra is of some importance; and the more as the Greek letters which have been employed as prefixes to the symbols of holo-symmetrical forms in order to represent their semiforms have received different significations from different authors.

In the Tetragonal system, for instance, the semiforms represented by Prof. Miller as $\lambda\,\{hkl\}$, $\kappa\,\{hkl\}$, and $\alpha\,\{hkl\}$ carry in the treatise of Prof. V. von Lang the symbols $\gamma\{hkl\}$, $\chi\{hkl\}$ and $\kappa\{hkl\}$ respectively.

The symbol π is alone in carrying by general consent a constant signification, namely, that the semiform it designates is diplohedral; but as employed in the Cubic and Tetragonal systems, for instance, it represents different ideas of symmetry.

To avoid confusion, therefore, a notation will be adopted in this treatise such that, in the cases where ambiguity has heretofore arisen, the letters employed as prefixes will recall by their sounds the nature of the symmetry that is not in abeyance and which therefore controls the extant form.

The following prefixes will accordingly be employed, in the case of hemi-symmetrical forms, to represent that the faces of these forms which are extant (or absent) are so symmetrically, *only*

(*as haplohedral and holo-systematic forms*)

1. to one or more zone-lines of symmetry .. .. **a**

2. to one or more such resultant zone-lines, and also

to the (proto-systematic or) S-planes	s
to the (deutero-systematic or) Σ-planes	σ
to the (trito-systematic or) C-plane, *see below*, ϕ.	
to the C-plane and the S-planes	x
to the C-plane and the Σ-planes	ξ
to the S-planes and the Σ-planes	ρ
(*as diplohedral and hemi-systematic forms*)	
to the S-planes and to a centre of symmetry	π
to the Σ-planes and to a centre of symmetry	ψ
to the C-plane and to a centre of symmetry	ϕ

The diplohedral character of the last three kinds of form is suggested by letters which involve the sound of π (opposite poles on the sphere separated by a distance π being extant in such forms), the meaning of the affix π being restricted to a special case; while the double letters x and ξ serve to recall the letters CS or $C\Sigma$ that represent those of the systematic planes which alone retain an actually symmetral character and thus determine the nature of the symmetry; ρ has been used by V. von Lang to represent hemimorphism in general, and is here retained for a frequent case of hemimorphism, that on the trito-systematic plane.

The usual mode of representing a tetarto-symmetrical form is that of uniting the prefixes corresponding to two out of the three pairs of hemihedra that may be constituted out of the faces of the four correlative tetartohedra.

Thus, in the Tetragonal system we have for the mero-symmetry of the general form $\{hkl\}$ (see Plate II) three hemi-symmetrical pairs of forms,

$$s\{hkl\},\ s\{khl\};\quad \sigma\{hkl\},\ \sigma\{hkl\};\quad \phi\{hkl\},\ \phi\{khl\}:$$

and these three sets of correlative semiforms may be produced by combining in distinct pairs four tetarto-symmetrical forms, which may be therefore indifferently designated as

$$s\sigma\{hkl\},\ s\sigma\{khl\},\ s\sigma\{\bar{h}kl\},\ s\sigma\{\bar{k}hl\};$$

or $s\phi$

or $\sigma\phi$

149. It results, as a consequence, from the law of mero-symmetry, that any holo-symmetrical independent form in the Cubic, Tetragonal, or Hexagonal systems may be considered, from a purely geometrical point of view, as a composite form capable of being analysed into various correlative pairs of hemi-symmetrical forms, while these again may undergo further analysis into tetarto-symmetrical forms; each stage in the analysis necessarily subtracting from the completeness of the symmetry of the figure, yet always so that the type of symmetry distinguishing the system is recognisable. Conversely, the various semiforms may be built up by a synthetical process of combining correlative tetarto-symmetrical forms in pairs; while again by the assemblage of the faces of either pair of corresponding semiforms the complete holo-symmetrical form will be constituted.

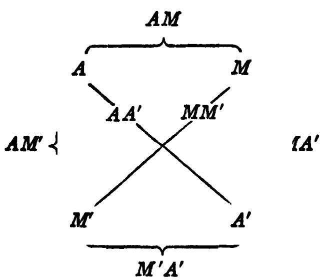


Fig. 56.

Thus, A, M, A', M', four correlative tetartohedra, form three pairs of correlative hemihedra AM and $M'A'$, AM' and MA', AA' and $M'M$, the faces of each correlative pair constituting if united the holo-symmetrical form. See also plates I to VII.

150. In discussing, by way of illustration, certain kinds of forms resulting from the disparting of holo-symmetrical polyhedra into constituent correlative hemihedra, the principle of the partition was not discussed. Evidently each semiform may appropriate one face from the pair belonging to every normal, or, on the other hand, may be built up of pairs of parallel faces belonging to alternate normals. But there are consequences depending on this character of the mero-symmetry which must not be lost to view. Thus, in the haplohedral holo-systematic case the original form is so disparted into two semiforms that, for each face of the one semiform, a face, parallel to it in the complete form, belongs to the other semiform. Whether the two semiforms can be brought into congruence will depend on whether their faces are severally capable of being so, metastrophically; and this, again, on the mode in which these faces are grouped.

From the general scalenohedron of a system may be derived two kinds of correlative semiforms. These may be such that the poles of one semiform lie in alternate systematic triangles—and so, therefore, that the faces of the one semiform are metastrophic to each other, but antistrophic to the faces of the other semiform: or, it may be that metastrophic are united with antistrophic faces in each of the semiforms. But it is impossible to partition the faces of a semiform of haplohedral character between two tetartohedral forms, otherwise than so as that each of the latter forms shall have all its faces antistrophic to the faces of the other correlative tetartohedron. The configuration of the one tetartohedron will then correspond to that of the other as seen in a mirror. In a word, the two tetarto-symmetrical forms are *enantiomorphous.*

And since antistrophic asymmetric faces on a crystal cannot be brought into congruence by the revolution of the crystal round any diametral line, it is not difficult to determine whether in any particular case correlative mero-symmetrical forms are enantiomorphous or tautomorphous; i.e. cannot be brought into congruence, or can be so brought by revolution round one or more zone-lines.

If the tetartohedra in Figure 56 that are represented by different letters are antistrophic and those with the same letters metastrophic to each other, of the correlative pairs of hemihedra formed by their combination those represented by union of different letters and formed by antistrophic tetartohedra will be tautomorphous, viz. AM with $M'A'$, AM' with MA', while the pairs represented by union of the same letters, viz. AA' and MM', are formed by metastrophic tetartohedra but are themselves enantiomorphous.

151. It has been stated in articles **135** and **136** that, whereas the hemi-symmetrical developement of a form implies in general the suppression of one of the correlative groups of its faces, the two groups may nevertheless be concurrent on the crystal; but that, whether they are so concurrent or are found only on separate crystals, they are often distinguishable from each other by differences in external feature or otherwise in physical character. It may be evidenced by variation in smoothness or lustre, or in roundness or plane character of surface, in the directions, depth and form of striations or of hollows, or in the relative magnitudes habitual with the faces of the respective semiforms; or, also, it may be associated with polarity of character in the physical properties of the correlative hemihedra, especially under pyroelectric excitement due to changing temperature. Such differences, then, may generally be held to indicate a mero-symmetrical habit, and they often impart a very marked contrast to the faces of the correlative semiforms. It may occur, on the other hand, that the distinctive features of concurrent mero-symmetrical forms may not be apparent or recognisable. That they should nevertheless exist follows as a consequence of the principles of crystallographic symmetry.

In the case of semiforms which are holo-systematic, and therefore haplohedral and not centro-symmetrical, each normal of a form would carry either a single face or else two parallel faces which differ in physical features; and the fulfilment of the conditions imposed by the mero-symmetry of the system is compatible with the supposition that the crystal is endowed with different—or also, in a polar sense, opposite—properties in opposite directions of any given line. This character has been designated by V. von Lang as Antihemihedrism.

When the crystal is at the same time hemi-systematic and haplohedral the forms are tetartohedral, and for two of the four quarter-forms which are tautomorphous the character of the distribution of the properties and of the features (as, for instance, of the poles) round corresponding lines in the crystal will be metastrophic, but will be antistrophic to that round corresponding lines in the two quarter-forms enantiomorphous to the former.

Along the principal axis of symmetry in particular crystals of the Tetragonal and Hexagonal systems, and along every direction in certain crystals of the Cubic system, a ray of plane-polarised light acquires rotatory polarisation. That this property should be confined to haplohedral crystals, hemi-symmetrical or tetarto-symmetrical, in which all the symmetral planes of the original holosymmetrical form are in abeyance, will be shown hereafter, when the physical characters of crystals are under discussion, to follow from the laws of symmetry.

152. In this chapter the subject of mero-symmetry has been treated as involving the presence or the absence of certain faces, consequent upon the abeyance of the actual symmetral character of planes which are otherwise potentially planes of symmetry. But in this treatment of the subject a symmetral influence has still been recognised as, so to say, latent in these dormant systematic planes; inasmuch as the zone-lines, which may be considered as becoming axes of symmetry by virtue of the original symmetral nature of those planes, generally retain this character in a greater or less degree, notwithstanding the abeyance of a direct symmetral influence in the planes themselves upon the forms of the crystal. By an inverse method of treating the subject of symmetry, it would have been possible to have evolved the laws of symmetry, and deduced those of mero-symmetry, from a discussion of the conditions regulating the degrees of symmetry possible round a zone-axis, and to have considered the various planes and groups of planes of symmetry, not as originating axes of symmetry so much as being the results of the symmetral character of such axes.

It is however evident that either method, and indeed that the whole treatment of crystallographic symmetry on the assumption of planes and axes of symmetry, actual or potential, represents a geometrical abstraction; an abstraction that needs for its developement and due explanation a complete science of position applied to the molecular mass-centres, competent to embrace not merely the relative distribution *inter se*—the *intermolecular* distribution—of the chemical molecules constituting the crystallised substance, but also the *intromolecular* arrangement of the atoms, or molecules of secondary order, whereof the molecules of the substance are themselves composed. Then the true significance of the ideal planes and axes of symmetry will be understood; and they will assuredly retain a place in the explanation of crystalline symmetry, since they rise into recognition directly from the fundamental principle of rationality of indices and are controlled by its consequences.

Section II.—On Composite and Twin Crystals.

153. Crystals have thus far been considered as single structures, each complete in itself: but this structural individuality is far from being the only or even the most frequent form of their occurrence. Whether as minerals or as products of the laboratory, crystals continually present themselves as aggregates; often, undoubtedly, united by no definite law, but often also so combined that, while corresponding faces of different individual crystals are quite or approximately parallel, they appear as belonging to a single crystal with its faces interrupted by recurring and by re-entering edges, or tesselated by surfaces not lying precisely in a plane. And a large proportion of the crystals that appear as single individuals are thus composite, built up of more or less numerous single but similar crystals, nearly but often not quite parallel in orientation, though continuous in their substance and in optical contact. In many crystals these component individuals exhibit faces, generally minute, to which it is only possible to assign symbols of a complex kind, that is to say, with indices that are high and often only approximate in their ratios; and to this cause numerous peculiarities have frequently to be traced, such as belong to forms of which the faces are rounded, tesselated, or terraced, or which exhibit ridged or 'drused' surfaces, wartlike protuberances, re-entrant edges, or hollows whereof the sides and base are facetted with crystallographic planes.

Instances of these peculiarities are familiar to the mineralogist in apophyllite, tourmaline, quartz, idocrase, galena, blende, diamond, calcite, etc.

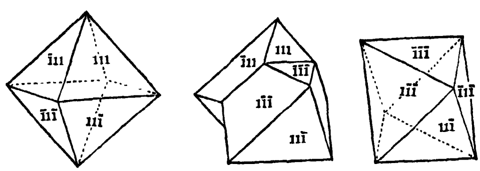
But there is another and somewhat more regular manner in which a composite structure asserts itself, that, namely, in which the corresponding faces of the united crystals, though not in the same plane, have a definite relative orientation. A law which, by the geometrical fiction of a rotation of one of the crystals, expresses the character of this orientation will be enunciated in the ensuing article. Such crystals as obey it are termed *hemitropic*, *macled*, or *twinned* crystals. Certain cases, however, which this law does not suffice to explain will be discussed when the forms and combinations presented under the different systems are described. They belong for the most part, though not exclusively, to symmetrically grouped tetartohedral forms.

154. Def.—*Twin-crystals.* Two crystals are said to be *hemi-tropic* or *twinned* when, presenting identical forms, they are united together in such a way that, if we conceive one of them as being turned through half a revolution round a particular line which will be termed *the twin-axis,* the two crystals would have identical orientation; that is to say, corresponding faces and edges in the two crystals would become parallel. Or, if the initial orientation of the crystals was the same, such a rotation would bring them into the relative position of twin-crystals. The plane to which the twin-axis is perpendicular is termed the *twin-face* or *twin-plane.*

The twin-axis may be (1) a face-normal, or (2) a zone-line, or (3) it may in certain cases be at once a face-normal and a zone-line: finally (4) there are crystals in which the twin-axis has been stated to be a line perpendicular to a zone-line and parallel to a face of the zone, while not itself either an edge or a normal. In fact, it is the result of observation that in the four first, or ortho-symmetrical, systems, in all of which it has been seen that three lines, at once normals and zone-axes, co-exist perpendicular to each other, the twin-axis is almost invariably a face-normal, and is, in some cases, at the same time a zone-axis. In the Mono-symmetric system the axis of twinning occurs as either a zone-axis or a face-normal, but by the conditions of the system cannot be at once a zone-axis and a normal, if the crystal be holohedral, since the single systematic axis is already an axis of diagonal symmetry. In the Anorthic system the twin-axis is found to be either a normal or a zone-axis; unless, further, as is maintained by some eminent crystallographers, there may be cases where it is neither of these, but forms with a normal and a zone-line perpendicular to each other a third line perpendicular to both. This question will recur for discussion in the section on the twins of the Anorthic system. In no case, however, can an axis of actual diagonal symmetry or one of higher symmetry, except in the case of a trigonal axis, be a twin-axis for the two crystals. Otherwise, the half rotation of either crystal round such a diagonally symmetral axis would only serve to bring the two crystals again into an identical aspect, and they would as at first merely present the condition of a parallel aggregation, not to be confounded with a twinned structure. It will also be evident that, a twin-axis being thus precluded from being an axis of diagonal symmetry, a face of one crystal can only be parallel to the corresponding face of the other, when it is either parallel to the twin-axis or perpendicular to it.

Where the twin-plane is a crystallographic plane, it must be parallel to an identical origin-plane or zone-plane for both the crystals, and will have the same symbol whether considered as belonging to one or the other crystal: the symbol, indeed, of the twin-plane will be the same for both crystals even in cases where this is not a crystallographic face and its symbol therefore has not rational indices.

In Fig. 57 the rotation of the regular octahedron *a* round the



a *c* *b*

Fig. 57.

normal of the face $11\bar{1}$ would bring that figure into the position *b*. The figure *c* exhibits the two octahedra *a* and *b* united so as to form a twin-crystal: it is the so-called *spinel-twin*, being of frequent occurrence in the mineral spinel.

Hemitropic crystals, in cases where they are merely juxtaposed (the cases to which the term hemitropic was more exclusively applied), are frequently but not invariably united at the surfaces of their common twin-plane: in fact, if we assume with Haidinger that the plane of junction must have the same relation to the two crystals, it will be clear from what has been said above in this article that the surface at which the two crystals are in contact must be either the twin-plane or a plane in the zone perpendicular to it. The face or plane in which two crystals are thus in contact is termed the *face* or *plane of union*, or also the *combination-plane*. There are cases in which this plane would seem to have irrational indices and thus not to be a face of the crystal.

155. In certain cases all or some of the faces of one crystal can have rational symbols when referred to the same axes and parameters as the other crystal. Such rationality is however confined to cases (1) where the crystal belongs to the Cubic system, or (2) where the crystal belongs to the Tetragonal or the Hexagonal system and the twin-plane is parallel or perpendicular to the morphological axis, i. e. is a face belonging to a zone containing planes of abortive symmetry in both crystals, or is perpendicular to that zone.

156. The two individual crystals which compose a twin-crystal never have their individuality merged in the resulting structure, the material of the one being never so intimately blended with that of the other that the two cease to coexist independently side by side, however minute the laminæ or intercalated parts may be which represent the several individuals; and there are cases in which the twin-structure recurs in successive parallel repetitions so numerous as to be represented by a series of laminæ of the utmost tenuity.

But the degree of intimacy in combination which twin-crystals may present is very varied.

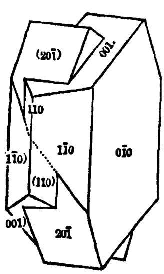


Fig. 58 *a*.

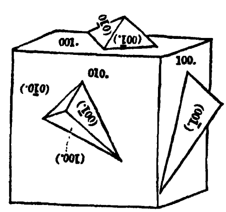


Fig. 58 *b*.

The two individuals may present a mere contact at a common surface or plane of union, as in the case of twins by hemitropic *juxtaposition;* such is the *juxtaposed* twin represented in Fig. 57 *c*, Article **154**: or there may be an *interlocking* of the crystals, each being bedded more or less deeply in the other, as in the (so-called Carlsbad) *embedded* twin of orthoclase, Fig. 58 *a*: or again, there may be a complete mutual *interpenetration* of the individuals, as in the Fig. 58 *b* of an *interpenetrant* twin of galena, generally with plane surfaces of junction, occasionally, however, the material of the one being so intercalated in that of the other that the individuals are united at surfaces with jagged, or waved, or otherwise irregular outlines. And in the case of *polysynthetic* twins several or almost innumerable hemitropic individual crystals may be combined—each individual being formed of a crystalline plate, and all being twinned on the same plane—so as to build up an essentially laminated structure, the alternate laminæ being in crystallographic orientation parallel to each other but hemitropic to the crystal-laminæ that intervene between them. Characteristic groovings or striations are often produced by the outcrop of these alternating laminæ where they present edges that are actually or nearly parallel in the two series, but are formed by faces which do not for either series coincide with a plane perpendicular to that of the laminæ. Such striations are thus due to ridges formed by the alternate crystal-layers, and they result in a corrugated surface which, when the laminæ are very thin, has the appearance of a crystalline face; as in Fig. 59 of a polysynthetic twin of albite and Fig. 60 of labradorite.

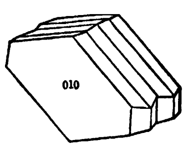


Fig. 59.

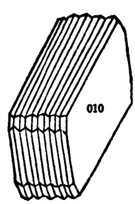


Fig. 60.

157. Where the twin-axis is a crystallographic line which in accordance with crystal-symmetry would be repeated in the crystals, it does not follow that the twinning process will be repeated for the other lines, whether normals or zone-axes, homologous to that line. Repetition of the twinning on similar twin-faces may indeed occur, and in certain kinds of zones the twinning often does recur, but not as a necessary result of the law of symmetry. Such crystals are triple, quadruple, &c. hemitropes (or triplings, fourlings, &c.). And crystals occasionally exhibit twinning of more than one kind; that is to say, one or more individuals twinned on a given crystal round one axis may be united with another individual or twin-group twinned on the same nuclear crystal round another axis not homologous with the former axis.

158. A noteworthy feature in hemitropic crystals is the mode in which the substance of the crystals is distributed in different directions; it being a frequent habit of twins that the thickness of the combined crystals in the direction of a twin-axis is no greater and is often less, relatively to the thickness along other directions, than would be the case for a single individual not twinned but otherwise corresponding in dimensions with the twin.

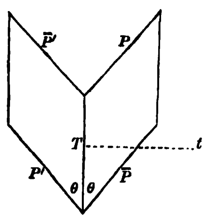


Fig. 61.

159. It will be seen that, in the case of two hemitropic diplohedral crystals, the twin-plane becomes *in a crystallographic sense* a plane of symmetry to the twin-structure; but at the same time each crystal in the hemitropic group retains its individuality, notwithstanding the mutual interpenetration of the crystals. From this it results that two faces of the different individuals may meet in an edge of which the internal angle is greater than π; or, in other words, the edge may be re-entrant, which is not the case in a simple crystal. In Fig. 61, if $P\bar{P}$ be the traces of two parallel planes of one individual, $P'\bar{P}'$ those of the corresponding planes of the other individual, the trace of the twin-plane being T and the twin-axis t, it will be seen that in the hemitrope position, as in the figure, one pair of the faces forms a re-entrant angle $\bar{P}'P = 2\,\theta$ and the other pair a salient angle $P'\bar{P} = 2\,\theta$.

Re-entrant edges of this kind then are highly but not exclusively characteristic of twinned crystals; at the same time they are not always to be met with upon them (see Fig. 62). If θ in Fig. 61 be a right angle, there can be no geometrical evidence, afforded by the faces $P\bar{P}'$ or $\bar{P}P'$, of the twin structure. It may happen that the twin-faces of the two individuals in cases where the plane of junction is perpendicular to these faces may form a single plane, or again, as in Fig. 62 of a crystal of hæmatite, that faces belonging to a zone, the zone-line of which is the twin-axis, thus fall into coincidence, and in the crystal in question, Fig. 62, both these conditions occur: but it is generally possible, at

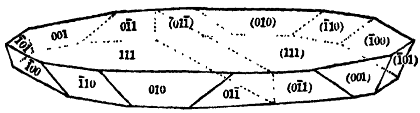


Fig. 62.

least in the latter case, to trace on the composite face the line of junction of its component faces; and frequently the needed evidence of twin-structure is afforded by the fact that the directions of the striations on these faces are symmetrically inclined on the line in question.

160. It follows from the foregoing statements that, in the case of diplohedral forms, whether holo- or hemi-symmetrical, a plane of actual symmetry to the individual crystals is precluded from being the twin-plane of their hemitropic union. It will be seen, however, that in the case of haplohedral crystals a face parallel to a plane of actual symmetry may be a twin-plane, since the normal of such a face is not an axis of diagonal symmetry.

In the case where a hemi-symmetrical crystal is thus twinned on a systematic plane, the poles of the two individuals will evidently fall into the same positions on the sphere as those of a simple holosymmetrical crystal. Hence it might be anticipated that cases would occur in which it might be a question whether we had to deal with a hemi-symmetrical crystal disguised by twin structure, or with a crystal, really holo-symmetrical, but exhibiting in its growth a composite structure simulating the characters of a twinned crystal. And further it might be suggested that in the case of a hemi-symmetrical crystal, on the forms of which the faces of both the correlative semiforms were concurrent, the resulting appearance of a holohedral developement may be due to the twinning of the correlative forms with each other. But in each of the supposed cases the question would in general be answered by the inspection of the crystal itself: since the two crystal-elements composing a twin can in fact in almost every case be distinguished from each other, owing to their retaining their individuality side by side, however minute the individuals may be and however completely they may exhibit mutual interpenetration.

In the case, for instance, of the haplohedral Cubic mineral blende, all the faces of the octahedron, that is to say, the faces o and the faces ω of the two correlative tetrahedra, may be present together on a crystal without twin structure, so that the geometrical character of the crystal would be holohedral; that the crystal is however of hemi-symmetrical character will be readily recognised by the faces of the octahedron in attingent octants being found to differ in physical character from those in the octants adjacent to them.

The diamond, on the other hand, offers an instance of a Cubic mineral presenting bright plane faces of the regular octahedron, or more or less rounded faces of other forms, developed in exactly the same manner geometrically and physically in all the octants, as if they belonged to a holohedral crystal. The edges, however, which lie in the proto-systematic planes, e.g. those of the octahedron, are frequently furrowed more or less deeply with re-entrant grooves, which are facetted with faces rounded when those adjacent to them are so, or in the case of the simple octahedron, are formed by plane faces parallel to the adjacent ones of the octahedron *.

A question of great difficulty has arisen whether diamond is to be considered as a crystal of tetrahedral (haplohedral) developement, twinned on the faces of the rhomb-dodecahedron,

* See Sadebeck, *Zeitschrift d. deutsch. geol. Gesell.* 1878.

as in the figure, one pair of the faces forms a re-entrant angle $\bar{P}'P = 2\theta$ and the other pair a salient angle $P'\bar{P} = 2\theta$.

Re-entrant edges of this kind then are highly but not exclusively characteristic of twinned crystals; at the same time they are not always to be met with upon them (see Fig. 62). If θ in Fig. 61 be a right angle, there can be no geometrical evidence, afforded by the faces $P\bar{P}'$ or $\bar{P}P'$, of the twin structure. It may happen that the twin-faces of the two individuals in cases where the plane of junction is perpendicular to these faces may form a single plane, or again, as in Fig. 62 of a crystal of hæmatite, that faces belonging to a zone, the zone-line of which is the twin-axis, thus fall into coincidence, and in the crystal in question, Fig. 62, both these conditions occur: but it is generally possible, at

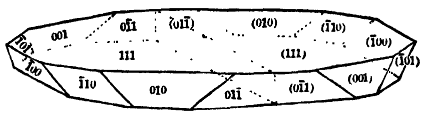


Fig. 62.

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* See Sadebeck, *Zeitschrift d. deutsch. geol. Gesell.* 1878.

or whether it is in fact a holohedral crystal, in which the furrowing of the octahedral edges is to be explained, not by a principle of twinning, but by a strong tendency to a lamellar developement during its growth; the crystal having, for so hard a body, a very facile cleavage parallel to the faces of the regular octahedron. The difficulty of answering this question is not diminished by the dissimilarity in habit of the diamond to other crystals; and in support of the latter of the two views the following considerations have been put forward:—first, the occurrence of a dodecahedron face as the twin-plane for haplohedral crystals of the Cubic system is otherwise rare, and when such twins occur they have a very marked character as interpenetrating twins of an entirely different habit from the crystals of diamond; which mineral, on the other hand, frequently occurs in twins of the spinel-twin variety: secondly, that, although simple crystals of diamond exhibiting indubitable tetrahedral habit (e. g. with forms of the hexakis-tetrahedron) do occur, yet they are among the rarest specimens*; and holosymmetrical crystals, such as magnetite, spinel, and gold, are also described as occurring with distinct tetrahedral habit: thirdly, that there is no evidence of any internal structure suggesting the idea of an aggregation of separate individuals: and finally, that there is clear evidence in the crystals of the diamond of a lamelliform structure similar to that which imparts to some unquestionably holo-symmetrical Cubic minerals, such as magnetite, steinmannite and cuprite, re-entrant octahedral edges and grooved surfaces on the planes of the rhomb-dodecahedron, somewhat similar to those which are met with in octahedra and rhomb-dodecahedra on the diamond.

In cases in which a hemi-symmetrical crystal is twinned on a systematic plane, the normal of which would be at least a dia-symmetral axis were its symmetry not in abeyance, it results that systematic planes of each group as well as equivalent lines will present parallelism in the two crystals. It will also be seen that any one of the systematic planes of the group to which the

* In the British Museum there are two diamonds in the form of hexakis-tetrahedra, and two others that are hexakis-tetrahedra with faces of the tetrahedron.

twin-plane belongs may be itself regarded as the twin-plane. Further, as stated above, the two crystals are mutually interpenetrant; sometimes, indeed, in a manner simulating the features of holo-symmetrical crystals.

DEF.—These combinations of correlative semiforms thus have the character of parallel growths and were termed by Haidinger *supplementary twins*. Certain correlative tetartohedral forms may be twinned in a manner analogous to that of supplementary twins; but they will obviously only build up a constructive hemihedral form, and the symmetry will be of the nature characterising that form.

161. There is a remarkable tendency in crystals to assume by the aid of twin-composition a higher degree of symmetry than that characterising the system to which they belong. And where the twin-plane, as in the case of haplohedral crystals, does not become a plane of symmetry, it will occasionally happen that the instinct, so to say, for symmetry will be satisfied by a face other than the twin-face, but perpendicular to it, being the plane of junction; this face becoming a plane of symmetry to the composite structure. Formerly Professor Groth, in his definition of twin-structure, virtually restricted it to '*symmetrical-twins*,' in which the individuals are symmetrically disposed with respect to a plane which is not a plane of symmetry for either of them; a definition the bearing of which will be considered in a future chapter.

162. The twin-law, though true in the form in which it has always been enunciated so far as concerns a representation of the nature of the union of hemitropic crystals, appears to permit of considerable divergence from precision in the relative orientation of the crystals subject to it; the angles obtained by measurement from faces on the different individuals giving in many cases results which accord less exactly with those obtained by calculation than is the case with the angles on the individual crystals. On hemitrope crystals of albite, the twin-axis of which was in each case normal to (010), Des Cloizeaux * found variations in the angle between a pair of faces theoretically parallel which amounted to from 40′ to 1° 40′.

* *Man. de Minéralogie*, t. i. p. 320.

163. It results from the law of hemitropy that each pair of corresponding faces on the two crystals lies in one zone with the twin-plane and that the faces make equal angles with it. So that, Fig. 63, if P, a pole on one of the individuals, correspond to P', a pole on the other individual, and a second pole Q on the first correspond to a pole Q' on the second, T being the pole of the twin-face, $TP = \frac{1}{2}PP',\ TQ = \frac{1}{2}QQ'$;
and if a pole L on either crystal can be found and determined in the same zone with P and P', and a pole R on the same crystal in a zone with Q and Q', the indices of T may be found, since $[QR]$ and $[PL]$ are tautohedral in T.

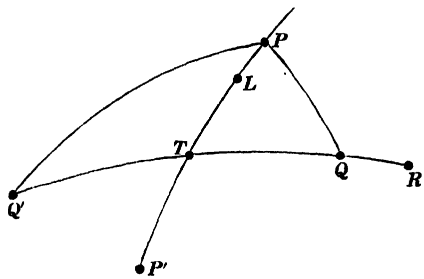


Fig. 63.

If great circles be drawn through PQ and PQ', we have

$$\cos PQ' = \cos TP \cos TQ' + \sin TP \sin TQ' \cos PTQ',$$

$$\cos PQ = \cos TP \cos TQ + \sin TP \sin TQ \cos PTQ,$$

and thus $\cos PQ' = 2 \cos TP \cos TQ - \cos PQ$;

an equation by means of which the angle between any two faces belonging to the two individuals may be found.

In order to find the indices of the twin-plane where they are not given directly by inspection or by the zone-rule from recognisable faces, it is necessary in the first place to determine, by measurements with the goniometer and by calculation, the *position* of a face of the second individual as referred to the axes of the first.

Thus if Q', Fig. 64, be the pole of a face of the second individual and Q, C, A be poles of known faces on the first, Q and Q' being recognisable as corresponding faces on the two individuals, the angles $Q'C$ and $Q'CA$ must be determined by measurement and calculation, so that we have the values of the arcs QC and $Q'C$ and the angle QCQ', for $QCQ' = Q'CA - QCA$: from the triangle QCQ' we then obtain QQ', and therefore $\frac{1}{2}QQ'$ or QT, and also CQQ'. Hence, further, from the triangle TCQ, in which QC, QT, and CQQ' are now known, the values of TC and TCA may be found, since $TCA = ACQ + TCQ$.

From these two values the indices of T can be obtained by methods which will hereafter be given for finding the indices of a face from the requisite data under each system.

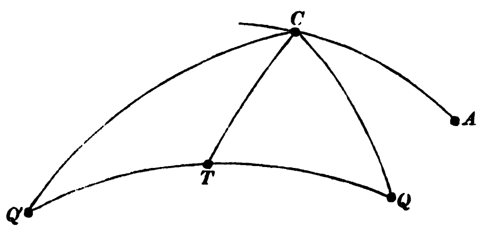


Fig. 64.

The values of $Q'C$ and $Q'CA$ can only be deduced from two measured angles, unless it should happen that a zone can be found on the first individual into which the pole Q' of the second individual falls, in which case the measurement of a single angle is sufficient; and where two zones can be found containing Q', no measured angle is required.

164. The practical method to be pursued in the investigation of a twin-crystal will thus be seen to consist, first, in determining the elements of one of the crystals, which of course involves our knowing those of both. Then, by having recourse to principles of symmetry and to the features characteristic of the faces of particular forms, we proceed to identify those faces which belong to some one or more forms on both the crystals, and to determine the symbols of the faces which lie in zones common to the two. Where a face can be recognised as belonging equally to the two individuals, or when a face on one crystal is parallel to a corresponding face upon the other, this may be the twin-face; or otherwise it is a face perpendicular to the twin-face, and therefore belongs to a zone the axis of which is the twin-axis. And often a careful inspection enables us to recognise such a plane either as the twin-plane or the plane of junction. Where these methods of attacking the problem wholly or partially fail, recourse must be had to others which may be suggested in each case by the character of the crystal itself, and which will depend on the principles laid down in the last article; but then especially must measurements be made of all zones containing faces of both crystals. A line traversing a face may be the trace of a plane of junction of two crystals, and often some physical character such as striation evinces differences on either side of the line, and perhaps also a symmetrical disposition in respect to it, thus serving as a guide to the discovery of the twin-plane. And the position and nature of re-entrant edges at one part of the structure, and of faces on another part combined in accordance with a higher symmetry than that which governs an individual crystal of the substance, may help in the solution of the problem. In the case of a transparent crystal, polarised light will often lend invaluable aid in establishing the existence of a composite structure, by shewing in a section or cleavage fragment, or even in an entire crystal, dark and light or differently tinted bands, representing regions in which the directions of the optical principal sections are not the same.

165. Many crystals belonging to the Ortho-rhombic and Mono-symmetric systems fall crystallographically into one of two groups in which the angles between the faces of the prevalent prism or of a dome have values which approximate either to a right angle, or to an angle of 60°. And such crystals, and particularly those of the latter group, often simulate, in the twin-structures they build up, the character of crystals endowed with much higher symmetry than their own.

Thus copper-glance, aragonite, witherite, and other ortho-symmetrical carbonates isomorphous with the latter minerals, occur as groups of crystals which are constructed by a repeated twinning on the prism plane: and they present a symmetry closely resembling that of a hexagonal crystal. Fig. 65 represents a triplet (or triple twin) of chrysoberyl (alexandrite), the character and relative orientation of the individual crystals that compose it being shewn in the three smaller crystals grouped around it. The lines t_1 t_2 are the twin axes, T_1 T_2 the traces of the twin-planes on which the

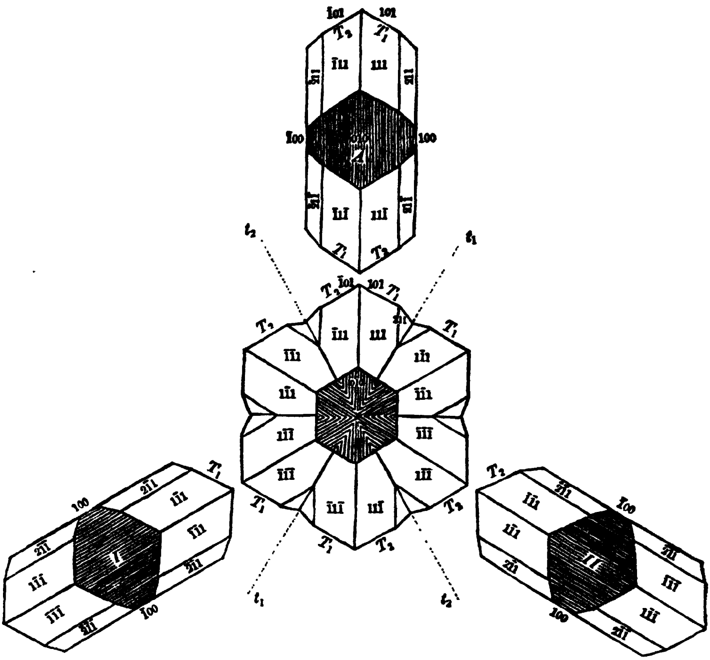


Fig. 65.

crystal *A* is twinned so as to assume positions parallel to those exhibited by the figures I and II respectively.

Faces of the dome $\{101\}$* are taken in the fig. as the twin-planes: the faces of this dome present the normal-angles 101 on $\bar{1}01 = 60^\circ 14'$, 101 on $100 = 59^\circ 53'$; the faces 010 of the different individuals

* The above was Kokscharow's interpretation of these triplets. Though offering a more simple illustration of the simulation of hexagonal forms, it is possibly less correct than the view of Hessenberg, as lately confirmed by Cathrein, viz. that a face of the form $\{301\}$ is both twin- and junction-plane (Groth's Zeitschr. 1882, p. 257).

are perpendicular to those of {101}, and unite to form a plane common to the crystals of the group. The striations of these faces parallel to the edges [010, 100] of the three individuals intersect in a feathered pattern on this plane.

166. Leucite offers another striking illustration of the simulation by twin-crystals of a symmetry higher than that of the simple individual. Long treated as a typical crystal of the Cubic system in which it represented the form {211}, (a variety of icositetrahedron formerly termed the leucitohedron), it was removed by the researches of Prof. vom Rath to the Tetragonal system, the crystals being viewed by him as twins in which two individuals presenting combinations of an isosceles (tetragonal) octahedron {111} and a scalene dioctahedron {421} are twinned on a plane parallel to a face of the form {201}, which is also the plane of union. The incomplete and anomalous character of the cleavage, a decided action on polarised light not easily explained in a cubic crystal, and pecularities in the striations on the faces, some of which occasionally exhibit a re-entrant angle, had previously induced doubt as to leucite being correctly ascribed to the Cubic system. M. Mallard has, subsequently to vom Rath's researches, further investigated crystals of leucite and as a consequence removes the symmetry of this mineral still further from that of the Cubic system, the result of his investigation going to shew that the crystals are mono-symmetrical, but capable of being referred to an axial system not widely removed from that of a cubic crystal. And in the same memoir M. Mallard has considerably extended the number of minerals in which he recognises the simulation of high symmetry by twinned crystals belonging to systems of comparatively simple symmetry. Among them is boracite, a mineral the forms and angular measurements of which are entirely in accord with the symmetry of a crystal of the Cubic system of haplohedral type: the cube being usually the most prominent form; and of this form the quoins are truncated by octahedral faces, of which the faces *o* of the form $\sigma\{111\}$ are large and bright; while the faces ω of the form $\sigma\{\bar{1}11\}$ are either absent or small and of no lustre, and acquire an electrical potential opposite to that of the faces *o* under changes of temperature; for, with a falling temperature down to 225° C., the smooth faces are positive, and then become negative till the temperature falls to about 120° C., when they again become positive until the ordinary temperature is reached: the rough tetrahedral faces exhibiting inverse phenomena. M. Mallard, guided in the case of boracite entirely by the phenomena presented by sections of a certain tenuity when examined both in parallel and in divergent polarised light, arrived at the conclusion that, notwithstanding their cubic haplohedral symmetry and the apparent accord of their measured angles with those of a cubic crystal, they are complicated structures in which twelve orthorhombic crystals are united into a single pseudo-cubic combination; each individual being a pyramid with its apex at the centre of the crystal and its base conterminous with a face of a pseudo-rhomb-dodecahedron.

The advance of crystallography has, indeed, for some time been marked by the gradual removal to lower types of symmetry of crystals previously assigned to systems of higher symmetry. And the tendency of exact investigation seems in the direction of continuing this process of removal; as well by establishing the existence of slight variations from what had previously been deemed to be rectangularity of the axes or equality of parameters, as also by asserting a more or less elaborately composite structure in crystals in which heretofore only simple forms and an almost typical representation of high types of symmetry had been recognised.

Recent investigations by Mallard and Klein respectively have led to the discovery that the optical characters of boracite and leucite undergo a sudden change at a temperature which, in the former case, is about 265°C., and, in the latter, is probably lower than the melting-point of zinc: above these critical temperatures the optical behaviour is in each case that of a cubic crystal. In the passage of boracite through the critical temperature there is an absorption or emission of heat amounting to 4·77 thermal units. Klein considers that the peculiar optical behaviour of these minerals at ordinary temperatures is of a merely secondary character, and a consequence of the contraction of the crystals on cooling.

CHAPTER VII.

THE SYSTEMS.

Section I.—**The Cubic (or Tesseral) System.**

A.—*Holo-symmetrical Forms.*

167. The symmetrical characters of a form belonging to the Cubic system have been already considered and the systematic triangle determined (Articles **125**, **126**, and **127**).

The distribution of the poles of the general scalenohedron, the tetrakis-hexahedron, and the cube is shewn on Fig. 66. Those of the octahedron, the rhomb-dodecahedron, and of the two remaining forms, the triakis-octahedron and icosi-tetrahedron, are exhibited on Fig. 75, Article **173**. Fig. 66 μ represents a systematic triangle of the system metastrophic with that containing the pole hkl; Fig. 66 a a systematic triangle antistrophic to the former.

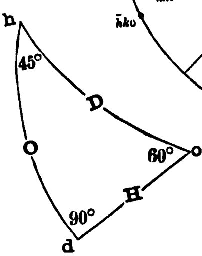
It is only requisite to state, in recapitulation, that a crystal belonging to this system, if it be holo-symmetrical, will exhibit symmetry to two groups of systematic planes, see Fig. 50, p. 145; viz. to three proto-systematic mutually perpendicular planes S, and to six deutero-systematic planes Σ; which latter planes intersect—

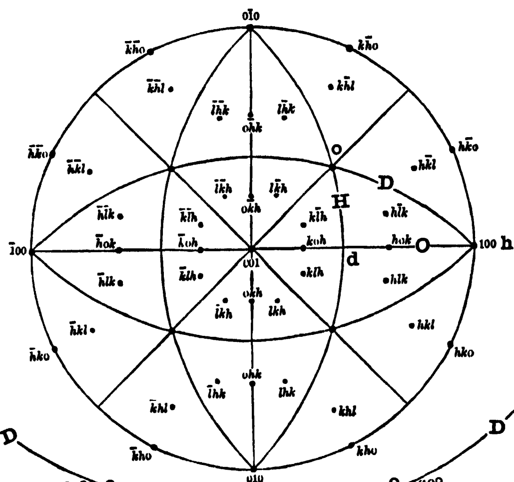
1. with each other in triads, in four trigonal axes **o**;
2. with each other in pairs, perpendicularly, and with the planes S in pairs at 45°, in the three tetragonal axes **h**;
3. and each individually, perpendicularly, with a plane S, there being one such intersection in each of the six ortho-symmetral axes **d**.

The letters **h**, **o**, **d** indicate at once the axial points for these axes and the angular points of the forty-eight systematic triangles into which the sphere of projection is divided by the planes S and Σ; and they also coincide with the poles—

h, of the three axial planes, i.e. of the proto-systematic planes S parallel to the faces of the regular hexahedron or cube {100},

o, of the parametral planes parallel to the eight faces of the regular octahedron {111}; and





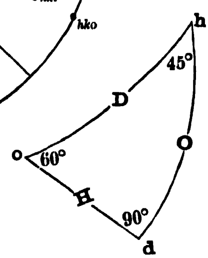


Fig. 66 *α*. Fig. 66. Fig. 66 *μ*.

d, of the planes, also parametral, parallel to the twelve faces of three similar and concurrent square prisms, which together constitute the form {110}, the rhomb-dodecahedron. These viewed as origin-planes are the deutero-systematic planes Σ.

The origin-planes indicated by the poles *o* are not symmetral planes; wherefore the symmetry on the axes **o** will only be trigonal or ditrigonal, and not hexagonal.

The symbols and mode of distribution of the forty-eight poles of the general scalenohedron $\{hkl\}$ have already been discussed. They are contained in the subjoined

TABLE A.

	i	iii	v	vii	ii	iv	vi	viii	
	hkl	$\bar{h}\bar{k}l$	$\bar{h}k\bar{l}$	$h\bar{k}\bar{l}$	$\bar{h}kl$	$h\bar{k}l$	$hk\bar{l}$	$\bar{h}\bar{k}\bar{l}$	
I μ	lhk	$\bar{l}\bar{h}k$	$\bar{l}h\bar{k}$	$l\bar{h}\bar{k}$	$\bar{l}hk$	$l\bar{h}k$	$lh\bar{k}$	$\bar{l}\bar{h}\bar{k}$	a II
	klh	$\bar{k}\bar{l}h$	$\bar{k}l\bar{h}$	$k\bar{l}\bar{h}$	$\bar{k}lh$	$k\bar{l}h$	$kl\bar{h}$	$\bar{k}\bar{l}\bar{h}$	
	hlk	$\bar{h}\bar{l}k$	$\bar{h}l\bar{k}$	$h\bar{l}\bar{k}$	$\bar{h}lk$	$h\bar{l}k$	$hl\bar{k}$	$\bar{h}\bar{l}\bar{k}$	
III a	khl	$\bar{k}\bar{h}l$	$\bar{k}h\bar{l}$	$k\bar{h}\bar{l}$	$\bar{k}hl$	$k\bar{h}l$	$kh\bar{l}$	$\bar{k}\bar{h}\bar{l}$	μ IV
	lkh	$\bar{l}\bar{k}h$	$\bar{l}k\bar{h}$	$l\bar{k}\bar{h}$	$\bar{l}kh$	$l\bar{k}h$	$lk\bar{h}$	$\bar{l}\bar{k}\bar{h}$	

where the poles in each column belong to a single octant and the blocks marked a contain the symbols of faces antistrophic to the faces of which the symbols lie in the blocks indicated by μ.

The six remaining forms are those of which the poles lie on one or other of the three sides of the systematic triangles, and those of which the poles lie at the angles of these triangles. They will be represented by symbols in which one or two of the indices are zero, or in which two of the indices are equal; and in the latter case the two equal indices may be greater or may be less than the third, that is to say, they may be represented as $\{lhh\}$ or $\{hkk\}$. In the parametral octahedron all the indices are equal.

The following are the seven holo-symmetrical forms of this system:—

1. $\{hkl\}$, the hexakis-octahedron or forty-eight scaleno-hedron.
2. $\{lhh\}$, the triakis-octahedron or the octahedrid pyramidion.
3. $\{hkk\}$, the icositetra-hedron or twenty-four deltohedron.
4. $\{hk0\}$, the tetrakis-hexahedron or the cube-pyramidion.
5. $\{110\}$, the rhomb-dodecahedron or twelve rhombohedron.
6. $\{111\}$, the regular or equilateral octahedron.
7. $\{100\}$, the cube or square hexahedron.

The names of these figures are more concisely expressed by the terms derived from the Greek numerals: the alternative English names here proposed for them have the single advantage of representing the contour of the figures by recalling certain characteristic features: the term pyramidion being employed in the case of forms in which a pyramidion or small pyramid composed of similar isosceles triangles surmounts every face of a simpler figure, the faces namely of a cube, octahedron, or dodecahedron: such a figure is then an isoscelohedron. An isoscelohedron, scalenohedron, deltohedron, rhombohedron, &c. are figures formed severally of similar and congruent (but not necessarily metastrophic) isosceles or scalene triangles, deltoids, rhombs, &c.: an equilateral octahedron is the regular octahedron bounded by equilateral triangles. For brevity, the term 'faced' in e.g. the forty-eight-faced scalenohedron, twenty-four-faced deltohedron, &c., usually employed in the nomenclature of these forms, is omitted since no confusion can arise from this adjectival use of the numbers.

It is obvious that names thus employed to represent some of the characteristics of a crystallographic figure can really only strictly describe it as an ideal figure constructed in equipoise; a figure that can have only an exceptional occurrence in nature.

If we bear in mind however the special characteristics of crystallographic symmetry, it will be seen that this is the simplest mode of considering a crystal-form; and consequently it will be employed in the following descriptions, in which the forms will be discussed in the order of the simplicity of their indices.

168. *The square hexahedron* or *crystallographic cube* $\{100\}$ consists of the six faces parallel to the three axial or proto-systematic planes S, the normals to which are the crystallographic axes and the tetragonal axes of symmetry **h**. The three planes S being similar, the pairs of faces parallel to them will be simultaneous in their occurrence and similar in their geometrical and physical characters.

The resulting form, see Fig. 69, when in equipoise, is the cube of geometry, a 'regular solid,' with equal squares for its faces; though as a crystal-form the faces are frequently rectangles. It has twelve crystallographically similar edges, the angle of each being 90°. And it has eight similar three-faced quoins, the edges of which meet at right angles, and which are symmetrical to the trigonal axes of symmetry **o**.

The symbols of the six faces of the form are

$$100 \qquad 010 \qquad 001$$
$$\bar{1}00 \qquad 0\bar{1}0 \qquad 00\bar{1}.$$

Among minerals presenting the cube form, galena (lead sulphide) is remarkable for the facility of its cleavage parallel to the faces of this form. The metals gold, silver and copper, common salt (Na Cl) and sylvine (K Cl), fluor spar, cuprite, occasionally diamond, are among the minerals which present themselves as simple cubes. Pyrites also occurs in cubes, but

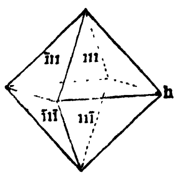


Fig. 67.

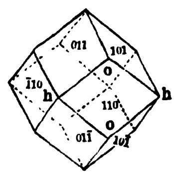


Fig. 68.

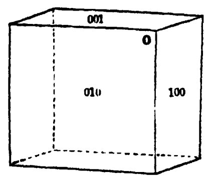


Fig. 69.

the striations often carried by the crystals prove them to be mero-symmetrical. See Fig. 55 (2), p. 165.

169. *The equilateral or regular octahedron* $\{111\}$, Fig. 67. The equilateral octahedron is the parametral form of the system. The intercepts of its faces are equal on all the axes; and the eight faces are equilateral triangles when the form is in equipoise. It then represents the regular solid from which it is named.

The twelve edges of the form are all similar and have the dihedral normal-angle of the regular octahedron, viz.

$$70^\circ\, 31{\cdot}7' = 2 \text{ arc } \mathbf{H}.$$

The distance of two alternate poles of the form, i. e. of poles lying in attingent octants or measured over a quoin, is the arc

$$2\,\mathbf{D} = 109^\circ\, 28{\cdot}3'.$$

The poles of the faces are the axial points **o** of the tri-symmetral axes, and its six quoins are symmetrical to the tetragonal axes **h** (in which lie the poles of the cube). The quoins are consequently formed by the meeting of four faces with plane angles of 60°, or of four edges of which the normal-angles are each 70° 31·7′.

The symbols of the eight faces of the octahedron are

$$111 \quad \bar{1}11 \quad 1\bar{1}1 \quad 11\bar{1} \quad 1\bar{1}\bar{1} \quad \bar{1}1\bar{1} \quad \bar{1}\bar{1}1 \quad \bar{1}\bar{1}\bar{1}.$$

Each edge of the cube subtends an arc formed of two consecutive arcs *H* of a zone-circle Σ, and each joins two poles of the octahedron: and, similarly, the edges of an inscribed octahedron join the poles of the cube; the quoins of the latter figure may thus be truncated by the faces of the octahedron, as may the quoins of the octahedron by the faces of the cube. The octahedron is represented as an isolated form by copper, silver, and by argentite, cuprite, senarmontite, fluor, spinel, chromite, periclase, magnetite, pyrochlore, franklinite, and, very rarely, the diamond.

170. *The rhomb-dodecahedron* {110}, Fig. 68.

A face with the symbol 110 will evidently be a parametral face parallel to the axis *Z*, and since the axis *Z* coincides with one of the tetra-symmetral axes **h**, the face will be repeated in four faces, $110 \ \bar{1}10 \ 1\bar{1}0 \ \bar{1}\bar{1}0$, forming a square prism.

Similarly, the faces $101 \ \bar{1}01 \ 10\bar{1} \ \bar{1}0\bar{1}$ will constitute a dome form or horizontal prism parallel to the axis *Y*, and the faces $011 \ 0\bar{1}1 \ 01\bar{1} \ 0\bar{1}\bar{1}$ a dome parallel to the axis *X*. And since the axes *X*, *Y*, and *Z* are identical with the tetragonal axes **h** and are similar to each other, the three square prisms concur in building a form with the twelve congruent rhombic faces

$$\begin{array}{cccc} 110 & \bar{1}10 & 1\bar{1}0 & \bar{1}\bar{1}0 \\ 101 & \bar{1}01 & 10\bar{1} & \bar{1}0\bar{1} \\ 011 & 0\bar{1}1 & 01\bar{1} & 0\bar{1}\bar{1}. \end{array}$$

And the axial points **d** of the axes of ortho-symmetry are at once the poles of the deutero-systematic planes Σ and of the faces of this form {110}.

Two contiguous poles *d* not lying on the same great circle *S* will be symmetrically situate with regard to an arc **D** of one of the great circles Σ. See projection, Fig. 50, Article **125**, p. 145.

The edge *D* of the corresponding faces will therefore lie in the plane of this great circle Σ, and will subtend an arc of it which is the side **D** of a systematic triangle.

These edges *D* will be twenty-four in number, and identical in feature. Contiguous faces with poles on the same great circle *S* will meet with two other contiguous faces with poles on a great circle *S* perpendicular to the former, in a quoin symmetrical to a tetragonal axis **h**. And the three contiguous faces symmetrical to an axis **o** will meet on that axis in a trigonal quoin.

The twenty-four edges of the dodecahedron will therefore meet in six four-faced quoins **h**, formed by identical acute plane angles, and in eight three-faced quoins **o**, of which the plane angles are also identical but obtuse.

The normal-distance of a pole *d* from a pole *h* being 45°, that between two adjacent poles *d* on different great circles *S*, for instance between $01\bar{1}$ and $\bar{1}10$ (see projection in Fig. 75), is obtained by the equation

$$\cos(01\bar{1}, \bar{1}10) = \cos^2 45^\circ = \cos 60^\circ :$$

the edge *D* has therefore a normal-angle of 60°.

And as in the zone-circle $[01\bar{1}, \bar{1}10]$ there lie the four other *poles* $\bar{1}01$ $0\bar{1}1$ $1\bar{1}0$ *and* $10\bar{1}$, *distant* 60° from each other, and as the edges of the faces corresponding to these six poles will be parallel to each other and to the zone-axis [111], it follows that if either of the trigonal axes **o** of the system, e.g. the axis {111}, be placed vertically, the rhomb-dodecahedron presents the character of a hexagonal prism *parallel to that axis, and terminated by the faces of a rhombohedron which in the case assumed has the faces*

at the upper end	110	101	011
at the lower end	$\bar{1}\bar{1}0$	$\bar{1}0\bar{1}$	$0\bar{1}\bar{1}$.

Lines joining contiguous trigonal quoins *o* will give the edges of a cube. Hence of such an inscribed cube the edges are the short diagonals of the rhomb-dodecahedron in the figure. And the diagonals of the faces of this cube are evidently each equal and parallel to a pair of the longer diagonals in the same rhomb-dodecahedron. So that the ratio of a shorter to a longer diagonal of a rhomb-dodecahedron = the ratio of the side to the diagonal of a square $= \sin 45^\circ = \frac{1}{\sqrt{2}}$.

Whence for the plane angles of the rhomb,

since $\qquad \cot\frac{o}{2} = \tan\frac{h}{2} = \frac{1}{\sqrt{2}},$

the angle $\qquad h = 70^\circ 31\cdot 7'$, and

the angle $\qquad o = 109^\circ 28\cdot 3'$,

which are also the normal-distances of two faces of the octahedron in adjacent and attingent octants respectively.

It will be seen from the position of the edges of the dodecahedron that its quoins are truncated by the faces of the cube and octahedron, and that its faces in turn truncate alike the edges of the octahedron and those of the cube.

The rhomb-dodecahedron is of frequent occurrence in combination with other forms, but is more rare in an isolated condition: it is however the habitual form of some varieties of garnet, sodalite, as well as of (haüyne or) lapis lazuli; while the minerals blende (ZnS) and lapis lazuli (or haüyne) present cleavages in six directions parallel to the faces of this form. Diamond, gold, and electrum are among the substances that occur in rhomb-dodecahedra.

171. Of the three simple forms hitherto considered, the faces are severally perpendicular to the three sets of axes of symmetry of the system, two or more zone-circles of symmetry being tautohedral in their poles.

The forms next to be considered are those of which the poles lie upon the several sides of the systematic triangles; the form $\{hk0\}$ having its poles on *the sides* **O** *of the systematic* triangles, while those of the form $\{hhl\}$ lie on *the sides* **H** and those of $\{hkk\}$ on the sides **D** of those spherical triangles: the edges of the faces of the several forms will consequently lie in the sectors of each zone-plane corresponding to the arcs **O**, **H**, or **D** on which their poles do not lie.

172. *The tetrakis-hexahedron,* or *the cube-pyramidion* $\{hk0\}$, Figs. 70–2. If the poles of the form lie on the arcs **O**, its faces will form zones of which the tetragonal axes **h** are the zone-lines, and they will have a zero for the first, the second, or the third index of their symbol, according as the zone-plane in which the pole lies is the plane S_3, or S_2, or S_1, Fig. 50, p. 145.

In any zone-circle *S*, a pole of the form $\{hk0\}$ lies on the

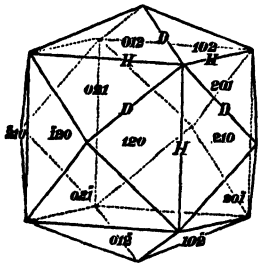


Fig. 70.

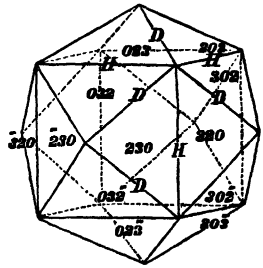


Fig. 71.

arc separating every two adjacent poles of the forms $\{100\}$ and $\{110\}$; eight such poles lying in each zone-circle. The form will therefore have twenty-four faces symmetrical in pairs to the poles of the forms $\{100\}$ and $\{110\}$; the edges of either of these latter forms would thus be bevilled by faces of the form $\{hk0\}$.

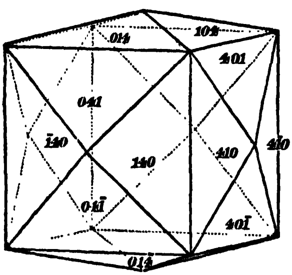


Fig. 72.

The figure presents the aspect of a cube each face of which is surmounted by an obtuse pyramid, and it may, on this account, be termed the cube-pyramidion. Since the faces are euthy-symmetrical to a plane *S*, the triangles that form them are crystallographically isosceles, and the figure is a twenty-four-faced isoscelohedron. The thirty-six edges of the form are of two kinds, viz. twelve similar edges *H* formed by faces which are symmetrical to and meet in the **H** sectors of the planes Σ, and which are therefore coincident in direction with the edges of the cube, and twenty-four edges *D* that are similar and are formed by faces symmetrical to the **D** sectors of the zone-planes Σ. These *D* edges meeting in fours form the six terminal quoins of the pyramidions of the figure symmetrical to the tetragonal axes **h**; while, of the H edges parallel to the edges of the cube, three, alternating with three of the D edges, meet with them to form a six-faced trigonal quoin o; and of these quoins there are eight.

The symbols of the faces of a form $\{hk0\}$ are arranged in

TABLE B.

$hk0$	$\bar{h}k0$	$\bar{h}\bar{k}0$	$h\bar{k}0$
$kh0$	$\bar{k}h0$	$\bar{k}\bar{h}0$	$k\bar{h}0$
$0hk$	$0\bar{h}k$	$0\bar{h}\bar{k}$	$0h\bar{k}$
$0kh$	$0\bar{k}h$	$0\bar{k}\bar{h}$	$0k\bar{h}$
$k0h$	$k0\bar{h}$	$\bar{k}0\bar{h}$	$\bar{k}0h$
$h0k$	$h0\bar{k}$	$\bar{h}0\bar{k}$	$\bar{h}0k$

where h and k may be any integer numbers. The distribution of their poles on the sphere is seen in the projection, Fig. 66, Article **167**.

In the cases of known crystals, the varieties of the cube-pyramidion include the following forms,

510, 410, 310, 520, 210, 320, 430, 540.

These are more or less frequent on several crystallised bodies; 510, for instance, on cuprite, 410 on silver and on gold, 520 on copper and on fluor, 320 on blende, 430 on perowskite, while 310 is a common variety of pyramidion. But the form {210} is by far the most frequent in occurrence, and presents the remarkable property that its H edges have the same dihedral angle as its D edges; see Fig. 70. In this case it will be proved in Chapter VIII that

$$\cos H = \cos D = \frac{4}{5},$$

$$\text{and} \qquad H = \quad D = 36^\circ\, 52'\, 11{\cdot}7''.$$

173. *The icositetrahedron, or twenty-four deltohedron* $\{hkk\}$, Figs. 73–4. This form has its poles situate on the arcs **D** of the systematic triangle. The edges consequently lie in the **H** and **O** sectors of the Σ and S planes, and they are of two kinds, viz. twenty-four edges H and twenty-four edges O. Of the twenty-six quoins, six are four-faced quoins h symmetrical to the tetragonal axes **h**, eight three-faced quoins *o* symmetrical to the trigonal axes **o**, twelve four-faced quoins *d* ortho-symmetrical to the axes **d**.

The twenty-four faces are euthy-symmetrical to the traces of the sectors **D**, and have thus the form of symmetrical trapezia or deltoids.

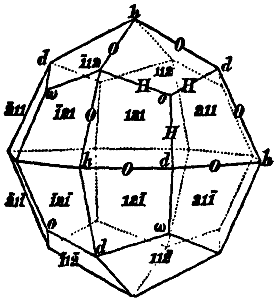


Fig. 73.

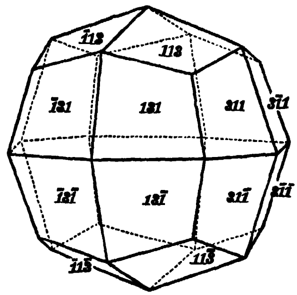


Fig. 74.

The icositetrahedron occurs rarely as an isolated form. Analcime, indeed, presents such crystals with the symbol {211}, and the attribution of analcime to the Cubic system, though it has been disputed, is now recognised*. The form {211}, Fig. 73, which is the commonest variety, is met with in a large number of crystals, replacing by its trigonal quoins those of the cube, and occasionally replacing the quoins of the octahedron by its tetragonal quoins *h*. Fluor, garnet, cuprite, and galena are among the minerals that exhibit the form {211}.

This particular form {211} was termed by Haidinger the leucitohedron, and the general form {*hkk*} the leucitoid, from the crystals of leucite having been, though erroneously, attributed to this form. The character of these crystals has already been discussed in Article **166**.

This variety of the icositetrahedron is however an important one, not only from its frequent occurrence in combination with other forms, but also from the circumstance that its poles lie in the edge zones, i. e. in the zone-circles perpendicular to the edges, of the rhomb-dodecahedron, so that its faces truncate the edges of that

* See Arzruni and Koch (Zeitsch. f. Kryst. 1881, p. 483), and Ben Saude (Jahrb. f. Min. 1882, p. 41).

figure. Thus the two adjacent poles of the dodecahedron (110) and (101) lie in the zone [$\bar{1}11$]; whence the poles lying on this zone-circle fulfil the condition $-h+k+l=0$, or $h=k+l$, which in the case of the icositetrahedron becomes $h=2k$ and the symbol is therefore {211}.

Another important icositetrahedron {311}, Fig. 74, occurs on the cubic native metals, on galena, pyrites, fluor, magnetite, and spinel, generally replacing the quoins of the cube by its trigonal quoins *o*.

The symbols of the faces of a form {*hkk*} are arranged in the following

TABLE C.

hkk	$\bar{h}\bar{k}k$	$\bar{h}k\bar{k}$	$h\bar{k}\bar{k}$	$\bar{h}kk$	$h\bar{k}k$	$hk\bar{k}$	$\bar{h}\bar{k}\bar{k}$
khh	$\bar{k}\bar{h}h$	$\bar{k}h\bar{h}$	$k\bar{h}\bar{h}$	$\bar{k}hh$	$k\bar{h}h$	$kh\bar{h}$	$\bar{k}\bar{h}\bar{h}$
kkh	$\bar{k}\bar{k}h$	$\bar{k}k\bar{h}$	$k\bar{k}\bar{h}$	$\bar{k}kh$	$k\bar{k}h$	$kk\bar{h}$	$\bar{k}\bar{k}\bar{h}$.

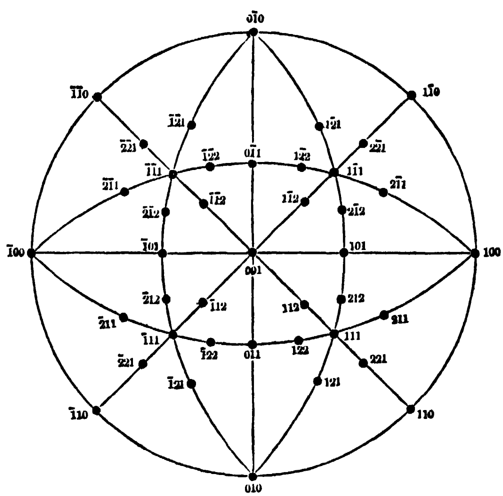


Fig. 75.

The distribution of the poles of the form {211} on the sphere is shown in the projection, Fig. 75.

174. *The triakisoctahedron,* or *octahedrid pyramidion* {*hhl*}, Figs.

76–7. The poles of this form lie on the arcs **H**, and its edges consequently in the sectors **D** and **O**: twenty-four in the former, and twelve longer than these in the latter. Hence the twenty-four faces are isosceles triangles, euthy-symmetrical to the traces on them of the sectors **H**.

The figure will have eight trigonal quoins *o*.

" " " six ditetragonal quoins *h*.

Its aspect is that of an octahedron, each of the faces of which forms the base of an obtuse pyramid, and the more acute the pyramidion the more nearly it approximates in aspect to a rhomb-dodecahedron.

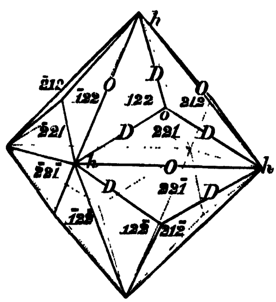


Fig. 76.

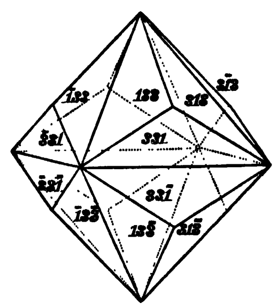


Fig. 77.

Hence it is a twenty-four-faced isoscelohedron, and may be termed the octahedrid pyramidion. The form {221} exists on the diamond, and this form, as also the form {331}, occurs associated with the cube, or bevilling the edges of the octahedron, on galena, and occurs also on argentite, spinel, fluor, magnetite, franklinite, and pharmacosiderite.

It is easily seen that the deltohedron which would truncate the edges of the form {211} must be that with the symbol {332}. In fact the addition of the symbols (211) and (121), which are those of adjacent faces symmetrical to the Σ-plane [001, 110], gives the symbol (332) of the face truncating the edge of the former faces. The form occurs on garnet.

The symbols of the faces of the form {*hhl*} are given in

TABLE D.

$$hhl\ \bar{h}\bar{h}l\ \bar{h}h\bar{l}\ h\bar{h}\bar{l}\ \bar{h}hl\ h\bar{h}l\ hh\bar{l}\ \bar{h}\bar{h}\bar{l}$$

$$lhh\ \bar{l}\bar{h}h\ \bar{l}h\bar{h}\ l\bar{h}\bar{h}\ \bar{l}hh\ l\bar{h}h\ lh\bar{h}\ \bar{l}\bar{h}\bar{h}$$

$$hlh\ \bar{h}\bar{l}h\ \bar{h}l\bar{h}\ h\bar{l}\bar{h}\ \bar{h}lh\ h\bar{l}h\ hl\bar{h}\ \bar{h}\bar{l}\bar{h}.$$

The distribution on the sphere of the poles of the triakis-octahedron $\{221\}$ is shown in Fig. 75.

175. There only remains to be considered the *hexakis-octahedron*, or *forty-eight scalenohedron* $\{hkl\}$, Figs. 78–80.

This form has received numerous designations, for the most part recalling the characters of the different figures which result from the indices receiving more or less widely differing relative values. Such are the terms *octakishexahedron*, *hexakisoctahedron*, *tetrakisdodecahedron*, which may be represented in an English form as the eight-on-six, six-on-eight, and four-on-twelve scalenohedron; and they in fact indicate the different aspects the form assumes, as it approximates to a cube with an eight-faced, an octahedron with a six-faced, or a dodecahedron with a four-faced pyramid on every face. In general these figures cannot be legitimately designated as true pyramidions, since the bases of the pyramids do not actually coincide with the faces of the cube, octahedron, or dodecahedron, respectively: in cases, however, where the indices present the ratio $h = k + l$ the basal edges coincide in position with those of the dodecahedron, and the figure, then, is a true *dodecahedrid pyramidion*.

The most general designation is the *tetrakontaoctahedron* (or, forty-eight-faced form); the most usual term is the *hexakis-octahedron;* the simplest designation for this, the most complex of crystallographic forms, is the forty-eight scalenohedron. Each of the forty-eight systematic triangles contains one pole of the form; in the case of some of the more frequent varieties of the form the poles lie either on great circles passing through a pole of the octahedron and bisecting the angles of inclination of two adjacent great circles Σ, or on great circles passing through two adjacent poles of the rhomb-dodecahedron, situate symmetrically in regard to a plane Σ.

The edges in which a face of the form meets the three faces adjacent to it will lie in the three systematic planes containing the sides of the systematic triangle; and the faces themselves will be crystallographically scalene triangles. Hence there will be three sorts of edges, namely, twenty-four edges *H* and twenty-four edges *D*, which alternating in triads with each other form eight ditrigonal quoins *o*, and twenty-four edges *O*, which alternating in fours with four of the *D* edges form six ditetragonal quoins *h*, and again alternating in pairs with pairs of the *H* edges form twelve ortho-symmetrical quoins *d*.

The symbols of the faces of a form $\{hkl\}$ have been set out in Table A, and the distribution of its poles on the sphere is seen in the projection, Fig. 66, Article **167**.

In considering the particular variety of this form, of which the poles lie on great circles bisecting the angles formed by each pair of great circles Σ, it will be seen that poles lying on such intermediate great circles correspond to those which in a Rhombohedral crystal may occur on the great circles of the deutero-systematic planes of the Hexagonal system. In the Rhombohedral type of that system the symmetral character of these deutero-systematic planes is in abeyance, and the poles lying on their great circles belong to forms of trigonal type as they are constrained to do in the Cubic system by the conditions of its symmetry.

The symbol of a form of this kind in the Rhombohedral system would be $\pi\{hil\}$, where $i = \frac{h+l}{2}$; and that this relation between the indices must hold good in the Cubic system also may be shewn by taking the value for the cosine of the arc between two poles in this system, as will be deduced in Chapter VIII, Section 2. Thus, for the adjacent poles *hkl* and *hlk* meeting in an edge *D*, it will hereafter be shewn that we have

and for the poles *hkl* and *khl* meeting in an edge *H*, we have

$2hk+l^2$

Whence the condition for the edges *D* and *H* to have the same dihedral angle is $h+l=2k$. Forms fulfilling this condition would be {321}, {531}, {543}, &c.

The other noticeable variety of the forty-eight scalenohedron is that the poles of which lie in the edge-zones of the rhomb-dodecahedron, the faces of which consequently bevil the edges of that form.

We have already seen, Article 173, that the poles of the form {211} lie in such an edge-zone-circle, its faces truncating the edges

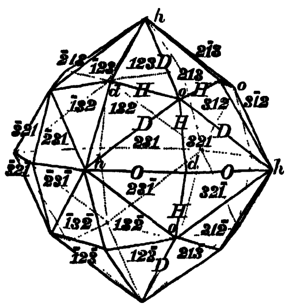


Fig. 78.

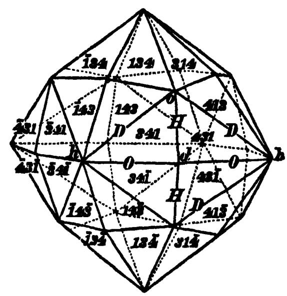


Fig. 79.

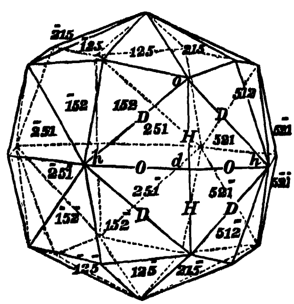


Fig. 80.

of the rhomb-dodecahedron: and it was also shewn in Article **173** that the indices of a plane lying in one of these zones must fulfil the condition that one of them equals the sum of the other two. The forms {321}, {532}, {431} are among those which belong to this variety of scalenohedron; the form {321}, Fig. 78, is thus remarkable as being common to both the mentioned varieties.

Since the faces of any form of the latter variety, e. g. those of the form {431} in Fig. 79, bevil the edges of the rhomb-dodecahedron, the complete form has the character of a pyramidion developement of the rhomb-dodecahedron, each face of the latter figure being surmounted by a rhomb-based pyramid, to which it forms a conterminous base. These therefore are the forms that may be correctly designated as *tetrakisdodecahedra* or *dodecahedrid pyramidions.*

The manner in which the quoins and edges of the cube, octahedron, and dodecahedron respectively are modified by association with the forms hitherto considered is exemplified in the following figures.

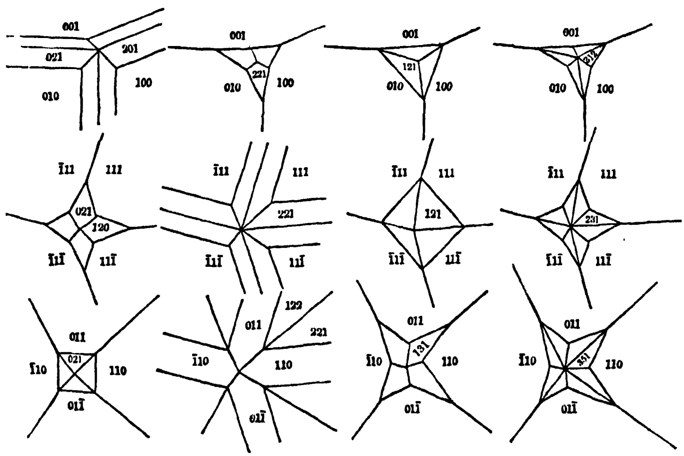


Fig. 81.

Cubic System. B.—*Mero-symmetrical Forms.*

176. In order that mero-symmetrical forms of the Cubic system may satisfy the law of mero-symmetry in respect to the two groups of systematic planes *S* and Σ, it is not necessary that either group retain its symmetral character, provided that the character of the symmetry peculiar to the system be preserved in the resulting forms. Where both groups thus fail in directly controlling the symmetry of the forms under consideration they in effect indirectly control it, inasmuch as the axes of symmetry in such a case must continue to regulate the symmetry of these forms. They can however do this, as far as the geometrical symmetry is concerned, only in the case of the general form of the system, the forty-eight scalenohedron $\{hkl\}$: for there must be six faces arranged round a trigonal axis and eight round a tetragonal axis in order that, if half the faces are suppressed, the remaining half may be grouped trigonally in the one case and tetragonally in the other.

Where only one group of systematic planes is operative as a group of actual planes of symmetry, the character of the resulting forms will be entirely different according as the group is that of the proto-systematic planes, the zone-axes of which are tetragonal in their symmetry, or is the deutero-systematic group, the planes of which intersect in threes, in zone-axes that are axes of trigonal symmetry.

If a group of six faces of the scalenohedron lying in one octant be considered, it will be evident either that all the six faces must be suppressed simultaneously, or that only those three can be so which, lying in alternate systematic triangles, are metastrophic to each other. In the former case the Σ or deutero-systematic planes may be actual planes of symmetry, in the latter case they can only be potentially such. And if an alternate suppression take effect in one octant, a similar suppression must take effect in every octant in the case of a semiform.

In the case however where all the six faces in one octant are suppressed, the only way in which the forty-eight scalenohedron can be divided into two correlative semiforms is by all the faces in any one octant, and in the three octants attingent to it, being simultaneously present or absent. A form of this kind is evidently haplohedral and holo-systematic, and its faces are symmetrical to the planes of the Σ group only; its symbol will be $\sigma\{hkl\}$ or $\sigma\{\bar{h}\bar{k}\bar{l}\}$.

Where, on the other hand, the alternate faces in a given octant undergo suppression, the faces belonging to the systematic triangles μ in, for example, the first octant, Table A, Article **167**, and

Fig. 82, might be associated (as extant or as absent faces) either with those belonging to the triangles μ or else with those belonging to the triangles a in the adjacent octants (ii, iv, and vi) and in the opposite octant viii; and in either case with the faces belonging to the triangles μ in the attingent octants (iii, v, and vii). The former case then, where all the absent or extant faces are those belonging to the triangles μ, yields a haplohedral form; the latter gives a diplohedral form. The former will be therefore holo-systematic, the latter hemi-systematic in character. The latter kind of partition divides the forty-eight-faced form into two correlative semiforms, the faces of each being symmetrical to the three proto-systematic planes, and not symmetrical to the six deutero-systematic planes: their symbols are $\pi\{hkl\}$ and $\pi\{khl\}$.

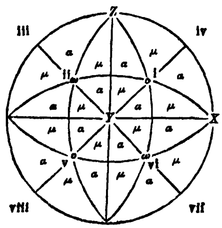


Fig. 82.

The former kind of partition is that first considered, in which the zone-axes **h**, **o**, and **d** operate as tetragonal, trigonal, and diagonal axes of symmetry, while neither the S-planes nor the Σ-planes are effective as planes of symmetry: the symbols of the two semiforms being $a\{hkl\}$ and $a\{lkh\}$.

Hence the cases of mero-symmetry of the forty-eight scalenohedron, of the three general twenty-four-faced forms of the system, and of the octahedron, are the following:—

I. Holo-systematic haplohedral forms; or holo-tesseral hemihedra.

1. *Asymmetric* forms. The pentagonal icositetrahedron or twenty-four-pentagonohedron,

$$a\{hkl\} \text{ and } a\{lkh\}.$$

2. The *tetrahedrid* forms:

(*a*) $\sigma\{hkl\}$ and $\sigma\{\bar{h}\bar{k}\bar{l}\}$: hexakis-tetrahedron.

(*b*) $\sigma\{hhl\}$ and $\sigma\{\bar{h}\bar{h}\bar{l}\}$: hemi-triakisoctahedron; or deltoid-dodecahedron, or twelve-deltohedron.

(*c*) $\sigma\{h\,k\,k\}$ and $\sigma\{\bar{h}\bar{k}\bar{k}\}$: hemi-icositetrahedron; twelve-isoscelohedron, or tetrahedrid pyramidion.

(*d*) $\sigma\{1\,1\,1\}$ and $\sigma\{\bar{1}\bar{1}\bar{1}\}$: the hemi-octahedron or tetrahedron.

II. Hemi-systematic diplohedral forms: hemi-tesseral diplohedra.

$\pi\{h\,k\,l\}$ and $\pi\{l\,k\,h\}$: the dyakis-dodecahedron, or twenty-four-trapezohedron: the diplohedron.

$\pi\{h\,k\,0\}$ and $\pi\{0\,k\,h\}$: the pentagonal dodecahedron, or twelve-pentagonohedron.

III. Hemi-systematic haplohedral forms: hemi-tesseral hemihedra; or, tesseral tetartohedra.

$\sigma\pi\{h\,k\,l\}$: the tetrahedrid twelve-pentagonohedron, or the tesseral tetartohedron.

The relations of these forms to one another and to the holosymmetrical forms from which they are derived are illustrated in the stereographic projections of their poles on Plate I.

177. B. I. Holo-systematic haplohedral forms. 1. *The asymmetric semiform.* Of the hemi-symmetrical forms in which the twenty-four normals of the general form $\{h\,k\,l\}$ are represented, each by a single face, the first to be considered is—

The *pentagonal icositetrahedron* (or *twenty-four-pentagonohedron*), $a\{h\,k\,l\}$ or $a\{l\,k\,h\}$, Figs. 83–4.

Here none of the systematic planes are planes of symmetry, and the faces of which the poles lie in alternate systematic triangles are alone extant.

The axes of symmetry however preserve the characteristics of the system, but in the absence of planes of symmetry they do so only by a gyroidal (or alternate) distribution of the poles.

The edges consequently are gyroidally grouped in triads G round the trigonal axes **o**; and in tetrads V round the axes **h**, while the axes **d** would intersect symmetrically a third sort of edge W, which again does not lie in any systematic plane.

It will be seen that the faces thus bounded are irregular pentagons, with no edge lying in a systematic plane. Their sides are formed of two contiguous edges *G*, two other contiguous edges *V*, and a fifth edge *W*, altogether sixty in number; namely, twenty-four edges *G*, twenty-four edges *V*, and twelve edges *W*.

The quoins are thirty-eight in number; viz. the six tetragonal quoins *h* in which four edges *V* meet, the eight trigonal quoins *o* formed by the meeting of three edges *G*, and twenty-four asymmetric quoins *k* in each of which an edge *W* meets an edge *V* and an edge *G*.

The symbols of the faces of a form $a\{h\,k\,l\}$ are contained in the first and fourth blocks of Table A, Article **167**, those of the

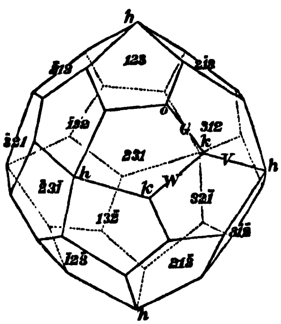


Fig. 83.

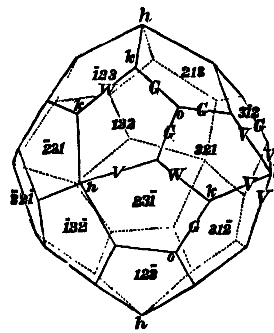


Fig. 84.

correlative form in the second and third blocks. Very few crystals have been observed to present this type of mero-symmetry (§ **188**).

178. I. 2. *Tetrahedrid mero-symmetry.* The second case of holosystematic hemi-symmetry, in which every normal is represented by a single face, is that in which the deutero-systematic planes Σ alone have a symmetral character. There being no centre of symmetry to the system, and the proto-systematic planes not being symmetral, the faces of the form are confined to four alternate (or attingent) octants.

Since the number of normals belonging to the semiform is the same as that belonging to the complete form with corresponding symbol, it will be twenty-four where the indices *h*, *k*, and *l* are all different, and twelve where two of them are equal, i.e. where the poles lie on the arcs **H** or the arcs **D** of the systematic triangle.

The number of normals will be four where all these indices are equal, and the resulting form will be constituted by four alternate faces of the octahedron {111}.

In the cases in which the poles of the form lie either on a zone-circle *S*, i.e. on an arc **O**, or at the angular points **h** or **d** of the systematic triangles, none of the faces of the holohedral form will be suppressed: they will however lose their crystallographic symmetry in respect to the planes *S*. Each face of the cube will nevertheless continue to be symmetrical to its two diagonals, as being the traces on it of two planes Σ; and therefore also, as in Fig. 55 (1), Article **147**, to its normal **h** as an axis of diagonal symmetry: and similarly, the faces of the dodecahedron will retain their physical symmetry in respect to their shorter but not to their longer diagonals. It is only in this sense, that is to

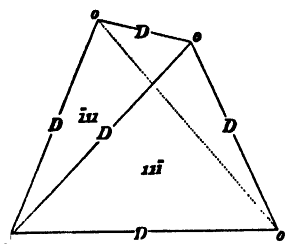


Fig. 85.

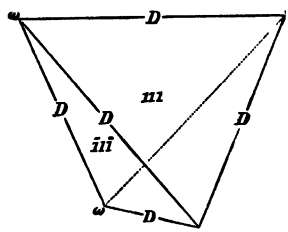


Fig. 86.

say in the sense of a physical hemi-symmetry, that we can speak of a cube, a rhomb-dodecahedron, or a cube-pyramidion presenting tetrahedrid hemi-symmetry.

The *tetrahedron* or *hemi-octahedron*, $\sigma\{111\}$ or $\sigma\{\bar{1}\bar{1}\bar{1}\}$, Figs. 85–6, is the simplest of the hemi-symmetrical forms belonging to the section under consideration, and as an isolated figure can only exist with equal faces.

In the tetrahedron the four alternate faces of the octahedron, those namely in alternate octants, are suppressed. The resulting semiform corresponds in all respects with the regular tetrahedron of geometry. Bounded by four equilateral triangles, as an isolated form it can only exist in equipoise. The symbols of its faces are—

for the form $\sigma\{111\}$, 111, $\bar{1}\bar{1}1$, $\bar{1}1\bar{1}$, $1\bar{1}\bar{1}$,

for the form $\sigma\{\bar{1}\bar{1}\bar{1}\}$, $\bar{1}\bar{1}\bar{1}$, $11\bar{1}$, $1\bar{1}1$, $\bar{1}11$.

The six edges of the figure will lie in the deutero-systematic planes **Σ**, each edge being the trace on the faces forming it of two consecutive **D** sectors of a **Σ**-plane.

The four quoins are trigonal; each axis o meeting a quoin *o* or *ω* on one side, and the pole of a face *ω* or *o* on the opposite side of the origin. The normal-distance between two faces is measured by two consecutive arcs **D**, and is therefore

$$2\,\mathbf{D} = 109^\circ\, 28{\cdot}3'.$$

Fahlore (or tetrahedrite), blende, eulytine, lead nitrate, and a variety of garnet from Elba are among the substances that present this form. Sometimes, as in helvine and in eulytine, the two correlative tetrahedra concur, but the faces of the one form are distinguished by being more largely developed than those of the other.

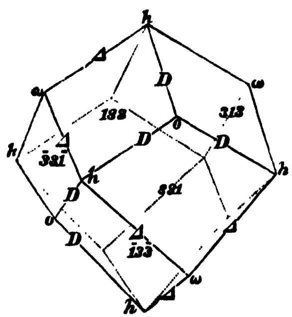


Fig. 87.

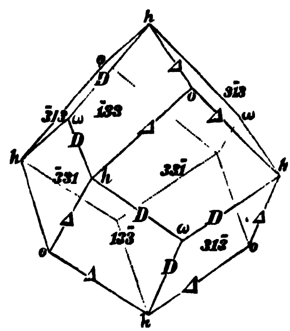


Fig. 88.

179. *The hemi-triakisoctahedron*, or *twelve-deltohedron*, $\sigma\{hhl\}$ or $\sigma\{\bar{h}\bar{h}\bar{l}\}$, called also the *deltoid-dodecahedron*, Figs. 87–8.

The tetrahedrid semiform of the triakisoctahedron $\{hhl\}$ will have three faces in each alternate octant corresponding in position and direction but not in form with the three faces with the same symbols belonging to the complete holohedral figure. Its poles therefore lie on the sides **H** of the systematic triangles.

Each face will form two edges *D* with the other two faces in its own octant, and two edges Δ with two faces lying in attingent octants; and each face being of course symmetrical to the plane of the great circle Σ on which its pole lies, the form of the face will be that of a deltoid.

If the edges D and Δ were similar the deltoid would become a rhomb, and the symbol become $\{110\}$, as the pole would then fall on the axial point d. The rhomb-dodecahedron is thus seen to be the limiting figure of the deltoid dodecahedron; as it is also of the holohedral form, the triakis-octahedron (or octahedrid pyramidion).

And of the fourteen quoins of the form $\sigma\{hhl\}$, four on alternate extremities o of the trigonal axes are dissimilar from the four on the extremities ω of the same axes, alternating with the former. And there are six four-faced quoins h on the tetragonal axes.

Of the twenty-four edges, the twelve D-edges meet in threes in the quoins o, and the twelve Δ-edges meet in threes in the quoins ω.

The symbols of the faces of the twelve-deltohedron $\sigma\{hhl\}$ are—

$$hhl \quad \bar{h}\bar{h}l \quad \bar{h}h\bar{l} \quad h\bar{h}\bar{l},$$
$$hlh \quad \bar{h}\bar{l}h \quad \bar{h}l\bar{h} \quad h\bar{l}\bar{h},$$
$$lhh \quad \bar{l}\bar{h}h \quad \bar{l}h\bar{h} \quad l\bar{h}\bar{h}.$$

Those of the correlative semiform $\sigma\{\bar{h}\bar{h}\bar{l}\}$ are the faces of the triakis-octahedron supplementary to these; they have an odd number of negative indices.

The semiform $\sigma\{332\}$ occurs on fahlore, and $\sigma\{221\}$ on blende.

180. *The hemi-icositetrahedron*, or *tetrahedrid pyramidion*, $\sigma\{hkk\}$ or $\sigma\{\bar{h}\bar{k}\bar{k}\}$, called also the *trigonal dodecahedron*, or *twelve-isoscelohedron*, Figs. 89, 90.

The suppression of the faces of an icositetrahedron (or twenty-four-trapezohedron) which lie in alternate (i.e. attingent) octants produces a pyramidion figure, namely, a tetrahedron with a three-faced pyramid on each of its faces.

Since the poles of the form lie on the **D** arcs of the systematic triangles, its twelve pyramidal edges H lie in the **H** sectors of the Σ planes, and the remaining six edges D coincide in position and direction with those of the tetrahedron.

The *H*-edges meet in threes in the four quoins *o* or *ω*; the remaining four quoins *ω* or *o* being each formed by the meeting

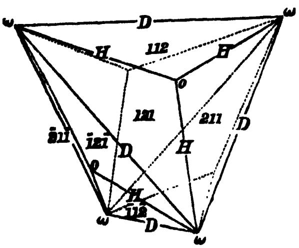


Fig. 89.

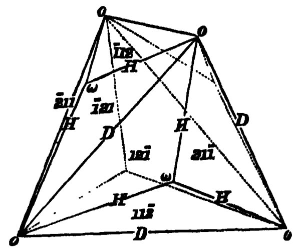


Fig. 90.

of three edges *H* with three edges *D*. The symbols of the faces of the form $\sigma\{hkk\}$ are

$$\begin{matrix} hkk & \bar{h}\bar{k}k & \bar{h}k\bar{k} & h\bar{k}\bar{k} \\ khk & \bar{k}\bar{h}k & \bar{k}h\bar{k} & k\bar{h}\bar{k} \\ kkh & \bar{k}\bar{k}h & \bar{k}k\bar{h} & k\bar{k}\bar{h}. \end{matrix}$$

Those of the form $\sigma\{\bar{h}\bar{k}\bar{k}\}$ are the symbols of the remaining faces of the trapezohedron; and in these the negative signs of the indices are one or three in number in each symbol.

The form $\sigma\{322\}$ occurs on tennantite; $\sigma\{211\}$ on fahlore and tennantite; $\sigma\{311\}$ on blende and fahlore.

181. *The hexakis-tetrahedron*, $\sigma\{hkl\}$ or $\sigma\{\bar{h}\bar{k}\bar{l}\}$, Fig. 91. The hemi-hexakis-octahedron, like the two preceding forms, presents generally a tetrahedrid aspect, each axis o meeting at opposite extremities quoins which are ditrigonal but dissimilar. The six faces in each octant are scalene triangles and meet in six edges, of which alternate triads *H* and *D* are similar, while the remaining edges of the figure represent broken edges of the tetrahedron and are similar; each successive pair meeting two edges *H* and *D* in an orthosymmetrical four-faced quoin *h*.

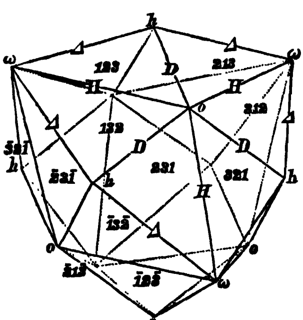


Fig. 91.

The form has therefore thirty-six edges; viz., twelve edges *D*, twelve edges *H*, and twelve edges Δ: fourteen quoins; four obtuse (or acute) ditrigonal quoins *o*, four acute (or obtuse) ditrigonal quoins *ω*, and six four-faced tetragonal quoins *h*.

The symbols of the faces of the form $\sigma\{hkl\}$ are given in the columns i, iii, v, vii; those of the form $\sigma\{\bar{h}\bar{k}\bar{l}\}$ in the columns ii, iv, vi, viii of Table A, Article **167**.

182. B. II. Hemi-systematic diplohedral forms; hemi-tesseral semiforms.

The second section of tesseral hemi-symmetrical forms is that under which the symmetry is hemi-systematic; i.e. half only of the normals of the integral form are represented by faces; but in the hemihedral case each of these normals carries its two parallel faces, so that the form is centro-symmetrical; at the same time the proto-systematic planes *S* are alone planes of symmetry; and the poles of extant faces of the scalenohedron $\{hkl\}$ circum-jacent to the trigonal axes must lie in alternate systematic triangles. Furthermore, no forms having their poles on zone-circles Σ can be hemihedral as regards the number of their faces, though every such form while presenting its full complement of faces will reflect in the distribution of their crystallographic (physical and geometrical) characters a symmetry analogous to that represented in the semiforms $\pi\{hkl\}$ and $\pi\{khl\}$; since in fact this abeyance of symmetrical conditions must be the consequence of the molecular structure of the entire crystal. The geometrically hemihedral forms of this class will therefore be confined either to such as have their poles lying on the zone-circles *S* and symmetrical to the proto-systematic planes, and they will thus be semiforms of the cube-pyramidion; or else they will be semiforms of the general scalenohedron $\{hkl\}$.

183. *The pentagon-dodecahedron*, $\pi\{hk0\}$ or $\pi\{kh0\}$, Figs. 92–5. The pentagon-dodecahedron represents the hemi-systematic form derived from the tetrakishexahedron or cube-pyramidion $\{hk0\}$. The symbols of the faces of one of the semiforms lie in alternate rows in Table B, Article **172**, those of the correlative semiform in the rows alternating with these.

Each face is divided euthysymmetrically by the trace of the systematic plane *S* in which its pole lies; and the edge *O*, formed by two adjacent faces of which the poles are on the same great circle, will lie in a second plane *S* perpendicular to the former.

The edges *G* of adjacent faces, the poles of which lie on different great circles *S*, will not lie in any systematic plane: each face of the semiform will be adjacent to four such faces, and will thus have five sides formed by one edge *O* and four similar edges *G*, as in Figs. 92 to 95.

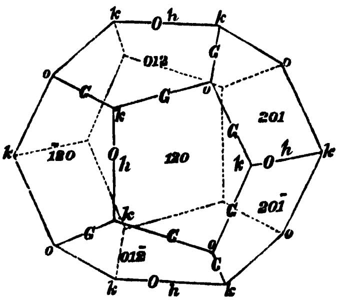


Fig. 92.

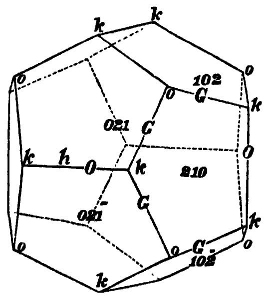


Fig. 93.

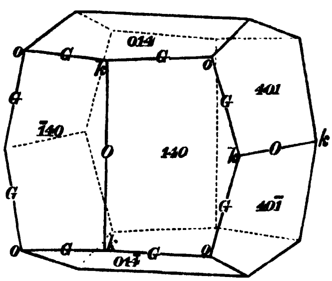


Fig. 94.

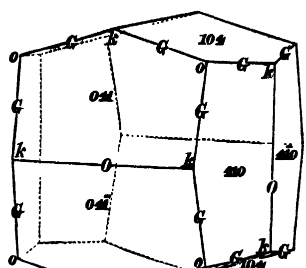


Fig. 95.

The semiform then has six similar edges *O* and twenty-four similar edges *G* gyroidally symmetrical in threes on the trigonal axes o: these meet in eight trigonal quoins *o* formed each by a triad of *G* edges, and twelve quoins *k* formed each by two edges *G* and one edge *O*.

Each face is thus an irregular pentagon symmetrical to the trace on it of a systematic plane *S* perpendicular to its edge *O*.

In the semiform $\pi\{hkl\}$ this trace of the systematic plane becomes an edge, each pentagonal face of the form $\{hk0\}$ being 'broken' into two trapezoidal faces. (Compare Figs. 92 and 96.)

The pentagon-dodecahedron approximates in character to the regular dodecahedron of geometry in proportion as the dihedral angles of its *G* and *O* edges approach equality, resulting in those edges also approximating to each other in length.

Were the pentagonal faces to become equilateral and equiangular the edges *O* and *G* would be equal, and thus the angle $(0hk, 0h\bar{k})$ would equal $(0hk, hk0)$; whence, as will be proved in Chapter VIII, Section II, $hk = h^2 - k^2$ and $\frac{h}{k} = \frac{1+\sqrt{5}}{2}$, which is irrational. The regular dodecahedron of geometry, thus impossible as a crystallographic form, is the limiting figure between the two classes of pentagon-dodecahedra, in which an edge *O* either is larger or is smaller in its dihedral angle than an edge *G*. In proportion as the length of the edge *O* increases, the figure approaches in form to the cube, the ratio $\frac{h}{k}$ becoming greater: where, on the other hand, this value approaches unity, i.e. as *h* approaches *k* in value, the figure approximates in form to the rhomb-dodecahedron $\{110\}$ and the length of the edges *O* is small relatively to that of the edges *G*.

The twelve-pentagonohedron is a very characteristic form of certain mineral species which belong to a small group, of which pyrites is a conspicuous member.

The forms $\pi\{230\}$, $\pi\{210\}$, $\pi\{310\}$ occur on pyrites, and the forms $\pi\{410\}$ and $\{210\}$ on cobaltine. Sixteen other varieties of this form have also been met with on the former mineral.

184. *The dyakis-dodecahedron* or *twenty-four-trapezohedron; the diplohedron* $\pi\{hkl\}$, Figs. 96, 97. The poles of this form lie in the alternate triad of systematic triangles in each octant, while in any two adjacent octants the systematic triangles containing extant poles of the form are those which are mutually symmetrical on a systematic plane *S*.

The edges *G* of the triad of planes in each octant are therefore similar, and, like those of the form $\pi\{hk0\}$, are gyroidal in their distribution round the trigonal axes; the edges *Ω* formed by pairs of contiguous faces in adjacent octants lie in the systematic planes *S*: and so also do the remaining edges *O*, which correspond to the 'broken' edge *O* of the form $\pi\{hk0\}$, and are formed by other pairs of faces likewise in adjacent octants, but the poles of which do not lie in contiguous systematic triangles. Figs. 96 and 97 represent the correlative semiforms $\pi\{231\}$ and $\pi\{321\}$.

The twenty-four faces of the semiform are therefore quadrilateral figures corresponding to 'broken' faces of the semiform $\pi\{hk0\}$, and, as in that figure, the similar adjacent edges *G* are not similar to the remaining two; and of the latter the one represented by an edge *Ω* is larger and the other represented by an edge *O* is shorter

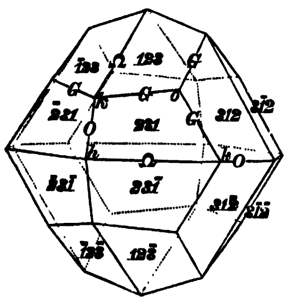


Fig. 96.

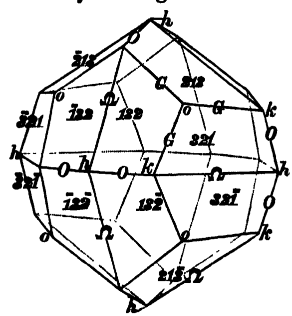


Fig. 97.

than an edge *G*. Each face is thus a trapezoid. The form has then twenty-four edges *G*, twelve edges *Ω*, and twelve edges *O*; and since the deutero-systematic planes fail of being symmetral, the axes **h** become axes not of tetragonal but of ortho-symmetral character, the six quoins *h* of the form being made up of two edges *Ω* alternating with two edges *O*: the eight quoins *O* correspond to those in the pentagonohedron and are made up of three edges *G*, and the remaining twelve quoins are formed by two edges *G* meeting an edge *Ω* and an edge *O*, both of which lie in the same systematic plane *S*.

The terms dyakis-dodecahedron and diplohedron have been employed to convey the idea of the form being a doubled or 'broken-faced' pentagon-dodecahedron.

The symbols of the faces of the form $\pi\{hkl\}$ are contained in

Blocks I and II, those of the form $\pi\{khl\}$ in Blocks III and IV of Table A.

A particular case of the diplohedron occurs where one of the edges G is parallel to an edge Ω. In such a case the form $\pi\{hkl\}$ will have e. g. its face hkl in the same zone with lhk and $\bar{l}hk$, i. e. in the zone $[0\bar{k}h]$, whence by the doctrine of zones

$$hl = k^2 \quad \text{or} \quad \frac{h}{k} = \frac{k}{l}.$$

The indices of the form $\pi\{421\}$ fulfil this relation. Such a form will have trapezia instead of trapezoids for its faces.

This form $\pi\{421\}$ occurs on pyrites. Other diplohedra are $\pi\{531\}$, $\pi\{321\}$, and $\pi\{543\}$, which are met with on crystals of pyrites, and occur also on those of hauerite and cobaltine.

185. B. III. Hemi-systematic haplohedral forms; tetartohedrism.

Another case of mero-symmetry in the Cubic system still remains to be considered; and it presents considerable interest from the association with this peculiar kind of mero-symmetry of the optical property of rotatory polarisation.

It is the case in which a hemi-systematic form is only haplohedrally developed, resulting in a tetartohedral form. Evidently such a form can only exist as a geometrically independent form in a single case, that of the general form $\{hkl\}$; the twenty-four normals characterising which are reduced to twelve. If therefore the half of the faces of each of the twenty-four-faced semiforms $\sigma\{hkl\}$ or $\pi\{hkl\}$ be mero-symmetrically suppressed, the resulting figure will be the twelve-faced tetartohedron in question.

And as this suppression can take effect only on three faces in an octant simultaneously, it must be that three extant faces in alternate octants will be those to be suppressed.

The resulting tetartohedral solid is the *tetrahedrid twelve-pentagonohedron* $\pi\sigma\{hkl\}$, or the *tesseral tetartohedron*, Figs. 98, 99.

Each of the forms $\pi\{hkl\}$ and $\pi\{lkh\}$ may be considered as built up of two tetartohedra, viz.

$\pi\{hkl\}$ of $\pi\sigma\{hkl\}$ and $\pi\sigma\cdot$ $\bar{}\,\bar{}\,\bar{}$

and $\pi\{lkh\}$ of $\pi\sigma\{lkh\}$ and $\pi\sigma\cdot$

and similarly for the forms $\sigma\{hkl\}$ and $a\{hkl\}$.

The faces of this solid are irregular pentagons presenting no symmetry. Their sides are formed by edges of three kinds, viz. two contiguous edges G, two contiguous edges Γ, and one edge V; the form presenting twelve edges G arranged in triads gyroidally round the alternate axial points **o** of the trigonal axes, twelve edges Γ similarly grouped round the axial points ω opposite to the points o of these axes, and six edges V through the centres of which the axes **h** pass. None of these edges lie in a systematic plane.

The quoins are also of three kinds, viz. four trigonal quoins o, four trigonal quoins ω, and twelve quoins k in each of which an edge V meets an edge G and an edge Γ.

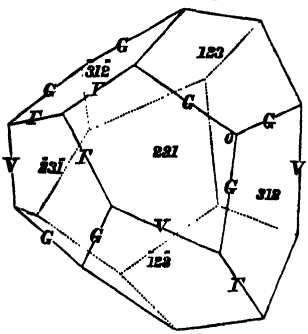


Fig. 98.

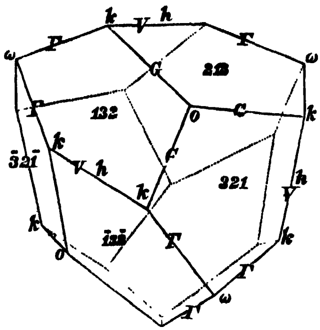


Fig. 99.

The symbols of the four correlative tetartohedra are contained each in one of the blocks in Table A, Article **167**, and Figs. 98 and 99 represent two of the four quarter-forms derived from the scalenohedron $\{321\}$, viz. $\pi\sigma\{231\}$ and $\pi\sigma\{321\}$.

Since the tetartohedral form may be derived from either of the three species of hemi-symmetric forms of the general scalenohedron $\{hkl\}$, it may equally well be represented by symbols $a\sigma\{hkl\}$ or $\sigma\pi\{hkl\}$, &c., or $\pi a\{hkl\}$, &c.

The relations of the tetartohedron to these hemihedra are represented in the projections of their poles in Plate I.

The tesseral tetartohedron has been met with on a very limited number of crystals.

Cubic System. C.—*Combinations of Forms.*

186. The combinations of forms already observed on crystals belonging to the Cubic system are indefinite in number and variety, and, owing to the completeness of their symmetry and the high number of the faces that consequently represent a form, such crystals are often very complex in their aspect.

The more frequent, and usually the prevalent forms, however, are those with the simplest symbols. In the Cubic, as indeed in every system, each mineral or artificially formed crystal has its own habit, certain forms being generally predominant; but this habit itself varies with the conditions under which the crystal has been formed; the nature of the parent solution or liquid from which the crystals have been deposited, possibly also the temperature and pressure under which the solidification has occurred, having an influence in determining the character and relative developements of the particular forms that the crystals present.

It results from this that a sort of individuality of type, or family likeness, as regards forms and even physical features, frequently characterises the crystals of a mineral coming from any given locality, and often approximately serves to indicate that locality; but, if such minerals afford abundant illustration of the variation of habit resulting from the varying conditions under which the crystals have originated, little, indeed almost nothing, is as yet known regarding those conditions or the methods by which they operate.

Generally, in tesseral crystals the faces of the three simple forms which have for their normals the axes of symmetry are the most frequent, and the most largely developed: and where two or all three of these forms concur the aspect of the crystal will vary much according as the cube, octahedron, or dodecahedron is the more predominant form. The figures 100 and 101 present two aspects of the combination of the cube and octahedron, Figs. 102 and 103 the combination of the cube and rhomb-dodecahedron, while in Figs. 104 and 105 the three forms concur.

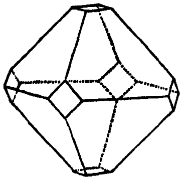


Fig. 100.

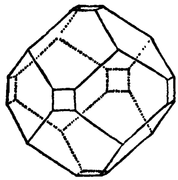


Fig. 102.

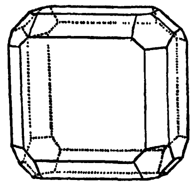


Fig. 104.

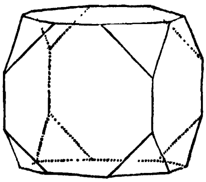


Fig. 101.

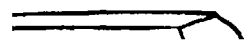


Fig. 103.

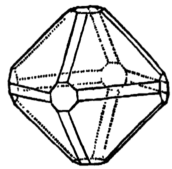


Fig. 105.

C. I. *Combinations of Holo-symmetrical Forms.*

187. The *octahedrid-pyramidion* (or *triakisoctahedron*) $\{hhk\}$ occurs only in combination, except in the case of the form $\{221\}$ which is stated to occur as a self-existent form on the diamond; although if that mineral could be considered as haplohedral this form would have to be explained either as a twin of a twelve-faced deltohedron $\sigma\{221\}$ twinned on a dodecahedral axis, or as a combination of two independent semiforms $\sigma\{221\}$ and $\sigma\{\overline{2}\overline{2}\overline{1}\}$.

Among the triakisoctahedrid forms met with in combination $\{332\}$ occurs on garnet with $\{211\}$ and $\{110\}$; $\{553\}$ on magnetite; $\{221\}$ on cuprite and galena, and $\{331\}$ with $\{111\}$ and $\{110\}$ on fluorspar and galena, and both the forms $\{221\}$ and $\{331\}$ concur with the cube and octahedron on crystals of cuprite.

The *icositetrahedron* $\{hkk\}$ is a more frequent form than the octahedrid pyramidion. And it occasionally occurs self-existent. The forms $\{211\}$ and $\{311\}$ are thus met with uncombined with other forms, the former in garnet, the latter in native silver, magnetite, and spinel. Otherwise the various icositetrahedra are usually met with in association with and subordinated in importance to the cube, octahedron, or rhomb-dodecahedron. Fig. 106, of a crystal of copper, exhibits these three forms associated with the form {411}. On crystals of magnetite from Traversella the form {311} is associated with the two last forms, the dodecahedron being predominant. The forms {322} and {433} occur on galena together with, but subordinate to, the cubo-octahedron. The form {833} is met with on fluorspar, as are the forms {311} and {211}, facetting and giving an obtuse aspect to the quoins of the cube. And a similar concurrence with the cube of the faces of the form {311} is met with on crystals of highly argentiferous gold from Transylvania, and of native silver from the Kongsberg

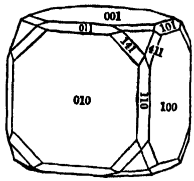


Fig. 106.

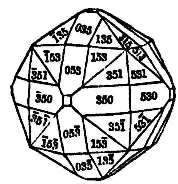


Fig. 107.

mine in Norway. On amalgam the form {211} is associated with {221}, respectively facetting the edges in which the faces of the octahedron meet those of the cube and dodecahedron.

The *tetrakishexahedron* or *cube-pyramidion* {*hk*0} occurs as a self-existent form in the case of the form {210} in native copper and native gold, and of the form {310} in fluorspar. Otherwise tetrakishexahedra are found only in association, and frequently so with the cube and with the rhomb-dodecahedron. Not infrequently a form {*hk*0} will be associated with the corresponding icositetrahedron {*hkk*}. Thus in Fig. 108, of a garnet from Dognaczka, the forms {210} and {320} are combined with the forms {211} and {321}. These forms {210} and {320} occasionally occur with the cube with their faces predominantly developed, but more usually they are the subordinate form, as seen in fluorspar.

The *forty-eight-scalenohedron* or *hexakisoctahedron* occurs as a self-existent form only on the diamond, and in that mineral with faces so curved and striated as to preclude an accurate determination of the symbols of the form or forms. The geometric characters of this form have been discussed, and it is only necessary to add that in combination it not unfrequently concurs with the particular icositetrahedron, of which the faces truncate its edges. Such is the case on the garnet, where the dodecahedrid pyramidion {321}, as also the pyramidion {431}, are associated with but subordinate to {211}, and with predominant faces of the dodecahedron {110}, which belong to the same zones.

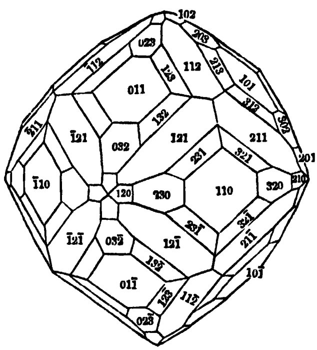


Fig. 108.

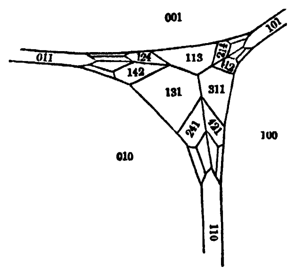


Fig. 109.

Fig. 107 represents a crystal in the British Museum of copper from Lake Superior exhibiting the forms {100}, {530}, and {531}.

Often, however, the icositetrahedra and triakisoctahedra concur with scalenohedra, to which they are not truncating forms, and the faces of the latter figures then facet the quoin-edges of the cube or dodecahedron with which they may be associated.

In fluorspar singularly elaborate and symmetrical modifications of the cube-quoins are effected by the faces of scalenohedra sometimes two or three in number, generally associated with icositetrahedra; often also narrow faces of the rhomb-dodecahedron are present truncating the cube-edges; and on this mineral both scalenohedra and the cube-pyramidia occasionally occur with symbols the indices of which rise to very high numbers. Fig. 109 represents one quoin of a crystal of fluor carrying the faces of the cube of the dodecahedron, and of the forms {311} and {421} associated with two other scalenohedra of which the symbols are indeterminable.

C. II. *Combinations of Hemi-symmetrical Forms.*

i. Holo-systematic haplohedral forms.

(*a*) *Asymmetric Forms.*

188. Though this type of hemi-symmetry has been long recognised as theoretically possible, its actual occurrence as a character of crystals has been only recently observed. Tschermak has described artificial crystals of ammonium chloride with small faces of the form *a* {785} replacing the edges of the form *a* {211}, the latter of which, as indicated in Art. **176**, has its full complement of faces. In Figs. 110 and 111 are represented crystals of cuprite

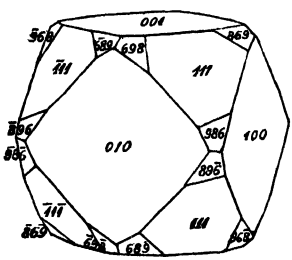


Fig. 110.

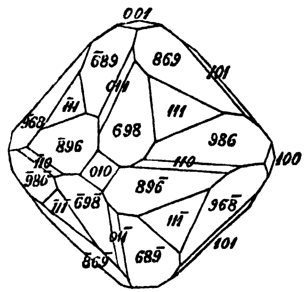


Fig. 111.

from Wheal Phœnix, Cornwall, in which the form *a* {986} is associated with the cube, octahedron, and dodecahedron, as described by Miers from specimens in the British Museum.

(*b*) *Tetrahedrid Forms.*

189. Of the tetrahedrid forms of the Cubic system the correlative semiforms are frequently concurrent on a crystal; but when they are it may be said that universally, even though the detection of the differences may be difficult, the different semiforms must be differently endowed in physical character if not in geometrical developement. For, the two directions of the same normal being in the case of no tetrahedrid form identical in property, it is not consonant with the idea of crystallographic similarity that the two faces with a common normal but belonging to different though correlative semiforms should be identical in feature and property. And this crystallographic axiom renders the tetrahedrid character of the diamond difficult of acceptance. The evidence in favour of the hemihedrism of the diamond rests on the character of some of the re-entrant edges of its crystals, and on the very rare but undoubtedly extant crystals which are tetrahedrid in form. The character of the last kind of evidence has been considered in Article **160**, and the further treatment of the question will find its place when the twinning of tetrahedrid forms on a dodecahedral axis is discussed.

The characteristic features that distinguish the faces of the *o* tetrahedron $\sigma\{111\}$ from those of the ω tetrahedron $\sigma\{\bar{1}11\}$ are very marked in the mineral blende. The faces now usually selected from the two correlative semiforms to represent the $\sigma\{111\}$ semiform are deeply striated parallel to the edges of the octahedron, and have a dull aspect as compared with the ω faces, which are brilliant in lustre and smooth. These differences are represented in Fig. 125, of a twin-crystal of blende, in which both the correlative tetrahedra are present in association with the faces of the cube.

In crystals where only one of the correlative forms occurs with the cube, the alternate quoins of the latter figure are truncated by the faces of the tetrahedron, an example of which is afforded by pharmakosiderite (cube-ore) from Cornwall. It occurs also on blende, see Fig. 55 (1). The faces of the ω tetrahedron in fahlore are large, and striated parallel to the edges of the tetrahedron; inverted therefore as compared with those of the *o* faces of blende, on which the striations are parallel to the edges of the octahedron, that is to say, the lines in which the striations meet correspond to the **H** sides of the systematic triangle in fahlore, and to the **D** sides of the triangle in blende; the *o* faces are very small in comparison with the ω faces.

The triakistetrahedra $\sigma\{hkk\}$ and $\sigma\{\bar{h}\bar{k}\bar{k}\}$ (tetrahedrid pyramidia), which are the tetrahedrid semiforms into which the icosihedron $\{hkk\}$ may be resolved, are only known to occur as self-existent forms in eulytine (bismuth-blende), and in helvine (or rather in achtaragdite, which appears to be a pseudomorph in the form of helvine); in each case as the form $\sigma\{211\}$.

Crystals which, like fahlore (or tetrahedrite), carry predominant faces of a pyramidion, always retain a tetrahedrid aspect. Associated with the striated tetrahedron $\sigma\{\bar{1}\bar{1}\bar{1}\}$ in that mineral the forms $\sigma\{\bar{2}\bar{1}\bar{1}\}$ striated parallel to the edges of the tetrahedron, $\sigma\{\bar{3}\bar{1}\bar{1}\}$, $\sigma\{\bar{9}\bar{5}\bar{5}\}$, $\sigma\{\bar{4}\bar{1}\bar{1}\}$ are met with, while the form $\sigma\{211\}$

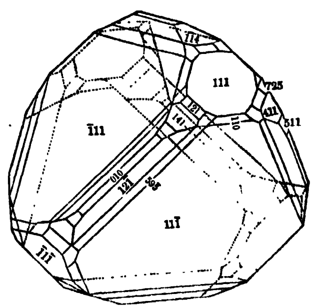


Fig. 112.

striated parallel to the edges of the occurring faces, $\sigma\{411\}$ and $\sigma\{511\}$ are associated with the ω tetrahedron $\sigma\{111\}$ in the alternate octants: and the rhomb-dodecahedron is frequently present. Fig. 112 is a representation of a crystal of fahlore (after Hessenberg) on which the forms $\sigma\{111\}$, $\sigma\{511\}$, $\sigma\{411\}$ and $\sigma\{211\}$ concur with $\sigma\{\bar{1}11\}$, $\sigma\{\bar{1}21\}$, $\sigma\{\bar{5}95\}$, $\sigma\{110\}$ and $\sigma\{100\}$. On blende the more dull and striated tetrahedron $\sigma\{\bar{1}\bar{1}\bar{1}\}$ is found associated with the forms $\sigma\{\bar{2}\bar{1}\bar{1}\}$ and $\sigma\{\bar{5}\bar{2}\bar{2}\}$, the faces of the latter form being somewhat rounded: occasionally also $\sigma\{\bar{2}21\}$ and $\sigma\{\bar{3}31\}$ are met with. The correlative tetrahedron, on the other hand, viz. $\sigma\{111\}$, has its faces more developed and more brilliant and is concurrent frequently with $\sigma\{311\}$, the faces of which carry striations parallel to their intersections with, and therefore to the edges of, the rhomb-dodecahedron: and the forms $\sigma\{411\}$ and $\sigma\{12\ 1\ 1\}$ with a hexakistetrahedron $\sigma\{431\}$ have in some few instances been observed. The dodecahedron is generally present, the cleavages of the mineral lying parallel to its faces, and occasionally it is accompanied by one or more of the forms $\sigma\{320\}$, $\sigma\{210\}$, or $\sigma\{410\}$.

Fig. 113 exhibits the forms $\sigma\{100\}$, $\sigma\{110\}$, $\sigma\{320\}$, $\sigma\{111\}$ with $\sigma\{311\}$; and $\sigma\{\bar{1}11\}$ with $\sigma\{\bar{2}11\}$.

The deltohedron $\sigma\{hhk\}$, which is the haplohedral form of the octahedrid pyramidion (triakisoctahedron), is unknown as a self-existent form.

Both correlative semiforms of the form $\{332\}$ co-exist in fahlore with $\sigma\{210\}$ and $\sigma\{110\}$, and the faces of $\sigma\{\bar{3}32\}$ which are smaller than those of $\sigma\{332\}$. In blende the negative semiforms $\sigma\{\overline{332}\}$ and $\sigma\{\overline{221}\}$ occur combined with the tetra-

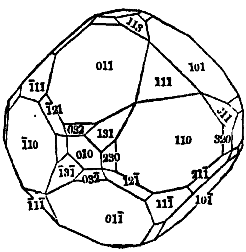


Fig. 113.

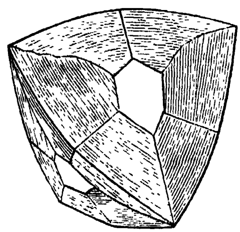


Fig. 114.

hedron $\sigma\{\overline{111}\}$, the dodecahedron $\sigma\{110\}$, and the tetrahedron $\sigma\{111\}$.

Of the *hexakistetrahedron* $\sigma\{hkl\}$ the minerals blende and fahlore offer the prominent examples. Boracite indeed exhibits the form $\sigma\{531\}$ in association with the tetrahedron $\sigma\{111\}$ and predominant faces of the cube truncated by those of the dodecahedron, while the forms $\sigma\{211\}$ and $\sigma\{\overline{111}\}$ facet the alternate quoins of the cube. And crystals of diamond occur which are hexakistetrahedra, but they are extremely rare. Four of these are in the British Museum, and two of them carry faces of a tetrahedron truncating the rounded and six-faced quoins of the hexakistetrahedron. Fig. 114 represents one of these. There are also three crystals in the same collection in which the hexakisoctahedron and octahedron are combined, but with the octahedron faces in the alternate octants smaller than the others. On fahlore, $\sigma\{\overline{3}\overline{2}\overline{1}\}$ is concurrent with $\sigma\{\overline{2}\overline{1}\overline{1}\}$ and $\sigma\{\overline{1}\overline{1}\overline{1}\}$ and the dodecahedron. And on blende the correlative forms $\sigma\{431\}$ and $\sigma\{\overline{4}\overline{3}\overline{1}\}$ are said to concur with $\sigma\{111\}$ and $\sigma\{\overline{1}\overline{1}\overline{1}\}$ and the faces of the cube which truncate the edges of the larger tetrahedron $\sigma\{111\}$.

C. II. *Combinations of Hemi-symmetrical Forms.*

ii. Hemi-systematic diplohedral forms.

190. The *twelve-pentagonohedron* or *pentagon-dodecahedron* $\pi\{hk0\}$, $\pi\{kh0\}$. Pentagonohedra are illustrated almost exclusively by the group of minerals of which pyrites (iron disulphide) is the most familiar member, and hauerite, cobaltite, and gersdorffite, severally the corresponding sulphides or arseno-sulphides of manganese $Mn\,S_2$, cobalt $Co\,(S, As)_2$, and nickel $Ni\,(S, As)_2$, are the other distinct minerals. Much discussion and laborious investigation have been devoted by Friedel, Gustav Rose, Brezina, Schrauf, and E. S. Dana to the endeavour to establish some connection between the thermo-electric or the pyro-electric properties of different crystals or parts of crystals of these minerals and their crystalline forms. The result arrived at from the whole series of these investigations may be summarised in the conclusions that (1) pyro-electricity, i.e. the production of opposite electric condition in different parts of a body, *while its temperature is changing*, has no place in the case of the pyritoid minerals; (2) that as regards thermo-electricity, evidenced by a current set up by a difference of temperature at two contacts of one substance with another, the character of the forms of the crystal has no influence on the thermo-electric sign (positive or negative potential); and (3) that, though the variations in potential do exist, they have nothing to do with crystallographic symmetry, but are dependent on the relative density, and probably on differences in chemical composition of the crystals or parts of crystals investigated; the less dense specimens generally representing the electro-positive, the more dense the negative varieties.

Of the different pentagonohedra known on the pyritoid minerals the only one that is self-existent is the 'pyritohedron' (of Haüdinger) $\pi\{210\}$ or $\pi\{120\}$, which is common in pyrites and cobaltite.

The correlative semiforms being superposable by a quadrant-revolution round a tetragonal axis no distinction can be drawn between the two semiforms as regards crystallographic development in alternate octants. It is, however, an exceptional occurrence when two pentagonohedra of symbols $\pi\{hk0\}$ and $\pi\{k'h'0\}$, the poles of which do not lie in the same systematic triangles, concur on a crystal: for usually, where two or more different forms occur together, their symbols are either all of the type $\pi\{hk0\}$ or all of the type $\pi\{kh0\}$. And the same observation holds regarding the concurrence of diplohedra represented by a general symbol $\pi\{hkl\}$ with those of which the symbol would be

On the other hand, crystallographers have recorded the occurrence of the correlative semiforms $\pi\{210\}$ and $\pi\{120\}$, $\pi\{320\}$ and $\pi\{230\}$, $\pi\{430\}$ and $\pi\{340\}$, $\pi\{540\}$ and $\pi\{450\}$, $\pi\{520\}$ and $\pi\{250\}$, $\pi\{650\}$ and $\pi\{560\}$; the symbol being determined by the association of the semiform with others belonging to octants adjacent to those in which the faces of the particular semiform lie.

Correlative diplohedra with symbols $\pi\{321\}$ and $\pi\{231\}$, $\pi\{421\}$ and $\pi\{241\}$, $\pi\{432\}$ and $\pi\{342\}$ have had their existence attested in a similar way by Romé de l'Isle, Haüy, Mohs and Strüver. Thus, too, Naumann and Zippe describe the union of $\pi\{123\}$ with $\pi\{435\}$. Strüver describes five other pentagonohedra of type $\pi\{hk0\}$ associated with $\pi\{120\}$. And finally G. Rose records $\pi\{435\}$ and $\pi\{324\}$ as concurrent with $\pi\{120\}$.

But these are few, and comparatively exceptional cases, met with among the very extensive and varied series of crystals that this group of minerals contributes to mineral collections.

The pentagonohedron presents in the symbols of the forms that have been observed a larger range than does the cube-pyramidion, of which the pentagonohedron is the semiform. Thus the forms $\pi\{530\}$, $\pi\{430\}$, $\pi\{540\}$, $\pi\{650\}$, and $\pi\{607\}$ are known of which the corresponding pyramidions have not been observed, whereas the hemihedron derived from the cube-pyramidion {520} has not yet been met with. Twenty-one distinct pentagonohedra have been recorded as existing on pyrites as well as seven kinds of tetrahedrid pyramidion, three of deltohedron, and thirteen of diplohedron.

A remarkable combination of pyritohedron and octahedron is a not uncommon form of pyrites and cobaltite. It resembles in aspect the regular icosihedron of geometry, and is formed by the union of the two forms $\pi\{210\}$ and $\pi\{111\}$ in equipoise. The eight faces of the octahedron are equilateral triangles, and the twelve faces of the pyritohedron assume also a triangular form; two sides of each of these triangles being formed by intersection

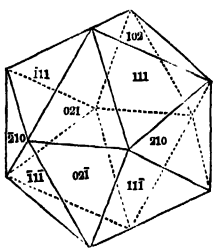


Fig. 115.

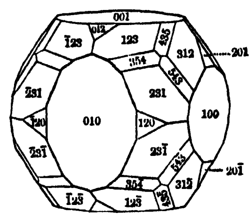


Fig. 116.

with adjacent octahedron faces, the third side being the edge between two adjacent faces of the pyritohedron. Consequently the latter are isosceles triangles, and the figure would be a regular icosihedron but for the shorter length of the third side as compared with the sides of the octahedron faces. This combination is represented in Fig. 115.

The *Diplohedron* $\pi\{hkl\}$ or $\pi\{khl\}$ is known to occur as a self-existent form on pyrites from Traversella with the symbol $\pi\{321\}$, and on that from Brosso with the symbol $\pi\{421\}$. The several other diplohedra occur in combination with one or more of the forms $\pi\{221\}$, $\pi\{211\}$, $\pi\{311\}$, the cube, octahedron, and dodecahedron. But the forms are so varied, so intricate in aspect, but, withal, so simple in their crystallography, that detailed description of them is quite unnecessary.

Fig. 116 represents a pyrites crystal with the forms $\pi\{120\}$, π {231}, π {543}, and π {100}; and Fig. 117 (*a*) a crystal after Strüver with the forms π {241}, π {5 11 2}, π {122} with π {120}, π {100}, and π {111}. Fig. 117 (*b*) represents a quoin of the last crystal drawn on a larger scale.

It is only requisite to observe, in regard to those complete forms of which the faces are grouped in triads but not ditrigonally round the trigonal axes, and which therefore do not admit of disparting into correlative semiforms, that these forms equally undergo hemi-symmetrical modification where they are concurrent with hemi-systematic semiforms. They are the cube, dodecahedron, icosi-tetrahedron, octahedrid pyramidion, and the octahedron itself. It has been mentioned in Article **147** that the cube may be striated in

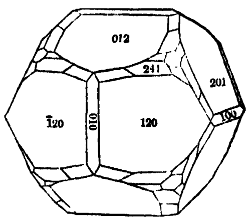


Fig. 117 (*a*).

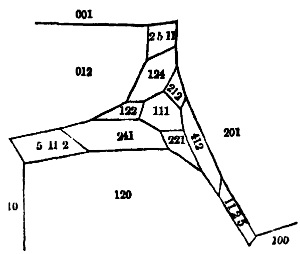


Fig. 117 (*b*).

only one direction, but symmetrically to the *S*-planes in the case of its association with diplohedral forms: and that in the case of haplohedral forms it is striated in one diagonal direction symmetrically to the Σ-planes only. Similar peculiarities of striation or of outline consequent on the abatement of some element of symmetry, and following the extant symmetry of the crystal, will be observed in all these forms. So that we are quite precluded from excluding from their symbols the characteristic σ or π which marks the symmetry or abatement of symmetry in the crystal.

C. III. *Combinations of Tetarto-symmetrical Forms.*

191. The tetartohedron of the Cubic system is unknown as a self-existent form; when met with it is always associated with other forms which are from the geometrical point of view either haplohedrally or hemi-systematically developed. The forms $\sigma\pi\{351\}$ and $\sigma\pi\{421\}$ occur on Barium nitrate. The original discovery by Marbach (*Pogg. Annalen*, Vols. 91 and 94) of the tetartohedral character and the property of rotatory polarisation belonging to Sodium chlorate, Sodium bromate, Sodium-uranyl acetate long stood by itself. Now, however, the number of substances tetartohedral in symmetry has been increased, and includes the nitrates of Barium, Strontium, and Lead; though in these the rotatory action on the plane of polarisation of plane polarised light either does not occur or is too feeble to be recognised.

The fact that there can only be a single kind of tetartohedron in the Cubic system, that namely resulting from a haplohedral hemi-systematic developement of the forty-eight-faced scalenohedron, does not preclude a crystal from having tetartohedral symmetry, even though its forms do not include one of the type $\sigma\pi\{hkl\}$. We find, for instance, the faces of a tetrahedron or of distinguishable correlative tetrahedra associated with those of a twelve-pentagonohedron in crystals of Sodium chlorate. Here the faces of a cube are associated with those of a tetrahedron $\sigma\{111\}$ and of the pyritohedron $\pi\{210\}$. Their proper symbols are therefore $\sigma\pi\{100\}$, $\sigma\pi\{111\}$, $\sigma\pi\{210\}$, since they afford evidence of the crystal being, concurrently, haplohedral for one form, and hemi-systematic for another form, where the faces of neither semiform group ditrigonally round the trigonal axes and cannot therefore be disparted into several tetarto-symmetric groups.

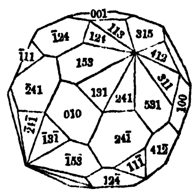


Fig. 118.

Fig. 118 represents a crystal of Barium nitrate (described by Prof. Lewis) which carries the faces of the octahedron, really of the two tetrahedra, those of the icositetrahedron $\{311\}$, which like the former is to be supposed resolved into two tetrahedrid pyramidions, and finally, the tetartohedra $\sigma\pi\{241\}$, $\sigma\pi\{\bar{2}41\}$, and $\sigma\pi\{351\}$, the last form being represented by the faces of but one of the four correlative tetartohedra.

Cubic System. D.—*Twinned Forms.*

192. The difficulty of describing and representing in a compact treatise on crystallography the multitude of combinations of forms, which the crystals of any system may exhibit, belongs also to any similar attempt in the case of the twins formed by these combinations. But these can be classified and discussed in general terms without entering on the details necessary in the practical work involved in the crystallographic study of any particular crystal.

Holo-symmetrical crystals in the Tesseral system cannot have a face of the cube or dodecahedron for a twin-plane, since these forms are parallel to planes of actual symmetry at the same time that their normals are axes of actual diagonal symmetry. The faces of the octahedron can be, and almost exclusively are, the twin-faces of holo-tesseral crystals; the only other recorded variety of such twins being in the case of certain crystals of galena, in which Sadebeck finds the plane of twinning to be a face of the octahedrid pyramidion $\{441\}$. In these crystals, however, the twinned individuals are intercalated as thin plates, the edges of which are seen on the faces of cleavage *.

In hemi-symmetrical crystals the *tetrahedrid* (haplohedral) semiforms are symmetrical only to the deutero-systematic planes Σ, and the normals of these planes though potentially axes of diagonal symmetry have their symmetral character in abeyance. This, however, is not the case as regards the normals of the proto-systematic planes parallel to the faces of the cube. There will then be no restriction on the character of the zone-axes and normals that may be twin-axes for such crystals, except that the actual crystallographic axes cannot be twin-axes. In the *diplohedral* semiforms again, the potentially tetragonal but actually diagonal axes of symmetry (the normals, namely, of the cube-faces), are the only crystallographic lines precluded from being twin-axes.

But though the idea of symmetry imposes no other restriction than that just mentioned on the kind and number of twin-laws that might exist in the Tesseral system, it is practically found that, excepting in the case of the galena twin alluded to, every twin in

* *Zeitschr. der deutsch. geol. Ges.* Band xxvi. S. 617.

this system may be referred either to a twin-law wherein a normal of an octahedron face is the twin-axis, or to a law by which the twin-axis is a normal of a dodecahedron face. The latter law affords twins which are always of the mutually interpenetrant kind. Twins under the former law may be either interpenetrant twins or juxtaposed twins, and in certain cases are embedded twins.

D. I. *Twin-law :—a face of the octahedron the twin-face.*

193. Among juxtaposed twins we may at the outset distinguish those in which the twin-face is also the plane of combination from such as have for the combination-plane a face perpendicular to the twin-face.

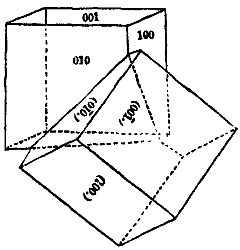


Fig. 119.

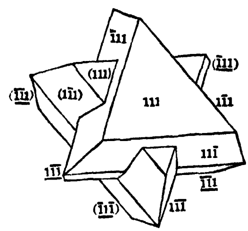


Fig. 120.

And again, in crystals of the former class we have the cases of simple twins composed of two individuals, and of complex twins formed by repeated twinning, while the complex twins are further to be subdivided into the cases of *parallel repetition* (polysynthetic twins), twinned in fact on the same face, and the cases of *inclined repetitions*, where the repetition takes place on different faces of the same form as twin-faces.

Holo-symmetrical crystals: twin-plane a face of the octahedron. Of the spinel-twin in which two crystals, on which the octahedron faces alone occur, are in juxtaposition with an octahedron-face for twin- and combination-face, a representation has been given in Fig. 57, Article **154**. Fig. 119 is an embedded twin of the cube, and Fig. 120 represents a double twin of diamond; the octahedra being compressed in the direction of the twin-axis. It is a frequent twin in the minerals of the spinel group including magnetite, and is distinctly a twin-law for the diamond; and for those tetrahedrid crystals which affect in the developement of their correlative tetrahedrid forms the appearance of diplohedral crystals, the spinel law holds also. Of this blende will be seen

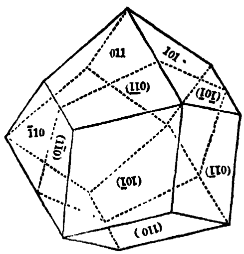


Fig. 121.

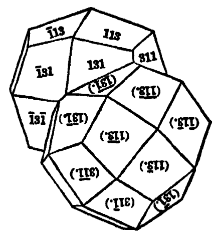


Fig. 122.

to be an example. Of other aspects of crystals twinned on an octahedron-face the following figures are illustrations. Fig. 121 is that of a twinned rhomb-dodecahedron of gold, and Fig. 122 represents a twin of the form {311}. Fig. 123 represents a twin

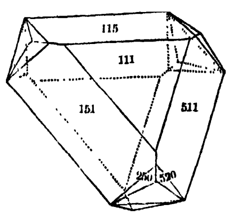


Fig. 123.

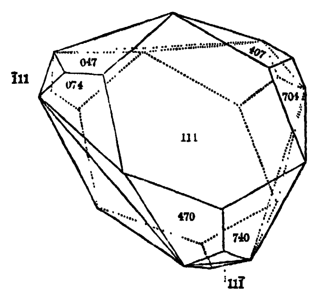


Fig. 124.

of copper from Lake Superior with the forms {111}, {511}, and {520}: fig. 124 is another twin of copper with forms {111} and {740}; both from crystals in the British Museum.

The cases in which the combination-plane is the twin-face are far more frequent than those in which the one plane is perpendicular to the other; the latter have in fact almost the character of exceptions in Tesseral crystals, and in them the combination-plane is a face of the form $\{211\}$.

Besides the twins so far considered, in which two crystals are juxtaposed at the face of an octahedron, or else of the form $\{211\}$ perpendicular to that face, there exist interpenetrant twins of this class. Of these, cubes of galena (of which Fig. 119 is an illustration) and dodecahedral crystals of sodalite are examples.

194. Haplohedral crystals: twin-plane a face of the tetrahedron. Of the simple twins of this kind we have characteristic examples in blende and fahlore.

Fig. 125 represents a twin of blende in which the twin-face $(11\bar{1})$ is also the combination-plane. It shows the faces of an ω and an o tetrahedron meeting in an edge in the twin-plane.

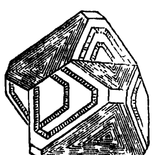


Fig. 125.

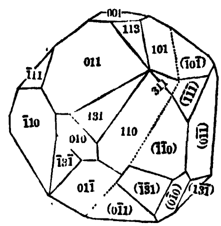


Fig. 126.

Fig. 126 represents a twin also of blende with the forms $\sigma\{100\}$, $\sigma\{110\}$, $\sigma\{311\}$, and $\sigma\{\bar{1}\bar{1}\bar{1}\}$ twinned on a face $(1\bar{1}\bar{1})$.

While blende usually presents a very holohedral aspect in consequence of the preponderance of the rhomb-dodecahedron faces in association with those of both correlative tetrahedra, fahlore, on the other hand, is a mineral with an essentially tetrahedrid aspect, as has already been explained. A twin of fahlore with a tetrahedron-face for its twin-plane is exhibited in Fig. 127.

Of twins by parallel repetition an example is given in Fig. 128, which represents a crystal of blende from the Binnenthal, the successive combination-planes being parallel to the same face of the tetrahedron $\sigma\{\bar{1}11\}$, which is at the same time the twin-face.

But as regards the crystals in which the combination-plane is perpendicular to the twin-plane, it is to be observed that such a plane can be simultaneously a face and a plane of symmetry to the twinned combination. In fact, Groth offered as a definition of a twin-plane that it may be a plane parallel to a face identical for the two crystals, which is not a plane of symmetry to either crystal, but in respect to which the two crystals are symmetrical. That such a face would be a plane of symmetry in the particular case where the plane in the zone [111] belongs to the form {211} will be seen if we consider a stereographic projection of a tesseral crystal on a plane parallel to a face of the octahedron, say to the face {111}. If, then, the crystal is such that the arrangement of the poles of any form is

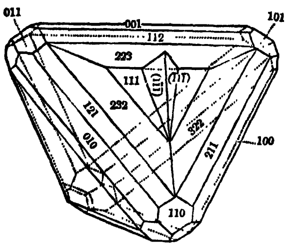


Fig. 127.

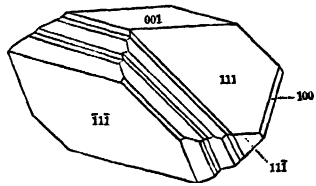


Fig. 128.

ditrigonal round the normal of any octahedron-face, for instance the face (111), and if we suppose the crystal to be twinned round such a normal as a twin-axis, the result will be a dihexagonal distribution of the *poles* of the *united* crystals. It will be seen that the faces ($1\bar{1}0$), ($\bar{1}01$), and ($01\bar{1}$) of the rhomb-dodecahedron, and the ($\bar{2}11$), ($1\bar{2}1$), ($11\bar{2}$) of the icositetrahedron {112}, and the faces severally parallel to them, are all in the zone [111] and are perpendicular to the face (111); and origin-planes parallel to these faces will be planes of symmetry for the poles of the twinned crystals.

If another face of the octahedron were taken as the plane of projection and of twinning, different groups of six faces of the forms {110} and {211} would become the planes of symmetry.

Taking the case first supposed; let now the plane ($11\bar{2}$) be the face of combination of two crystals twinned round an axis [111]. The face ($1\bar{1}0$) for both crystals is perpendicular to their common face $(11\bar{2})$ and an origin-plane parallel to it, being a deutero-systematic plane, is a plane of symmetry for each of the crystals. Hence any pole or face A (say, hkl) of the first crystal, repeated on the second crystal by twinning in A', will be symmetrical in respect to the plane $(11\bar{2})$, with a pole B', i. e. (khl) on the second crystal, which itself is symmetrical with A' in respect to $(1\bar{1}0)$, or both A and A' are symmetrical with a pole B on the first crystal in respect to the planes $(1\bar{1}0)$ and $(11\bar{2})$. Therefore corresponding faces, i. e. faces with the same indices (here, hkl) on the two crystals, are those which are symmetrical to an identical face (here, khl); the one symmetrical to it over the combination plane $(11\bar{2})$, the

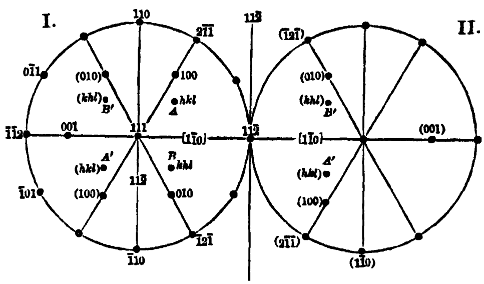


Fig. 129.

other symmetrical to it over the plane $(1\bar{1}0)$ perpendicular to the latter. Figs. 130 and 131 (after Sadebeck) represent a crystal of galena twinned on (111) and with combination-plane perpendicular to (111).

The very rare occurrence of crystals in which this kind of twinning is observed, would seem to preclude the definition of a twin that is drawn from it from being admissible as a general statement for twin-crystals.

Interpenetrant twins, with an octahedron-face for twin-plane, occur with haplohedral crystals in the case of fahlore. Of this Fig. 127 is a representation in the case of a crystal in which the tetrahedron $\sigma\{111\}$ is associated with the pyramidions $\sigma\{322\}$ and $\sigma\{211\}$, and with the faces of the rhomb-dodecahedron $\sigma\{110\}$ in subordinate developement. It exhibits the peculiarity that the second individual is smaller than the first, which it penetrates, and does not carry the pyramidion forms but only the faces of the tetrahedron and dodecahedron.

Between the interpenetrant twins and the juxtaposed twins there are different degrees of interpenetration represented by more or less entirely embedded twins; and of these an example is cited by Sadebeck in the case of fahlore, in which the composition-plane $(\bar{2}1\bar{1})$ is perpendicular to the face of twinning $(11\bar{1})$; but instead of the two crystals being in contact merely at the former face, they are so united that a portion of each is, as it were, lost in the other

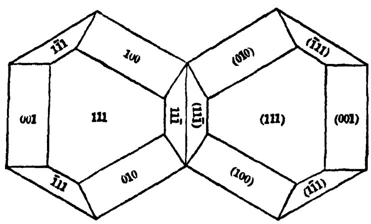


Fig. 130.

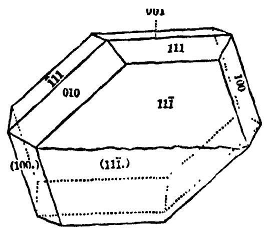


Fig. 131.

but symmetrically to the plane of their junction. Fig. 131 illustrates such a growth in the case of a twin-crystal of galena.

D. II *Twin-law:—a face of the rhomb-dodecahedron the twin-face.*

195. Haplohedral crystals. Examples of holohedral forms twinned on an axis of actual diagonal symmetry being impossible, we have only to deal with the cases of (1) haplohedral crystals twinned round a normal of the form $\{110\}$, and (2) of a diplohedral crystal so twinned. Examples of the former kind are extremely rare: eulytine has been cited as one, and pseudomorphous crystals after fahlore from Bieber in Hanau are instanced by Kopp* and by Groth†. Such twins are interpenetrant, and

* *Neues Jahrb. für Min.* 1877, p. 62.

† *Strassburg Catalogue*, p. 4.

Fig. 132 represents a twin of eulytine, presenting the pyramidion $\sigma\{211\}$ of Fig. 89, twinned according to this law. Doubts however as to the tesseral symmetry of eulytine have been brought forward by Bertrand, who would refer this mineral to the Hexagonal system.

It is under the head of twins of tetrahedrid forms twinned on a face of the rhomb-dodecahedron that the discussion falls regarding the hemi-symmetrical character of the diamond.

The grounds urged in favour of the traditional view that the diamond is a hemi-symmetrical crystal, so far as they rest on the occurrence of simple hemi-symmetrical forms and on certain peculiarities of conformation in the crystals, have been detailed in Article **160**. Of the latter, the most important feature is the very symmetrical furrows or re-entrant edges that are met with

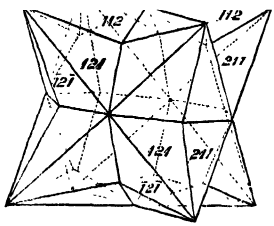


Fig. 132.

on certain crystals which carry the faces of the octahedron, or of that form in association with a hexakisoctahedron, as in Figs. 133, 134.

These re-entrant furrows occur at the edges of the octahedron, and channel the crystal in the directions of these edges where the form is an octahedron, or in the directions of the corresponding edges of the hexakisoctahedron, where the latter form is also present. In the former case they are often, as in Fig. 133, merely step-formed, and seem to indicate a lamellar structure resulting in the successive layers of each octahedral face not being completely developed up to the edges of the face. Oftener they are facetted on each side of the furrow with faces usually rounded and repeated symmetrically to a proto-systematic (S) plane, to which the furrow and the crystal are alike symmetrical; of this the diamond figured in Fig. 134 is an illustration. But these faces—though it is very difficult to allot them their symbols—can be recognised as belonging to forms that differ from, and in fact are shallower in their contour than, the hexakisoctahedral forms extant in association with the octahedral faces in the octants immediately adjacent to them.

Now to explain this singular, indeed unique, type of crystals, it is supposed that they are in fact composed of correlative tetrahedra or of hexakistetrahedra, twinned on a dodecahedral axis,

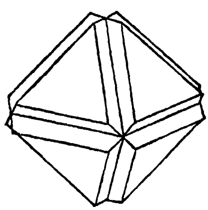


Fig. 133.

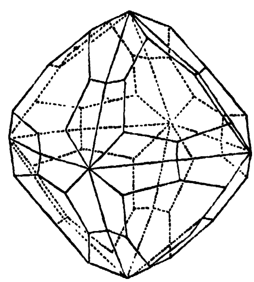


Fig. 134.

but so that faces in the octants containing one form, or set of forms, invariably cover and obliterate the faces lying in the adjacent octants; only a small fringe, as it were, of the latter remaining unobliterated and forming the narrow margin of the re-entrant furrow. If it be asked where corroborative evidence is to be found in favour of the existence of these otherwise unseen semiforms, the answer is, that there are crystals—few indeed in number but indubitable in their character—which are actual hexakistetrahedra, sometimes with, sometimes without, the faces of the octahedron belonging to the same octants. And these semiforms are furthermore shallower than the usual forms on the crystals of hexakisoctahedral aspect, and are never channelled by re-entrant edges. Of such crystals, four have been described, and one figured, Fig. 114, Article **189**, from the Collection at the

British Museum. And to these are to be added, as instances of hemihedral developement, from the same collection, one crystal with forms $\{111\}$ and $\{hkl\}$, showing salient edges and faces of the octahedron less developed in alternate octants; another crystal similarly developed but with re-entrant edges, and a third similar to the second but with one of the octahedral faces in an alternate octant comparatively large, indicating a transition to the furrowed octahedron.

So far, then, the explanation seems complete which treats the channelled crystals as twins on a $\{110\}$ face, of a crystal presenting forms $\sigma\{111\}$, $\sigma\{hkl\}$ (belonging to the *o* octants), with $\sigma\{\bar{1}11\}$ and $\sigma\{\bar{h}'k'l'\}$ forms (belonging to the ω octants) but almost obliterated by the former.

It is, nevertheless, difficult to accept this explanation. The re-entrant edges are peculiar to certain forms. The hexakisoctahedron is a frequent form in the diamond, but its edges would seem only to be furrowed when it is united with the faces of the octahedron. The cube and the rhomb-dodecahedron exist as distinct forms; but the edges of the cube are not furrowed, and the dodecahedron is entirely regular in aspect.

Out of 250 crystals in the British Museum there are eight simple octahedra with sharp edges that are entirely holo-symmetric in aspect and present none of the peculiar furrows that have been alluded to. Four of the crystals are cubes and thirteen crystals carry cube with cube-pyramidion forms, but of these none shows a re-entrant edge. There are ten with combined forms of octahedron and rhomb-dodecahedron or only the latter form that are without re-entrant edges, but with evident lamellar structure. Of the octahedrid pyramidion, or this form combined with the octahedron, there are nine crystals with evident lamellar structure; six being twinned on an octahedral face.

Of crystals in which the octahedron is combined with the hexakisoctahedron, there are twenty-two that present no re-entrant edges, seven that show undoubted lamellar structure in a step-formed layer of material deposited on the octahedral faces; and twenty-three such crystals show the same in a less obvious way. And of crystals with these forms that show re-entrant edges in the way described there are sixteen in the collection.

On the other hand, that the diamond, viewed as a holosymmetrical crystal, is capable of undergoing the ordinary kind of twinning is seen in the many examples of it that occur in the form of the spinel twin. The British Museum collection contains six crystals with forms $\{111\}$ $\{hhl\}$, and ten of the hexakis-octahedron thus twinned, but without any of the facetted channellings, though many of them are distinctly lamellar in structure. One large twin octahedral crystal from the Cape has all the faces of a peculiar dull aspect, but the faces of different octants are absolutely indistinguishable from each other. It is of course to be recognised that a collection of the kind referred to does not represent the average character of the crystals of a mineral, but consists for the most part of crystals selected on account of the perfection of their crystallisation or of peculiarities which they illustrate.

In all the many crystals thus seemingly holo-symmetrical, if we except the three crystals above alluded to, we look in vain for any difference in the faces belonging to adjacent octants, such as ought not unfrequently to manifest itself if these were faces belonging to correlative tetrahedrid forms. Of lamellar structure, and successive step-formed deposition of the laminæ that build up the crystal, there is abundant, almost universal, evidence. The question is, whether the furrowed re-entrant edges that seem peculiar to crystals composed of the octahedron, or octahedron and forty-eight-scalenohedron combined, may not be looked on as results of a lamellar structure in which the facetting of the sides of the furrows has been an incident of the manner in which the later-formed laminæ of the crystal have been deposited.

196. Diplohedral crystals. The examples of diplohedral crystals twinned on a face of the rhomb-dodecahedron have to be looked for in the case of the minerals of the pyritoid group. As regards the twinning round such an axis, the normal of a Σ-plane, it will be seen that the poles of either of the correlative semi-forms of a diplohedron or a pentagon-dodecahedron will, by the process, fall into the positions of the poles of the other correlative semiform. But with the relative disposition of the planes themselves this is not the case. In fact, the result in this, as in every case of twinning on a [110] axis, is an interpenetrant combination. An example of such a twin is illustrated in Fig. 135, which represents a crystal of pyrites with the forms $\pi\{100\}$, $\pi\{111\}$,

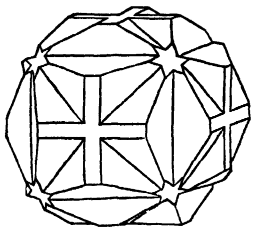


Fig. 135.

197. Tetartosymmetrical forms. Crystals of Sodium chlorate, presenting the features of supplementary twins, have been described by Groth. They show only the faces of the form $\{332\}$. Fig. 136 represents a simple crystal with the faces of the form $\sigma\{332\}$, Fig. 137 the crystal described by Groth as a twin: and the two interpenetrant individuals are found by him to

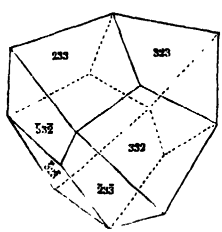


Fig. 136.

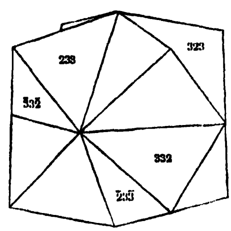


Fig. 137.

present opposite characters as regards rotatory influence on the plane of polarisation, and are on that account treated as tetartosymmetrical with the symbol $\sigma\pi\{332\}$. The two quarter-forms will be enantiomorphous if of opposite, tautomorphous if of the same rotatory character. But if light incident on the crystal is plane polarised, the particles of luminiferous ether within the crystal will at any instant be arranged in either a right-handed or left-handed helix according as the individual possesses right or left rotatory polarisation. A right-handed helix is repeated over a diagonal axis of symmetry as a right-handed helix, whereas it is repeated antistrophically in a left-handed helix if it be reflected on a plane of symmetry. Two quarter-forms of opposite rotatory character cannot therefore be twins by rotation through 180° round a twin-axis. And it may be reasonably assumed that this difference as regards congruence in physical deportment corresponds to a congruence or incongruence in the molecules themselves.

In individual crystals of Sodium chlorate, the forms associated with the right-handed rotatory action are such as would be extant if we look on the tetartohedra $\sigma\pi\{hkl\}$ and $\sigma\pi\{\bar{h}\bar{k}\bar{l}\}$ as dextro-rotatory, and $\sigma\pi\{khl\}$ and $\sigma\pi\{\bar{k}\bar{h}\bar{l}\}$ as lævo-rotatory. Consequently the association of $\sigma\pi\{332\}$ with $\sigma\pi\{111\}$ and $\sigma\pi\{210\}$ is dextro-rotatory; so is that of $\sigma\pi\{\bar{3}\bar{3}\bar{2}\}$ with $\sigma\pi\{\bar{1}\bar{1}\bar{1}\}$ and $\sigma\pi\{\bar{2}\bar{1}0\}$. The lævo-rotatory combinations would be $\sigma\pi\{111\}$ with $\sigma\pi\{332\}$ and $\sigma\pi\{120\}$; and $\sigma\pi\{\bar{1}\bar{1}\bar{1}\}$ with $\sigma\pi\{\bar{3}\bar{3}\bar{2}\}$ and $\sigma\pi\{\bar{1}\bar{2}0\}$.

The question here arises, as to such enantiomorphous combinations, whether they are to be designated as twins or not; that is to say, how far twins following the ordinary law but formed by two dextro-rotatory or else by two lævo-rotatory quarter-forms are to be included under the same definition with combinations in which the two individuals have opposite rotatory characters. The latter being enantiomorphous cannot be brought into a position of parallelism; nor, indeed, can they be strictly described as identical molecular structures. The two kinds of structure cannot, in fact, be included under the same definition; and two alternatives offer themselves: either we may class the tautomorphous combinations in a separate category as parallel growths and define the enantiomorphous combinations as twins formed by repetition on a systematic plane or planes—in this case the Σ-systematic planes—with abeyant symmetry; or, on the other hand, we may adhere to the definition of twins as structures formed round an axis or axes of diagonal symmetry—in this case the normals of the rhomb-dodecahedron

—*and refer the enantiomorphous combinations to the class of parallel growths having similar systematic lines or planes in parallelism but with the two antistrophic kinds of molecules forming separate portions of the crystalline aggregate.*

The definition on which the former view is founded, viz. *that a twin is a structure symmetrical to a plane parallel to a possible face which is not a plane of symmetry for the separate individuals*, fails in completeness, since the growths of many holo-symmetrical, and with few exceptions of all haplohedral, crystals undistinguishable from twins have to be excluded from it, as for instance is the case with the twins on a zone-axis in the Anorthic system that will be hereafter considered.

On the other hand, it is not possible to include all the structures that have been designated as twins, under a definition founded on the law of a half-rotation round a twin-axis, even if such structurally enantiomorphous combinations as those just considered be excluded.

As a fact, with a single exception, all the other varieties of twin-structure at present known and described as such can be completely and in every case would seem to be more simply explained as rotation-twins. But the single exception of a particular growth of crystals of the tetragonal mineral copper-pyrites precludes this definition from possessing universality of application. That growth, and certain other cases not so difficult of explanation by this definition, will be considered under the systems to which they belong.

The case of interpenetrant twin-structure is not different in kind from that of juxtaposition; there has been an intergrowth of each crystal with the other, involving a mutual overlapping of their material as the crystal has grown from the original nucleus of departure for the developement of the separate individuals.

Section II.—The Tetragonal System.

A.—*Holo-symmetrical Forms.*

198. This system is characterised by a morphological axis which is the zone-axis of two distinct pairs of systematic planes, and is perpendicular also to a systematic plane. The planes of each pair are perpendicular to each other and inclined at 45° on those of the other pair; they constitute, the one pair, the proto-systematic planes S, the other pair the deutero-systematic planes Σ. The equatorial plane, the zone-plane $[S\Sigma]$, is the trito-systematic plane C (Fig. 138).

The points in which the normals of these planes meet the sphere of projection are the angular points of the systematic triangles; a systematic triangle being composed of two sides S and Σ, which are quadrants, and a third side C, having the value $\frac{\pi}{4}$; the angles of the triangle being $s = \sigma = 90^\circ$ and $c = 45^\circ$.

The morphological axis cc' and the normals ss', ss' are taken for the crystallographic axes, and a plane with its pole on an arc Σ of a systematic triangle for the parametral plane. The axial system is therefore represented by the expressions

$$\xi = \eta = \zeta = 90^\circ,$$
$$a = b \gtrless c,$$

and offers only a single variable element, namely, the parametral ratio $\frac{a}{c}$. Of the sixteen systematic triangles which cover the sphere two will lie in each octant.

The distribution of the poles of the various forms under this type of symmetry has been discussed in Chapter V. p. 129. It remains however to consider the nature of the forms themselves, and also of the semiforms which the system admits of under the law of mero-symmetry.

The seven holo-symmetrical forms of the system will be designated as—

1. $\{hkl\}$, the scalene dioctahedron, or ditetragonal scaleno-hedron. Its poles lie on no arc of a systematic triangle.

2. $\{hk0\}$, the ditetragonal prism. Its poles lie on the arcs C.

3. $\{h0l\}$, the isosceles proto-octahedron, or the S- or axial isosceles octahedron. Its poles lie on the arcs S of the systematic triangles.

4. $\{100\}$, the tetragonal proto-prism, or the S- or axial square prism. Its poles lie at the axial points in which the arcs S and C intersect.

5. $\{hhl\}$, the isosceles deutero-octahedron, or the Σ- or diagonal isosceles octahedron. Its poles lie on the arcs Σ of the systematic triangles.

6. $\{110\}$, the tetragonal deutero-prism, or the Σ- or diagonal square prism. Its poles lie at the intersections of the arcs Σ and C.

7. $\{001\}$, the tetragonal (basal or C-) pinakoid. Its poles are the axial points in which the arcs S and Σ intersect.

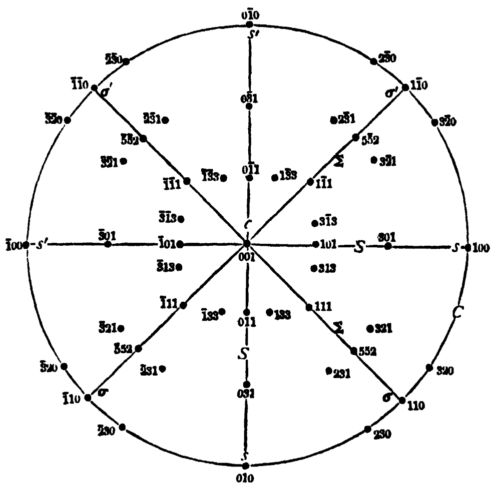


Fig. 138.

The relative positions of the poles belonging to a form of each kind are exhibited in the projection, Fig. 138, of a crystal in which the parametral ratio $\frac{c}{a} = 0{\cdot}6724$ (that of cassiterite), and consequently the angle $101, 001 = 33^\circ 55'$: the particular forms are $\{100\}$, $\{320\}$, $\{110\}$, $\{001\}$, $\{101\}$, $\{301\}$, $\{111\}$, $\{552\}$, $\{313\}$, and $\{321\}$.

199. 1. *The scalene dioctahedron* $\{hkl\}$, Fig. 139.

The sixteen poles of this form are contained in

TABLE E.

μ	hkl	$\bar{h}\bar{k}l$	$\bar{h}k\bar{l}$	$h\bar{k}\bar{l}$	$\bar{h}kl$	$h\bar{k}l$	$hk\bar{l}$	$\bar{h}\bar{k}\bar{l}$	a'
a	khl	$\bar{k}\bar{h}l$	$\bar{k}h\bar{l}$	$k\bar{h}\bar{l}$	$\bar{k}hl$	$k\bar{h}l$	$kh\bar{l}$	$\bar{k}\bar{h}\bar{l}$	μ'

They lie each within a systematic triangle, and unsymmetrically therein, that is to say, on none of the great circles bisecting the angles; hence the faces of this triangle must be scalene triangles, and the edges that are the sides of these triangles will lie in the systematic planes, and will be of three kinds.

There will thus be eight similar edges S lying in the proto-systematic planes S; eight similar edges Σ lying in the deutero-systematic planes Σ; and eight similar basal edges C lying in the trito-systematic plane C.

The pyramidal edges S and Σ will meet in two quoins c, ditetragonal in symmetry on the morphological axis, forming the vertices of the figure; and there will be four quoins s, in each of which an adjacent pair of C-edges will meet two S-edges symmetrical on the plane C, and there will also be four quoins σ in each of which two Σ-edges meet other two adjacent edges C. The quoins of these two latter groups are ortho-symmetrically grouped on their axes ss' and $\sigma\sigma'$ respectively.

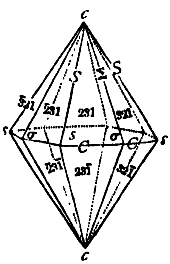


Fig. 139.

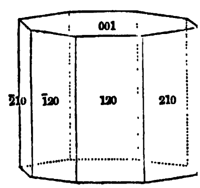


Fig. 140.

A section of the tetragonal scalenohedron perpendicular to the morphological axis at any point is a symmetrical (never a regular) octagon.

In Fig. 139 is represented the scalene dioctahedron $\{321\}$ corresponding to the above parametral ratio, and in Fig. 138 are shown the positions of its poles on the sphere of projection.

200. 2. *The ditetragonal prism* $\{hk0\}$, Fig. 140.

If in the symbols of a series of dioctahedra the values of h or k do not alter, while that of l varies, all sections perpendicular to the tetragonal axis will be similar symmetrical, but not regular, octagons; if the index l becomes zero, the S and Σ edges of the figures become parallel to the vertical axis, and the form is that of a ditetragonal prism, the poles of which lie on the zone-circle C.

Such a prism not being a closed figure can have no quoins and can only exist in combination; as, for instance, with the pinakoid $\{001\}$, with which it forms quoins of two kinds (Fig. 140). Its faces are all similar, but the two edges of each face are dissimilar, since adjacent faces are antistrophic to each other. Four similar edges lie in the proto-systematic planes S, and the four similar edges alternating with them lie in the deutero-systematic planes Σ. The faces are euthy-symmetrically divided by the trace on them of the trito-systematic plane C. The symbols of the faces are—

$$hk0 \quad \bar{h}k0 \quad h\bar{k}0 \quad \bar{h}\bar{k}0$$
$$kh0 \quad \bar{k}h0 \quad k\bar{h}0 \quad \bar{k}\bar{h}0.$$

201. *The isosceles octahedra.* The terms proto- and deutero-pyramid have been applied by various writers somewhat ambiguously to the diplo-pyramidal figures, or, in crystallographic language, pyramids, which have been here termed isosceles octahedra. It would seem more appropriate to apply, as above suggested, the term *proto-octahedron* to the form the poles of which lie on the proto-systematic zone-circles, and the term proto-prism to the figure of which the faces are parallel and perpendicular to the proto-systematic planes. But Naumann applied these terms, respectively, to the octahedron $\{hhl\}$ which has its poles on the deutero-systematic zone-circles, and to the prism $\{110\}$ which has its faces parallel to the deutero-systematic planes.

To avoid ambiguity we may adopt a nomenclature which recognises the relative positions of the two classes of forms on the crystal, while recalling also their distinctive character as compared with the analogous forms of the Cubic and Ortho-rhombic systems. The former or proto-systematic forms may be designated the *axial*, and the latter forms the *diagonal* isosceles octahedron and square prism; or for brevity they may be distinguished as the S- and Σ-octahedron and the S- and Σ-prism.

The square prisms, whether axial or diagonal, will evidently be the limiting forms where the l index in a symbol $h0l$ or in a symbol hhl becomes zero by the merging into a single pole of the poles $h0l$ and $h0\bar{l}$ in the one case, and of the poles hhl and $hh\bar{l}$ in the other case, the resulting forms having the symbols {100} and {110}. If, on the other hand, the h index becomes zero, the symbols $h0l$ and hhl alike become 001, and represent a face of the basal pinakoid.

202. 3. *The axial isosceles octahedron*, or *S-octahedron*, Figs. 141–2, is the form $\{h0l\}$, of which the poles lie on the systematic great circles S (Fig. 138), and the edges in the deutero-systematic planes Σ and in the trito-systematic plane C. Fig. 141 represents the form {301} corresponding to the assumed parametral ratio.

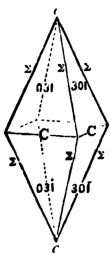


Fig. 141.

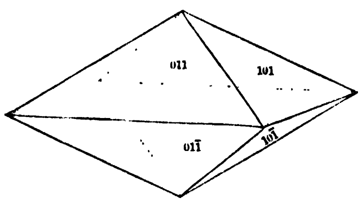


Fig. 142.

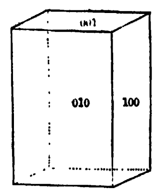


Fig. 143.

The faces are euthysymmetrically divided by the trace on them of the systematic planes S, and therefore are metastrophic similar isosceles triangles bounded by sides which are edges of two sorts, namely, eight edges Σ culminating in two quoins c, tetragonally symmetrical on the Z-axis, and four edges C lying in the equatorial plane and meeting in four quoins σ orthosymmetrical on the systematic axes $[\Sigma C]$.

The symbols of the faces of the form are

$$h0l \quad \bar{h}0l \quad h0\bar{l} \quad \bar{h}0\bar{l}$$

$$0hl \quad 0\bar{h}l \quad 0h\bar{l} \quad 0\bar{h}\bar{l}.$$

The form {101} is parametral (Fig. 142), giving the value of the one variable element of the system the parametral ratio $\frac{a}{c}$.

203. 4. *The tetragonal axial prism, axial square prism, or the S-prism* $\{100\}$, Fig. 143.

This prism is formed of the four faces parallel to the proto-systematic planes. They build a right square prism the faces of which have the symbols 100, 010, $\bar{1}00$, $0\bar{1}0$; it can only be present in combination, as in Fig. 143.

204. 5. *The diagonal isosceles octahedron* or *Σ-octahedron* $\{hhl\}$, Figs. 144–5.

The poles of the diagonal octahedron lie on the deutero-systematic great circles Σ and the edges of the form in the proto-systematic and trito-systematic planes *S* and *C*. It is an analogous form to that of the axial octahedron, but the quoins lying in the equatorial plane are ortho-symmetrical on the axes [*CS*]. Of the two kinds of octahedra however it is to be noted that an axial and a diagonal

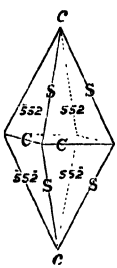


Fig. 144.

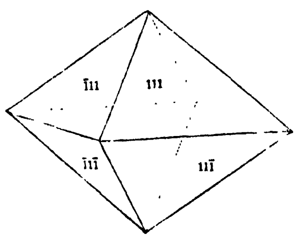


Fig. 145.

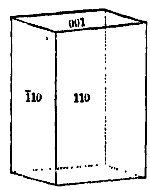


Fig. 146.

octahedron can in no case be congruent: were they so, adjacent faces of the two forms would be symmetrical to a new plane of symmetry bisecting the angle formed by two planes *S* and Σ, a condition incompatible with the principles of crystalloid symmetry.

Indeed it is readily shewn that the symbol of a diagonal octahedron $\{h'h'l'\}$, the poles of which are equidistant with those of a form $\{mon\}$ from the poles $\{001\}$ and $\{00\bar{1}\}$, will be $\{h, h, \sqrt{2}l\}$, where h and l are zero or integral; this is only possible as a form of the system in the former case, namely, where h is zero and the form is the pinakoid $\{001\}$, or where l is zero and the form is the diagonal prism $\{110\}$.

The symbols of the diagonal octahedron are

$$hhl \quad \bar{h}\bar{h}l \quad \bar{h}h\bar{l} \quad h\bar{h}\bar{l} \quad \bar{h}hl \quad h\bar{h}l \quad hh\bar{l} \quad \bar{h}\bar{h}\bar{l}.$$

The form {552} corresponding to the above parametral ratio is shewn in Fig. 144, and the arrangement of its poles in Fig. 138. Where the indices *h* and *l* are equal the form becomes the parametral octahedron {111} (Fig. 145).

205. 6. *The diagonal square prism* or *Σ-prism* {110}, Fig. 146. The character of the faces of the diagonal square prism is identical with that of the axial prism, except that, as its poles lie on the great circles Σ, the edges of the form lie in the proto-systematic planes, and the faces of the form are ortho-symmetrical on the zone-lines [Σ *C*]. The symbols of the faces of the Σ-prism are

$$110 \quad \bar{1}10 \quad 1\bar{1}0 \quad \bar{1}\bar{1}0.$$

Fig. 146 shews the prism {110} in combination with the form {001}.

206. 7. *The tetragonal pinakoid* {001} comprises only the two faces 001 and 00$\bar{1}$ parallel to the trito-systematic plane *C*.

It can only exist in combination with other forms, as in Figs. 140, 143, and 146, and must always, where the crystal is holo-symmetrical, present tetragonal or ditetragonal symmetry in regard to its normal.

Tetragonal System. B.—*Mero-symmetrical Forms.*

207. In the Tetragonal system the law of mero-symmetry may take effect (1) in the abeyance of symmetral character in either of the dual groups of systematic planes *S* or Σ; or, (2) in that of the equatorial systematic plane *C*, resulting in hemimorphous forms; or again (3) the symmetral character of all these systematic planes may be dormant, giving rise to asymmetric haplohedral semiforms of which the faces are grouped round a tetragonal (in lieu of a ditetragonal) axis and round diagonal (in lieu of orthogonal) axes of symmetry. And again (4) where the trito-systematic plane *C* retains its symmetral character, mero-symmetry requires that no other systematic planes shall be planes of symmetry.

Since, in the Tetragonal system diplohedrism results alike from centro-symmetry and from an actual symmetral character in the equatorial plane *C*, it follows that holo-systematic forms may result from the first three of the above mero-symmetrical conditions, while hemi-systematic semiforms only take their rise from the last condition.

The mero-symmetrically distributed poles of a scalene dioctahedron are those of—

I. Eight faces corresponding to the eight normals: holo-systematic haplohedral forms; holotetragonal hemihedra.

II. Eight faces corresponding to four of the normals: hemi-systematic diplohedral forms; hemitetragonal diplohedra.

III. Four faces corresponding to four of the normals: the tetartohedral case, namely that of hemi-systematic haplohedral forms; tetragonal tetartohedra.

208. The following are the mero-symmetrical forms of the scalene dioctahedron and isosceles octahedra.

I. Holo-systematic haplohedral forms; holotetragonal hemihedra:—

1. *Asymmetric forms;*

$a\{hkl\}$ and $a\{khl\}$, the tetragonal trapezohedron.

2. *Tetrahedrid or sphenoidal forms;*

(*a*) $s\{hkl\}$ and $s\{khl\}$, the tetragonal proto- or *S*-disphenoid.

(*b*) $s\{h0l\}$ and $s\{0hl\}$, the tetragonal axial or *S*-sphenoid.

(*c*) $s\{101\}$ and $s\{011\}$, the parametral axial sphenoid.

(*d*) $\sigma\{hkl\}$ and $\sigma\{\bar{h}\bar{k}\bar{l}\}$, the tetragonal deutero- or Σ-disphenoid.

(*e*) $\sigma\{hhl\}$ and $\{\bar{h}\bar{h}\bar{l}\}$, the deutero- or Σ-sphenoid.

(*f*) $\sigma\{111\}$ and $\sigma\{\bar{1}\bar{1}\bar{1}\}$, the parametral deutero-sphenoid.

(*g*) $\rho\{hkl\}$ and $\rho\{\bar{h}\bar{k}\bar{l}\}$, the hemimorphous dioctahedron.

II. Hemi-systematic diplohedral forms; hemitetragonal diplohedra:—

$\phi\{hkl\}$ and $\phi\{khl\}$, the tetragonal hemi-dioctahedron.

III. Hemi-systematic haplohedral forms; tetragonal tetartohedra:—

1. $s\sigma\{hkl\}$, the hemidisphenoid.
2. $a\rho\{hkl\}$, the hemimorphous scalenohedron.

209. I. Holosystematic haplohedral forms; holotetragonal semiforms.

1. *The asymmetric semiform. The tetragonal trapezohedron* $a\{hkl\}$ or $a\{khl\}$, Fig. 147 (*a*), (*c*).

The defalcation of alternate planes round each of the axes of symmetry of the system involves the abeyance of the symmetral character of all the systematic planes, and is only possible in the case of the general form, the ditetragonal scalenohedron $\{hkl\}$. The morphological axis $[S\Sigma]$ from being an axis of ditetragonal

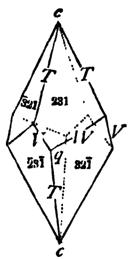


Fig. 147 (*a*).

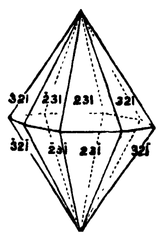


Fig. 147 (*b*).

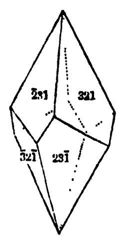


Fig. 147 (*c*).

symmetry becomes an axis of tetragonal symmetry only, and the four ortho-symmetral axes $[s]$ and $[\sigma]$ become axes of diagonal symmetry.

The symbols of the faces of the semiform $a\{hkl\}$ are those lying in the blocks μ and μ' of Table E, Art. **199**, viz.

$$hkl \quad \bar{h}\bar{k}l \quad \bar{h}k\bar{l} \quad h\bar{k}\bar{l}$$

$$\bar{k}hl \quad k\bar{h}l \quad kh\bar{l} \quad \bar{k}\bar{h}\bar{l};$$

those of the semiform $a\{khl\}$ are the remaining symbols of the Table.

The eight faces of an asymmetric semiform (Figs. 147 *a* and *c*) will be quadrilateral figures formed each by two of the similar edges T that meet on the morphological axis, and two dissimilar edges V and W which like the former lie in no systematic plane. The edges V are traversed at their points of bisection by the crystallographic axes X and Y, and the edges W are similarly traversed by the secondary axes normal to the deutero-systematic planes Σ. There are two tetragonal quoins c on the axis Z formed by the edges T, and eight similar three-faced quoins q formed by the meeting of an edge T with an edge V or an edge W.

The faces of either form are, by the principle of the construction, metastrophic trapezoids, and those of the correlative form are antistrophic to them, so that the two figures are enantiomorphous.

210. 2. *The tetrahedrid or sphenoidal mero-symmetry.*

The eight normals of the tetragonal scalenohedron may carry the eight faces of a semiform in other ways than asymmetrically as regards the systematic planes. Thus the faces may be extant or absent in alternate pairs, symmetrical either on the proto- or on the deutero-systematic planes: and, in accordance with the notation adopted in Art. **148**, p. 167, these will be denoted as $s\{hkl\}$ if the proto-systematic planes S are the planes of symmetry, and as $\sigma\{hkl\}$ where the planes of symmetry are the deutero-systematic planes Σ. In neither case is the form symmetrical to the trito-systematic plane or to a centre.

The axis $[S\Sigma]$, which is an axis of ditetragonal symmetry in the holo-symmetrical forms, is here one of ortho-symmetry; the ortho-symmetrical axes σ or $[\Sigma C]$ become diagonal axes, and the axes s or $[SC]$ lose their symmetral character. These forms are termed sphenoidal, from their wedge-like contour.

211. (*a*) *The tetragonal proto-* or *axial* or *S-disphenoid* $s\{hkl\}$ or $s\{khl\}$, Figs. 148 (*a*), (*c*).

The symbols for the faces of the first semiform are

$$hkl \quad h\bar{k}l \quad \bar{h}kl \quad \bar{h}\bar{k}l$$
$$kh\bar{l} \quad k\bar{h}\bar{l} \quad \bar{k}h\bar{l} \quad \bar{k}\bar{h}\bar{l},$$

those of the semiform $\{khl\}$ are

$$khl \quad k\bar{h}l \quad \bar{k}hl \quad \bar{k}\bar{h}l$$
$$hk\bar{l} \quad h\bar{k}\bar{l} \quad \bar{h}k\bar{l} \quad \bar{h}\bar{k}\bar{l}.$$

The eight faces bounding either semiform (Figs. 148 *a* and *c*) will be triangles and will have two sorts of edges, S and S', lying in the systematic planes S, and grouped ortho-symmetrically on the axis of form. The remaining four edges U are bisected by the diagonal zone-axes $[C\Sigma]$ and lie in zigzag across the equatorial systematic plane.

The quoins are of two kinds; two ortho-symmetrical quoins formed by the meeting of the edges S and S' on the Z-axis, and four lateral four-faced quoins in which two of the edges U meet an edge S and an edge S'.

The two correlative semiforms are tautomorphous.

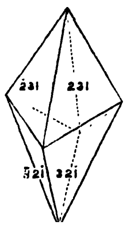


Fig. 148 (*a*).

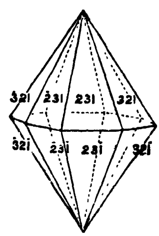


Fig. 148 (*b*).

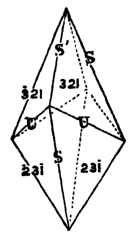


Fig. 148 (*c*).

(*b*) *The tetragonal axial* or *S-sphenoid*, $s\{hol\}$ or $s\{ohl\}$, Figs. 149 (*a*), (*c*), 150 (*a*), (*c*).

The faces of the disphenoid being symmetrical in pairs on the planes S, it is evident that a semiform of which the poles lie on the great circles S will accord with the same condition of symmetry. Such a semiform is the tetragonal axial sphenoid $s\{hol\}$, the faces of which are represented by the symbols

$$hol \quad \bar{h}ol \quad oh\bar{l} \quad o\bar{h}\bar{l},$$

the correlative S-sphenoid being composed of the faces

$$ohl \quad o\bar{h}l \quad ho\bar{l} \quad \bar{h}o\bar{l}.$$

The sphenoids resemble in general features the tetrahedron of the Tesseral system, and have a double-wedge-like aspect; see Figs. 149 *a* and *c*. The four faces are isosceles triangles (whereas those of the tesseral tetrahedron are equilateral), and the two bases of the triangles are edges perpendicular in their directions, being respectively parallel to the X and Y axes. These crystallographic axes pass through the faces of the form but are not axes of symmetry, while the axes $[\Sigma C]$, no longer orthosymmetral, retain diagonal symmetry and bisect the four equal edges. The morphological axis becomes only an axis of ortho-symmetry, traversing the points of bisection of the basal edges. The four quoins are

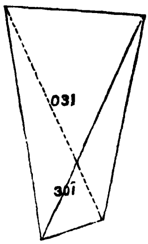


Fig. 149 (*a*).

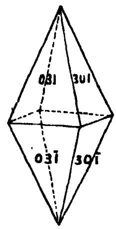


Fig. 149 (*b*).

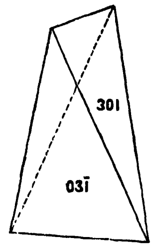


Fig. 149 (*c*).

similar and three-faced, each being formed by the meeting of two of the four equal edges with one of the basal edges.

(*c*) When the indices *h* and *l* are equal, the semiforms are the parametral axial sphenoids shewn in Figs. 150 (*a*) and (*c*).

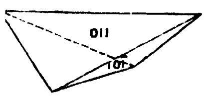


Fig. 150 (*a*).

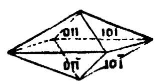


Fig. 150 (*b*).

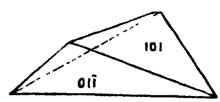


Fig. 150 (*c*).

212. (*d*) *The deutero-* or *diagonal* or *Σ-disphenoid* $\sigma\{hkl\}$ or $\sigma\{\bar{h}\bar{k}\bar{l}\}$, Figs. 151 (*a*), (*c*).

The Σ- or diagonal disphenoid is *mutatis mutandis* identical

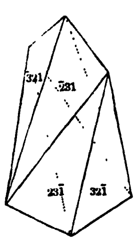


Fig. 151 (*a*).

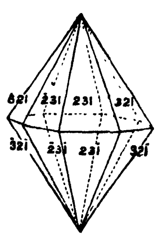


Fig. 151 (*b*).

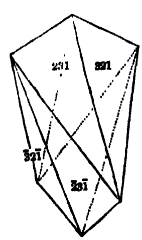


Fig. 151 (*c*).

in its general features with the S-disphenoid: but its pyramidal edges lie in the Σ-planes which are the planes of symmetry in place of the S-planes, and the zigzag edges are bisected by the axes X and Y. The symbols of the faces of the Σ-disphenoid $\{hkl\}$, corresponding to the left half of Table E, Art. **199**, are

$$hkl \quad \bar{h}\bar{k}l \quad \bar{h}k\bar{l} \quad h\bar{k}\bar{l}$$

$$khl \quad \bar{k}\bar{h}l \quad \bar{k}h\bar{l} \quad k\bar{h}\bar{l}.$$

Those of the correlative semiform $\sigma\{\bar{h}\bar{k}\bar{l}\}$ are contained in the right half of Table E, viz.

$$\bar{h}kl \quad h\bar{k}l \quad hk\bar{l} \quad \bar{h}\bar{k}\bar{l}$$

$$\bar{k}hl \quad k\bar{h}l \quad kh\bar{l} \quad \bar{k}\bar{h}\bar{l}.$$

(*e*) *The tetragonal diagonal sphenoid or Σ-sphenoid*, $\sigma\{hhl\}$ or $\sigma\{\bar{h}\bar{h}\bar{l}\}$, Figs. 152 (*a*), (*c*), 153 (*a*), (*c*), presents similar features to the S-sphenoid, but its poles lie on the great circles Σ which

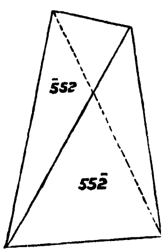


Fig. 152 (*a*).

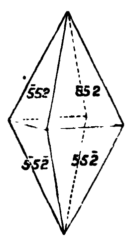


Fig. 152 (*b*).

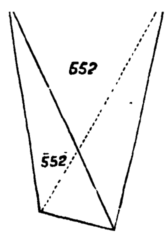


Fig. 152 (*c*).

retain their symmetral character, while that of the proto-systematic planes S is in abeyance. The two wedge-shaped basal edges are parallel to the zone-lines $[C\Sigma]$.

The symbols of the faces of the form $\sigma\{hhl\}$ are

$$hhl \quad \bar{h}\bar{h}l \quad \bar{h}h\bar{l} \quad h\bar{h}\bar{l};$$

those of the form $\sigma\{\bar{h}hl\}$ are

$$\bar{h}hl \quad h\bar{h}l \quad hh\bar{l} \quad \bar{h}\bar{h}\bar{l}.$$

The crystallographic axes penetrate and bisect the edges of the form. The morphological axis, as in the S-sphenoid and disphenoid, becomes an axis of ortho-symmetry, the $[s]$ or $[SC]$

axes become axes of diagonal symmetry, but the symmetral character of the [σ] axes is in abeyance.

(*f*) When h equals l, the figures are the parametral deutero-sphenoids shewn in Figs. 153 (*a*), (*c*).

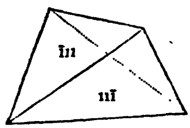


Fig. 153 (*a*).

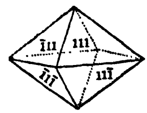


Fig. 153 (*b*).

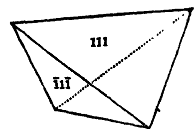


Fig. 153 (*c*).

Since proto- and deutero-sphenoidal forms cannot concur on a crystal, and it is at the option of the crystallographer to take either pair of similar axes as crystallographic axes, the analogy with tetrahedrid forms in the Cubic system points to the selection of an axial system such that, when sphenoidal semiforms occur on a crystal, they are treated as deutero-sphenoidal: and in usage therefore they are termed simply sphenoidal semiforms.

213. (*g*) Another variety of holotetragonal mero-symmetry is presented in the case of the *hemimorphous dioctahedron* $\rho\,\{hkl\}$ wherein the systematic plane C fails of being symmetral.

And since this kind of hemi-symmetry permits of the other systematic planes being symmetral, we may have hemimorphous forms in which the indices h and k are equal, or one of them is zero, and of which therefore the poles lie on one of the great circles S or Σ, or on both. Such semiforms comprise those with the symbols $\rho\,\{hhl\}$, $\rho\,\{111\}$, $\rho\,\{h0l\}$, $\rho\,\{101\}$, $\rho\,\{001\}$.

None of these hemimorphous pyramids or prisms can exist except in association with other forms.

214. II. Hemisystematic diplohedral forms; hemi-tetragonal diplohedra. *The tetragonal isosceles octahedron* or *hemidioctahedron* $\phi\,\{hkl\}$ or $\phi\,\{khl\}$, Figs. 154 (*a*), (*c*).

In a hemi-systematic form of the Tetragonal system four normals are absent, and the extant four are diplohedral: a distribution which can only occur in one manner, so that there is only one type of hemi-systematic diplohedral form. Of the faces, as grouped round the morphological axis, only alternate faces are extant, but the adjacent faces of the form symmetrical on the trito-systematic plane *C* are concurrent with them, since these are the faces parallel to the former and thus have normals in common with them. The only plane of symmetry is the trito-systematic plane *C*, the symmetral character of the *S* and Σ planes being in abeyance; and the form is obviously centro-symmetrical. The symbols of the correlative semiforms are, of

$$\phi\,\{hkl\},\quad hkl \quad \bar{k}hl \quad \bar{h}\bar{k}l \quad k\bar{h}l,$$
$$\bar{h}\bar{k}\bar{l} \quad k\bar{h}\bar{l} \quad hk\bar{l} \quad \bar{k}h\bar{l};$$

and of $\phi\,\{khl\}$,

$$khl \quad \bar{h}kl \quad \bar{k}\bar{h}l \quad h\bar{k}l,$$
$$\bar{k}\bar{h}\bar{l} \quad h\bar{k}\bar{l} \quad kh\bar{l} \quad \bar{h}k\bar{l}.$$

The form has the character of a tetragonal octahedron, Figs. 154 (*a*), (*c*), its section perpendicular to the *Z*-axis being a square, the sides

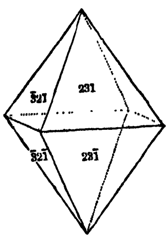


Fig. 154 (*a*).

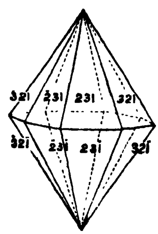


Fig. 154 (*b*).

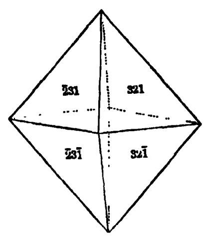


Fig. 154 (*c*).

of which are however not parallel to those of the axial or diagonal forms; the pyramidal edges are gyroidally grouped in tetragonal symmetry round the *Z*-axis and are similar; so again are the basal edges. Hence the faces have the character of metastrophic isosceles triangles. The quoins are of two kinds, two gyroidal tetragonal quoins symmetrical on the morphological axis, and four similar four-faced quoins geometrically but not crystallographically ortho-symmetrical on two lines passing through the origin.

215. *The hemidiprism* $\phi\,\{hk0\}$ or $\phi\,\{kh0\}$, Figs. 155 (*a*), (*c*).

The alternate faces of the ditetragonal prism (Fig. 155 *b*) produce a prism with a square section which is the limiting form of a series of hemidioctahedra having the same ratio for $\frac{h}{k}$; they are tetragonally instead of, as in the diprism, ditetragonally disposed round the morphological axis.

The faces of the form $\phi\,\{hk0\}$ (Fig. 155 *c*) are,

$$hk0 \quad \bar{k}h0 \quad \bar{h}\bar{k}0 \quad k\bar{h}0,$$

and of the form $\phi\,\{kh0\}$ (Fig. 155 *a*),

$$kh0 \quad \bar{h}k0 \quad \bar{k}\bar{h}0 \quad h\bar{k}0.$$

Their edges are similar, but symmetrical to no plane except the trito-systematic plane to which the faces are euthysymmetrical. And it will be seen that the sections of the forms $\phi\,\{hkl\}$ and $\phi\,\{hk0\}$ perpendicular to the morphological axis do not accord in the character of their angles with a crystallographic square.

The association of the hemidioctahedron and hemidiprism with holo-symmetrical forms imparts also to the latter a quasi-mero-

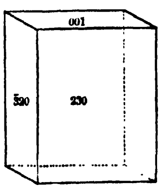


Fig. 155 (*a*).

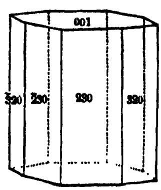


Fig. 155 (*b*).

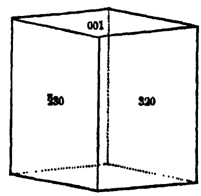


Fig. 155 (*c*).

symmetral character, since they thereby lose the special symmetry which characterises their edges; and, as in the analogous case in the Cubic system, it cannot be doubted that this is only the external exponent of an interruption in the symmetry of the physical characters throughout the crystal.

In this sense such forms as $\phi\,\{hhl\}$, $\phi\,\{h0l\}$, $\phi\,\{001\}$, $\phi\,\{100\}$ and $\phi\,\{110\}$ may be recorded as existing on a crystal.

216. III. Tetartohedral forms; tetragonal tetartohedra. The tetartohedra of the Tetragonal system are limited to two kinds, necessarily derived, as distinct geometrical forms, from the groups of faces forming the dioctahedron. As regards the eight normals of this form, there is only one way of suppressing four of them: ditetragonally disposed round the $[S\Sigma]$ axis, they must be reduced by hemi-systematic suppression to four normals tetragonally disposed round that axis. But the single face corresponding to a normal may be extant or absent alternately on one or the other side of the trito-systematic plane, or all four faces may be simultaneously extant or absent on one side of that plane. Hence the two following forms may be originated:—

(1) *The hemidisphenoid,* $s\sigma\{hkl\}$.

The faces common to each pair of forms of a proto- and a deutero-disphenoid combine in producing one of the four quarter-forms

$$s\sigma\{hkl\},\quad s\sigma\{h\bar{k}l\},$$
$$s\sigma\{khl\},\quad s\sigma\{\bar{k}hl\},$$

each of which is a sphenoid figure with triangular faces which are geometrically but not crystallographically isosceles; their symbols being,

for	$s\sigma\{hkl\}$,	hkl,	$\bar{h}\bar{k}l$,	$\bar{k}h\bar{l}$,	$k\bar{h}\bar{l}$ (Fig. 156 *c*);
	$s\sigma\{h\bar{k}l\}$,	$h\bar{k}l$	$\bar{h}kl$	$kh\bar{l}$	$\bar{k}\bar{h}\bar{l}$ (Fig. 156 *f*);
	$s\sigma\{khl\}$,	khl	$\bar{k}\bar{h}l$	$\bar{h}k\bar{l}$	$h\bar{k}\bar{l}$ (Fig. 156 *a*);
	$s\sigma\{\bar{k}hl\}$,	$\bar{k}hl$	$k\bar{h}l$	$hk\bar{l}$	$\bar{h}\bar{k}\bar{l}$ (Fig. 156 *d*).

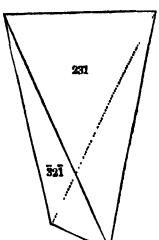


Fig. 156 (*a*).

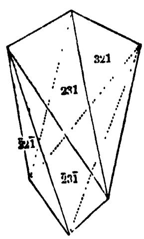


Fig. 156 (*b*).

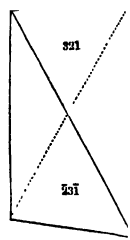


Fig. 156 (*c*).

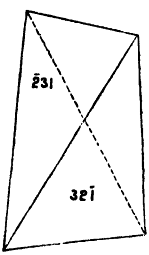


Fig. 156 (*d*).

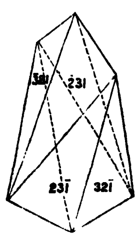


Fig. 156 (*e*).

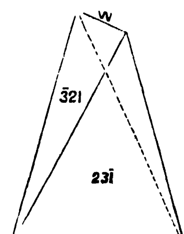


Fig. 156 (*f*).

The wedge-like edges W of the sphenoid parallel to the XY plane are not parallel to the axes X, Y, but lie obliquely to them, as in the case of the diagonals of the hemidiprism.

The $[S\Sigma]$ zone-axis is an axis of diagonal symmetry.

Taken in pairs these tetartohedral forms reconstitute the two corresponding disphenoids; the faces of the forms whose symbols in the above Table are in alternate rows constituting a diagonal, those whose symbols are in the two upper or two lower contiguous rows, an axial-disphenoid; the symbols in the first and fourth rows constitute the hemidioctahedron $\phi\{hkl\}$, those in the second and third rows the correlative form $\phi\{khl\}$, (see Plate II).

(2) *The hemimorphous hemidioctahedron.* The symbols of the faces belonging to the several quarter-forms are;

for	$a\rho\{hkl\}$,	hkl	$\bar{k}hl$	$\bar{h}\bar{k}l$	$k\bar{h}l$,
	$a\rho\{khl\}$,	khl	$\bar{h}kl$	$\bar{k}\bar{h}l$	$h\bar{k}l$,
	$a\rho\{\bar{h}\bar{k}\bar{l}\}$,	$\bar{h}\bar{k}\bar{l}$	$k\bar{h}\bar{l}$	$hk\bar{l}$	$\bar{k}\bar{h}\bar{l}$,
	$a\rho\{\bar{k}\bar{h}\bar{l}\}$,	$\bar{k}\bar{h}\bar{l}$	$h\bar{k}\bar{l}$	$kh\bar{l}$	$hk\bar{l}$.

Each of them is formed by four alternate faces of the dioctahedron lying above or lying below the trito-systematic plane C, and can only occur in combination with those of other forms. Fig. 157 represents the hemimorphous hemidioctahedron $a\rho\{321\}$ in combination with the forms $a\rho\{100\}$ and $a\rho\{\bar{1}\bar{1}\bar{1}\}$.

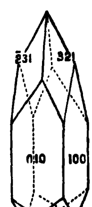


Fig. 157.

Those faces of which the symbols lie in the first and fourth rows would concur to build up a trapezohedron $a\{hkl\}$, those in the second and third, the form $a\{khl\}$; those in the two upper, or in the two lower rows, would constitute a hemimorphous dioctahedron $\rho\{hkl\}$ or $\rho\{\bar{h}\bar{k}\bar{l}\}$. Taken alternately, the first with the third or the second with the fourth row, the faces represented by these symbols would by their union produce a hemidioctahedron $\phi\{hkl\}$ or $\phi\{khl\}$ respectively (see Plate III).

There is as yet no proof that either of the above tetartohedral developements has been observed on tetragonal crystals: Werther, indeed, has described crystals of urea which presented a single pinakoid plane {001} in combination with a square prism {110} and a sphenoid σ {111}, but the occasional absence of a single plane can scarcely be regarded as sufficient evidence of the actual occurrence of tetartohedral crystals.

Tetragonal System. C.—*Combinations of Forms.*

217. The simpler character of the symmetry of the Tetragonal system as compared with that of the Cubic system is conspicuous in even the most complex combinations, since the faces of all the forms are seen grouped round a morphological axis with much regularity, and frequently with an approximation to equipoise developement.

The following figures illustrate the dispositions of the faces of some prominent forms when combined with each other.

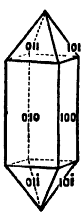


Fig. 158.

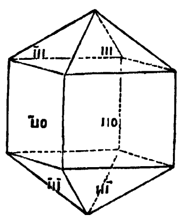


Fig. 159.

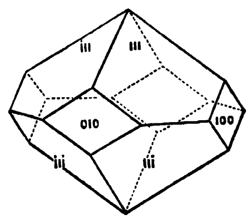


Fig. 160.

Fig. 158 represents the combination of the axial prism with the axial octahedron.

Fig. 159 shews the diagonal prism and octahedron in combination, and Fig. 160 the axial prism and diagonal octahedron.

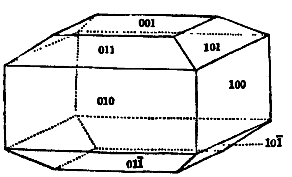


Fig. 161.

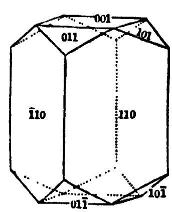


Fig. 162.

In Figs. 161–165 are illustrated combinations of several other simple forms.

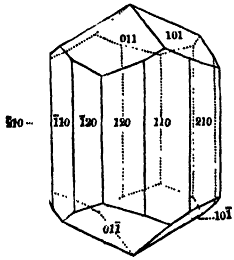


Fig. 163.

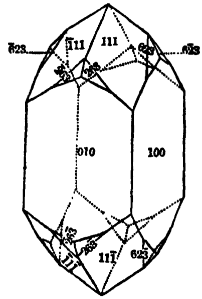


Fig. 164.

Figs. 166, 167, 168 represent respectively crystals of anatase,

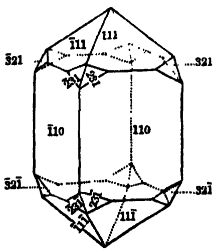


Fig. 165.

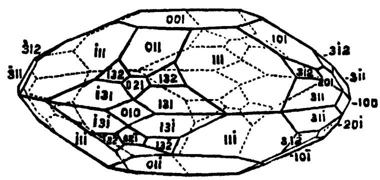


Fig. 166.

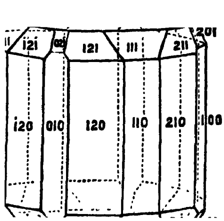


Fig. 167.

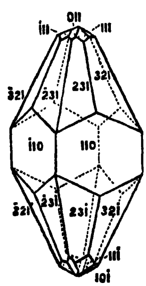


Fig. 168.

cromfordite, and cassiterite which have been actually observed, and they illustrate the concurrence in various combinations of the diprism, the dioctahedron, the axial and diagonal forms, and the pinakoid.

218. (*a*) Of the mero-symmetrical forms of the Tetragonal system the asymmetric forms are not known to be represented on crystals, but in the case of hexa-hydrated strychnine sulphate, by a process of etching on the faces, Baumhauer developed traces of faces grouped asymmetrically.

(*b*) Of the other types of hemisymmetrically developed forms, the tetrahedrid (or sphenoidal) is illustrated by the minerals copper-pyrites and edingtonite. Copper pyrites occurs as a deutero-sphenoid $\sigma\{111\}$, which differs from the regular tetrahedron by the normal-angle between two faces of the sphenoid, $(111, \bar{1}\bar{1}1)$ being $108^\circ 40'$, and $(111, \bar{1}1\bar{1})$ being $109^\circ 53'$; each of the corresponding angles in the regular tetrahedron being $109^\circ 28'$.

Again, the angle (001, 111) is

in copper pyrites $54^\circ 20'$,

in the Cubic system $54^\circ 44'$.

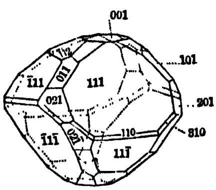


Fig. 169.

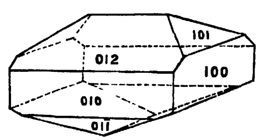


Fig. 170.

The *o* and *ω* sphenoids often concur on copper pyrites as they do on blende, the positive or *o* sphenoid $\sigma\{111\}$ having its faces dull from a brown incrustation of copper suboxide, and striated parallel to their intersections with the two faces of the form $\{201\}$ tautozonal with and adjacent to them: they are also larger than the *ω* faces belonging to the form $\sigma\{\bar{1}\bar{1}\bar{1}\}$, which on the other hand are lustrous and without striation. The crystals therefore shew considerable analogy to those of blende. They sometimes carry also the faces of the deutero-disphenoids $\sigma\{316\}$ and $\sigma\{312\}$, besides other forms the poles of which lie on the systematic zone-circles. Fig. 169 represents a crystal of this mineral.

Edingtonite (Fig. 170) and strontium-uranyl acetate crystallise in forms which, owing to the manner in which the axial system has been selected, represent the proto-sphenoidal type $s\,\{hkl\}$.

(*c*) The hemitetragonal but diplohedral type is represented by several crystalline bodies. Scheelite (calcium tungstate), repre-

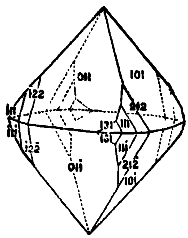


Fig. 171.

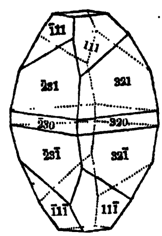


Fig. 172.

sented in Fig. 171, stoltzite (lead tungstate), and wulfenite (lead molybdate) occur in crystals of this type, as do also fergusonite, shewn in Fig. 172, and sarcolite (Fig. 173), a mineral belonging to the same group with scapolite, which itself exhibits mero-symmetry of this kind.

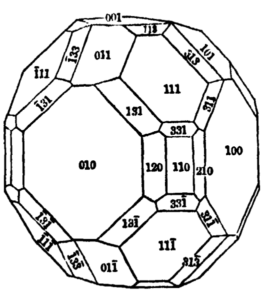


Fig. 173.

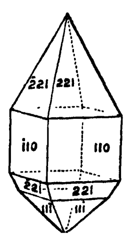


Fig. 174.

(*d*) A hemimorphous developement of a holosymmetrical form has been observed by Groth on crystals of an artificial compound, iodosuccinimide ($C_4H_4NO_2I$), illustrated in Fig. 174.

Tetragonal System. D.—*Twinned Forms.*

219. I. *Holosymmetrical crystals.* The axis of form [001], as being an axis of tetragonal symmetry, and the horizontal axes [100] [010] and [110] [$\bar{1}$10], as being axes of diagonal symmetry to the system, are precluded from being twin-axes for holosymmetrical forms. In fact, the only twin-axes known to occur are such as lie in one or other of the systematic planes but are not normals of such planes. Thus, a face of the form {101} is a frequent twin-plane.

Fig. 175 is a twin of zircon twinned on a plane of the form {101}, which is also the combination-plane; and similar twins are common in cassiterite (Fig. 176) and rutile.

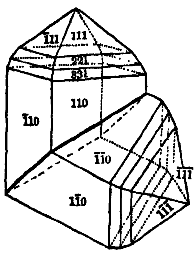


Fig. 175.

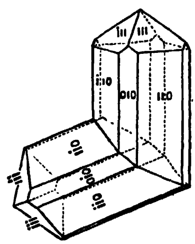


Fig. 176.

Reiteration of the twinning is common in rutile and cassiterite; sometimes on faces parallel with the first twin-face, resulting in geniculated prolongations of the crystals, as in Fig. 177, which represents a twin-growth of the latter mineral.

Sometimes the twinning is repeated on each of the eight faces of the form {101}; or again the crystals are successively twinned, each succeeding crystal on the previous one in the series.

Pseudo-hexagonal twin-forms may thus arise, as in the case of cassiterite, whereof six individuals are sometimes grouped round a line which is perpendicular to a plane containing the morphological axis of each of the six crystals (Fig. 178); adjacent axes of five of the individuals being inclined to each other at an angle of 112° 10′, while the axis of the sixth individual, which is twinned on the fifth but not on the third, meets the axis of the latter at an angle of 139° 10′.

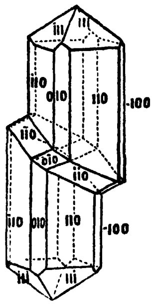


Fig. 177.

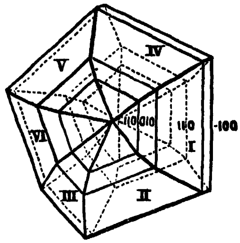


Fig. 178.

II. *Hemisymmetrical crystals.* As in the Cubic, so in the Tetragonal system, a twin upon a face of the form {110} or of the form {100}, though impossible in holo-symmetrical crystals, may occur where the normal on such a face is not an axis of diagonal symmetry. Twinning upon a face of the form {110} is not possible, therefore, in the case of a semiform of asymmetrical type $a\,\{hkl\}$, but it is possible with each of the other hemisymmetrical types of the Tetragonal system.

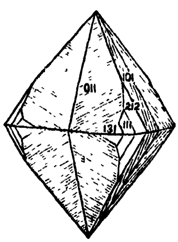


Fig. 179.

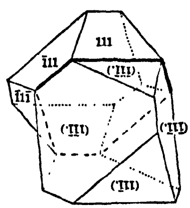


Fig. 180.

(*a*) *Twins of Diplohedral crystals.* The hemitetragonal diplohedral type of crystal (Fig. 171) occurs in the twinned condition in certain of the growths of scheelite (calcium tungstate). The twin-plane is a face of the prism {110}, which is also the combination-plane, and the crystals interpenetrate (Fig. 179). Or again, each is represented by a half-crystal divided by the combination-plane, the two half-crystals being then in juxtaposition.

220. (*b*) *Twins of Haplohedral crystals.* Under the type of holotetragonal haplohedral symmetry copper pyrites affords the only illustrations of twinned forms, but these exhibit considerable variety. Three twin-laws have been ascribed to this mineral; the twin-plane being in the several cases, (1) a face of the form $\{111\}$, (2) a face of the form $\{110\}$, and (3) a face of the form $\{101\}$.

(1) Twin-law; the twin-plane a face of the form $\{111\}$.

Crystals of copper pyrites twinned under this law present considerable analogy to those of blende similarly twinned; the bright faces ω or $\sigma\,\{\bar{1}\bar{1}\bar{1}\}$ of the one individual are seen in juxtaposition with the duller o or $\sigma\,\{111\}$ faces of the other crystal, the combination-plane being parallel to the twin-plane. Thus a revolution of one crystal, from parallel orientation, round an axis normal to the face $11\bar{1}$ would bring the two individuals into a relative juxtaposition analogous to that seen in blende (Fig. 125): such a twin is illustrated in Fig. 180.

(2) Twin-law; the twin-plane a face of the form $\{110\}$.

This variety of twin-structure is so rare as almost to rest for its authority on the early description given of it by Haidinger in his classical memoir on the crystallisations of copper pyrites. The twin is interpenetrant and is very similar to Fig. 133.

(3) Twin-law; the twin-plane a face of the form $\{101\}$.

The true character of the crystal-growths that obey this law has been obscured, in part as a consequence of a certain ambiguity in the language in which it was described sixty years ago by Haidinger in one of his papers 'On the Regular Composition of Crystallised Bodies,' but chiefly from the difficulty of finding crystals complete enough, or with faces affording sufficiently good reflections to enable accurate measurements of the angles to be made. Mr. Fletcher has recently pointed out the real significance of the words in which Haidinger described the twin, and by subjecting the crystals in the British Museum collection to a careful examination has established the true character of the growth. That collection possesses several excellent examples of the twin, including a specimen which had been described and figured by Haidinger himself as an illustration of the law.

The correct description of this particular combination of crystals of copper pyrites possesses additional importance from the fact that the interpretation of its character, following from these observations, either involves so strained an application of the ordinary twin-law as to be quite unsatisfactory, or leads to the alternative that these apparent twins hold an entirely exceptional position among such crystal-growths. They, in fact, present the sole known illustration of a 'symmetrical twin' of Groth (Article **161**), which cannot be completely explained either by the ordinary twin-law as applicable to mero-symmetrical crystals, or by a principle of parallel orientation of equivalent directions where the union is one of enantiomorphous tetarto-forms.

Haidinger, in 1822[1], described the law of these twins as being twin-plane, a face of the octahedron {101}, combination-plane perpendicular to the twin-plane; but, in a subsequent recital of his earlier conclusion[2], he described the law as 'regular composition parallel to a face of the octahedron {101} or perpendicular to the terminal edges of the octahedron {111}.' That Haidinger implied the twin-plane of the regular composition, and not the combination-plane, by this alternative statement is evident from his referring immediately afterwards to the above as a single kind of regular growth. And such in fact it is, crystallographically speaking, if the twin-plane be in view, for then these statements are identical in their results, as is seen from the following considerations.

The growth due to twinning about the plane (101) is such that both of the normals to the planes (010), (101), which are at right angles to each other, are axes of orthosymmetry to the combination: hence a third line at right angles to both, and therefore parallel to an edge of the octahedron {111}, will likewise be an axis of orthosymmetry to the combination, though not to the separate individuals; and the same combination will thus be produced by a rotation of one of the individuals through two right

[1] Memoirs of Wernerian Society, vol. iv. p. 1, 1822.
[2] Edinburgh Journal of Science, vol. iii. p. 68, 1825.

angles around this edge as by a similar rotation round the normal to the plane (101).

As to the combination-plane, it is, according to Haidinger's view, *perpendicular* to the face (101): if this be true, its indices are irrational and the plane is not a possible face of the crystal. It is now found, however, that an erroneous interpretation placed on the above statement of Haidinger by Naumann, and after him by Sadebeck and since generally accepted on their authority, more nearly represents the nature of the combination. This interpretation gives the law as—twin-plane a face of the form (101) and combination-plane parallel to the twin-plane—if the crystals of copper pyrites were holo-symmetrical, this statement of the law would be an accurate description of the twin, and would accord with the results of the measurements recently made. Or, if the faces of the second individual that are brought into contiguity with the faces of the first individual by a rotation round, say, the normal of $(10\bar{1})$ presented a contrast in respect to the characteristics of striation, brilliance and magnitude that distinguish the faces of the semiforms $\sigma\{hhl\}$ or $\sigma\{hkl\}$ (the *o* forms) from the faces of the correlative (ω) semiforms $\sigma\{\bar{h}\bar{h}\bar{l}\}$ or $\sigma\{\bar{h}\bar{k}\bar{l}\}$, the explanation would be complete. But it is indubitable that this is not the case: that in fact it is an *o*- and not an ω-face that is found contiguous to an *o*-face throughout (Fig. 181).

The explanation by Naumann thus fails as a simple statement of the twin-law. It would require, in order to make it complete, that the original orientation of the two individuals gave *correlative* faces the same aspect towards space; such as would arise if from a position of identical orientation one crystal was revolved through two right angles round an axis normal to a face of the form {110} —similar, namely, to the revolution under the previously considered twin-law.

Corresponding systematic planes would be parallel in such an orientation, but the faces with corresponding aspects would belong to opposite octants, in the two individuals.

Assuming such an initial revolution round a normal of the form {110}, a second revolution of one of the crystals would have to take place round a face-normal of the form {101} in order to give such an association of the crystals as that under consideration; and the plane of combination would be parallel to the plane of twinning. Evidently, in such a case, this plane of combination is physically, as well as geometrically, a plane of symmetry for the two crystals, the faces of either crystal being reflected in it with identical features. Fig. 182 shows the relative positions of the

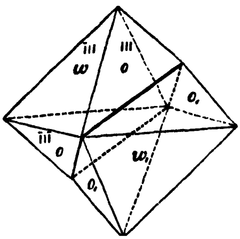


Fig. 181.

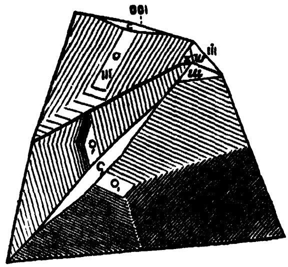


Fig. 182.

o and *ω* planes of an actual twin-growth according to this law; for the sake of clearness the growth is represented as having been rotated through a right angle, about the vertical axis, from the position shown in Fig. 181.

Such a combination represents the 'symmetrical twin' of Groth, and it is as a symmetrical twin that it is most simply described.

SECTION III.—**The Hexagonal System.**

A.—*Holo-symmetrical Forms.*

221. Crystals belonging to the Hexagonal system are symmetrical to a morphological axis (the axis $[S\Sigma]$), which in the case of complete symmetry is the zone-axis of the two triads of systematic planes, that is to say, of the three proto-systematic planes S (S_1 S_2 S_3), and of the three deutero-systematic planes Σ (Σ_1 Σ_2 Σ_3), and is perpendicular to the trito-systematic plane C which is the plane of the zone-circle $[S\Sigma]$, (Fig. 183).

The crystallographic axes OX, OY, OZ lie in the proto-systematic planes and are equally inclined on the axis of form, but are not axes of symmetry. The parameters being by the conditions of symmetry necessarily equal, the axial system is represented by the expressions

$$\xi = \eta = \zeta \lesseqgtr 90°,$$
$$a = b = c.$$

The zone-axis $[S\Sigma]$ is an axis of dihexagonal symmetry, and is termed, for brevity, the hexagonal axis.

The mutual inclinations of the planes of either triad of systematic planes are 60°, the inclinations of a plane of one group on the adjacent planes of the other being 90° and 30°.

The normals of the one group coincide with the lines in which the planes of the other group intersect the trito-systematic plane C, and are axes of ortho-symmetry.

The systematic triangle of the Hexagonal system is formed by two quadrantal arcs S, Σ on great circles S and Σ, and an arc C' with the value $\frac{\pi}{6}$ on the great circle C; its sides being

$$S = \Sigma = \frac{\pi}{2}, \quad C = \frac{\pi}{6};$$

its angles $\sigma = s = 90°, \quad c = 30°.$

222. The relations of two diplohedral plane-systems symmetrical, the one to a trigonal, the other to a hexagonal axis, have already been considered (Chapter V, Cases 3 and 4), and it has been seen that the conditions which need to be fulfilled by the axial elements are identical for the two systems. Further, in treating of the Tesseral system (Case 5), we had to deal with diplohedral forms presenting symmetry of a trigonal type which, under the conditions of that system, had to be considered as holo-systematic forms.

In the system now under discussion the trigonally symmetrical forms which are possible are numerous; but they will have to be treated in the character of hemi-symmetrical forms of the Hexagonal system, rather than as representing an independent type of symmetry. The semi-independent character of a diplohedral trigonal system is, however, kept continually in view in the Hexagonal system by the composite character of the symbols of certain of its forms, and conveniently also in some degree by a nomenclature that recalls the relations of the two systems: indeed this quasi-trigonal character is found to underlie the symmetry of a large number of the most important of the substances which crystallise in this system in forms of rhombohedral type: of these calcite is a conspicuous example.

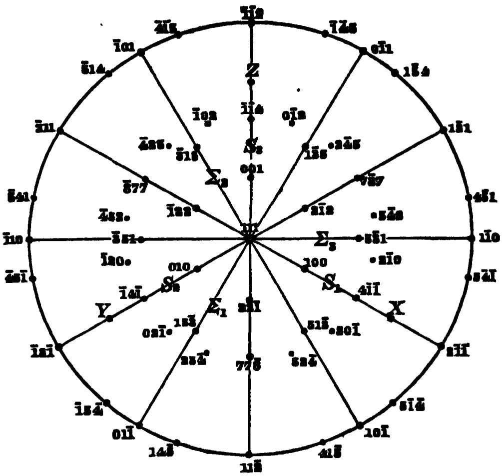


Fig. 183.

223. The surface of the sphere of projection is divided by the seven systematic planes into twenty-four systematic triangles, each of which would contain, in unsymmetrical position within it, a pole of the general scalenohedral form (the discalenohedron) of the system, the symbol for which is composite, namely, $\{hkl, efg\}$. It is thus a twenty-four-faced figure.

The seven holo-symmetrical forms of the Hexagonal system will consist of such as have their poles situate

1. on no systematic great circle, the general independent form,

2. on the trito-systematic great circle C only, $\{pqr\}$;
3. on the proto-systematic great circles S only, $\{hkk, eff\}$;
4. on the deutero-systematic great circles Σ only, $\{min\}$;
5. at the intersection of the great circles S and C, $\{2\bar{1}\bar{1}\}$;
6. at the intersection of the great circles Σ and C, $\{10\bar{1}\}$;
7. at the intersection of the great circles S and Σ, $\{111\}$.

The relations of the indices in the several symbols of these forms have already been established in the chapter dealing with crystalloid symmetry.

In fact the indices of the general symbol $\{hkl,\ efg\}$ are connected by the relation

$$e = 2(h+k+l)-3h,$$
$$f = 2(h+k+l)-3k,$$
$$g = 2(h+k+l)-3l;$$

where therefore $e+f+g = 3(h+k+l)$: $h\,k\,l$ are in the order of decreasing magnitude.

For the symbol $\{p\,q\,r\}$ of a plane in the zone [111] the indices have to satisfy the condition $p+q+r = 0$, where the magnitudes of the indices taken absolutely are in the order $p > q > r$; and in the symbol $\{min\}$ the index i is the arithmetic mean of the other two indices (i.e. $i = \frac{m+n}{2}$), of which m is assumed to be algebraically the greater.

The seven holo-symmetrical forms will be designated as follows:

1. $\{hkl, efg\}$, the dihexagonal scalenohedron or discalenohedron.
2. $\{pqr\}$, the dihexagonal prism or diprism.
3. $\{hkk, eff\}$, the (proto-dihexahedron or) dirhombohedron.
4. $\{min\}$, the (deutero-dihexahedron or) hexagonal pyramid.
5. $\{2\bar{1}\bar{1}\}$, the hexagonal proto-prism or S-prism.
6. $\{10\bar{1}\}$, the hexagonal deutero-prism or Σ-prism.
7. $\{111\}$, the hexagonal pinakoid.

The positions of the poles of such forms on the sphere of projection are illustrated in Fig. 183 for the case where

$$\xi = \eta = \zeta = 112°33'$$

(as in the mineral willemite). In Table F the poles of the discalenohedron are grouped in triads; those of the semiform $\{hkl\}$ (to which the axis $[S\Sigma]$ is an axis of ditrigonal symmetry) taken in the order hkl are indicated by p, and in the order hlk by p': the poles of the correlative semiform $\{efg\}$ are indicated in a similar way by q and q'.

224. 1. *The discalenohedron* $\{hkl,\ efg\}$, Fig. 184. The symbols of the faces of the general independent scalenohedron of the Hexagonal system are contained in

TABLE F.

$p \ldots hkl$	lhk	klh	hlk	khl	$lkh \ldots p'$,
$q \ldots efg$	gef	fge	egf	feg	$gfe \ldots q'$,
$\bar{p} \ldots \bar{h}\bar{k}\bar{l}$	$\bar{l}\bar{h}\bar{k}$	$\bar{k}\bar{l}\bar{h}$	$\bar{h}\bar{l}\bar{k}$	$\bar{k}\bar{h}\bar{l}$	$\bar{l}\bar{k}\bar{h} \ldots \bar{p}'$,
$\bar{q} \ldots \bar{e}\bar{f}\bar{g}$	$\bar{g}\bar{e}\bar{f}$	$\bar{f}\bar{g}\bar{e}$	$\bar{e}\bar{g}\bar{f}$	$\bar{f}\bar{e}\bar{g}$	$\bar{g}\bar{f}\bar{e} \ldots \bar{q}'$;

wherein $h > k > l$ and $h + k + l \gtrless 0$, and any of the literal indices may represent either positive or negative values independently of the sign in the general symbol. Fig. 184 represents the form $\{20\bar{1}, \bar{4}25\}$ corresponding to the above parametral angle.

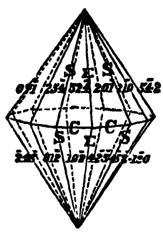


Fig. 184.

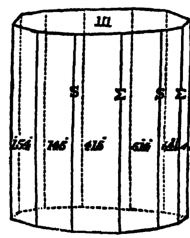


Fig. 185.

The twenty-four alternately metastrophic triangles which comprise the form are grouped round the morphological axis, and form two pyramids united by a common base in the horizontal plane *C*, a figure which, though doubly terminated, is by crystallographic usage termed a pyramid. Any section of this figure perpendicular to the morphological axis will be a symmetrical hexagon, of which only the alternate angles are equal.

Two dihexagonal quoins form the vertices of the pyramids and are composed by edges *S* and Σ alternating with each other, adjacent edges representing dihedral angles of different magnitudes.

The twelve basal edges *C* common to the two pyramids are similar; but the six lateral quoins which these edges form by meeting pairs of edges *S* are dissimilar from the six alternating quoins in which they meet pairs of edges Σ. These lateral quoins are ortho-symmetrical, the former six on the axes [*SC*], the latter on the axes [Σ*C*].

225. 2. *The dihexagonal prism* or *hexagonal diprism* $\{pqr\}$, Fig. 185. Since discalenohedra with any given dodecagonal base in common may be indefinite in number, but will differ from each other in the acuteness of the apex of their pyramids, we may consider the case in which the angle formed by the pyramidal edges with the morphological axis $[S\Sigma]$ vanishes, and the edges themselves become parallel with this axis.

In such a case the twenty-four edges of the discalenohedron will become twelve, from the coincidence of each pair of edges symmetrical to the trito-systematic plane; and the resulting form will have twelve faces parallel to the hexagonal axis $[S\Sigma]$. It will therefore be a symmetrical (not regular) dodecagonal prism, an open form, the section of which perpendicular to its edges is the same as that of the discalenohedra of which it is the limiting form. But it is to be observed that it is, so far as its geometrical characters are concerned, also the limiting form of *any* twelve of the faces of the same figure if they be so selected that no pair is symmetrical on the equatorial plane C.

If hkl be a pole of one of the group of discalenohedra having a common section with the diprism $\{pqr\}$, a zone-circle passing through the poles (111) and (hkl) will contain corresponding poles of every discalenohedron of the group and also a pole of the form $\{pqr\}$; and the symbol of the pole (pqr), in which the zone-circles $[111, hkl]$ and $[111]$ are tautohedral, will be

$$(2h-k-l \quad 2k-l-h \quad 2l-h-k),$$

a symbol which fulfils the conditions $p+q+r = 0$ and $p > q > r$, since $h > k > l$. If the symbol be derived from the correlative pole efg or the poles $\bar{h}\bar{k}\bar{l}$, $\bar{e}\bar{f}\bar{g}$, the two first of these three poles will give the symbol opposite to the face pqr. Thus the zones $[efg, 111]$ and $[111]$, that is to say,

$$[2(k+l)-h \quad 2(l+h)-k \quad 2(h+k)-l, \ 111] \text{ and } [111]$$

are tautohedral in the face

$$(-2h+k+l \quad -2k+l+h \quad -2l+h+k).$$

The hexagonal diprism will have two sorts of edges alternating with each other, and lying respectively in the proto- and in the deutero-systematic planes. Fig. 185 represents the diprism $\{5\bar{1}\bar{4}\}$.

An important feature of the hexagonal diprism results from its faces belonging to a zone, all the planes of which, except the systematic planes S and Σ, are planes of abortive symmetry. They in fact present a stereotyped feature of unchangeability whether for crystals with different parameters or for different temperatures.

The following are the normal-angles between the face (pqr) of some of these prisms and the face $2\bar{1}\bar{1}$. The angle $(2\bar{1}\bar{1}, pqr)$, where pqr is

$5\bar{2}\bar{3} = 6^\circ 35'$, $3\bar{1}\bar{2} = 10^\circ 54'$, $7\bar{2}\bar{5} = 13^\circ 54'$, $4\bar{1}\bar{3} = 16^\circ\ 6'$;
$5\bar{1}\bar{4} = 19^\circ\ 6'$, $6\bar{1}\bar{5} = 21^\circ\ 3'$, $\bar{7}16 = 22^\circ 27'$, $8\bar{1}\bar{7} = 23^\circ 25'$;
$\bar{9}18 = 24^\circ 11'$, $10\bar{1} = 30^\circ$, $11\bar{2} = 60^\circ$, $01\bar{1} = 90^\circ$.

226. 3. and 4. *The isosceles dodecahedra.* The culminating edges of the discalenohedron being of two kinds, the one symmetrical to the proto-, the other to the deutero-systematic planes, the faces of a form truncating the edges of one kind will be dissimilar from those which truncate the edges of the other kind. Both the forms will be twelve-faced and their faces isosceles triangles, meeting in two vertical quoins symmetrical to the hexagonal axis. Perpendicular to that axis the section of a figure of either kind is a regular hexagon. Thus they are isosceles dodecahedra or hexagonal pyramids.

The form of which the poles lie on the proto-systematic zone-circles has for its symbol $\{hkk,\ eff\}$, and may be regarded as composed of two correlative rhombohedra, formed by planes having in their symbols the indices $h\,k$ and $e\,f$ respectively. And since the rhombohedral type of hemi-symmetry, to which the scalenohedron also belongs, is of preponderating importance among all the types of this system, the relation of the hexagonal to this trigonal type is recalled by the use of a terminology for the hexagonal forms corresponding to the composite nature of their symbols.

The hexagonal proto-pyramids will therefore be termed dirhombohedra, while the others, of which the faces truncate the Σ-edges of the discalenohedron, will be designated simply as hexagonal pyramids, since they belong equally to the hexagonal and trigonal types. The general symbol for the hexagonal pyramids will be $\{min\}$; a symbol in which, as has already been seen, the index

i is the arithmetic mean of the other two: as in fact is evident, since the symbol for a systematic plane Σ, which is also a zone-plane containing four of the poles of a form $\{min\}$, is that of a plane of the form $\{1\bar{2}1\}$, and thus $m-2i+n=0$.

227. 3. *The dirhombohedron* $\{hkk,\ eff\}$, Figs. 186–7. The symbols of the faces of the isosceles dodecahedron $\{hkk,\ eff\}$ are

$$\begin{array}{ccc} hkk & khk & kkh, \\ eff & fef & ffe; \\ \bar{h}\bar{k}\bar{k} & \bar{k}\bar{h}\bar{k} & \bar{k}\bar{k}\bar{h}, \\ \bar{e}\bar{f}\bar{f} & \bar{f}\bar{e}\bar{f} & \bar{f}\bar{f}\bar{e}; \end{array}$$

where $e=-h+4k$, $f=2h+k$, since eff is the pole in which the zones $[11\bar{2},\ khk]$ and $[01\bar{1}]$ are tautohedral; that is to say, is $(4k-h,$

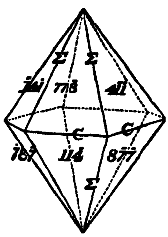


Fig. 186.

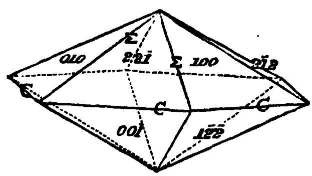


Fig. 187.

The dirhombohedron may also be treated as the form truncating the S-edges of a discalenohedron. If hkl, hlk be two adjacent faces of such a discalenohedron, since the symbols are symmetrical we obtain the symbol of the truncating face by the addition of their indices, i. e. the face in question is $(2h\ \ k+l\ \ k+l)$.

The poles of a dirhombohedron $\{hkk,\ eff\}$ which lie on the zone-circle

$$\begin{array}{rl} \text{or } [01\bar{1}] \text{ are} & hkk \quad eff \quad \bar{h}\bar{k}\bar{k} \quad \bar{e}\bar{f}\bar{f}, \\ [S_2] \text{ or } [10\bar{1}] \text{ are} & khk \quad fef \quad \bar{k}\bar{h}\bar{k} \quad \bar{f}\bar{e}\bar{f}, \\ \text{or } [1\bar{1}0] \text{ are} & kkh \quad ffe \quad \bar{k}\bar{k}\bar{h} \quad \bar{f}\bar{f}\bar{e}. \end{array}$$

Figs. 186 and 187 represent the dirhombohedra $\{4\bar{1}\bar{1}, \bar{8}77\}$ and $\{100, \bar{1}22\}$ corresponding to the given parametral angle.

228. 4. *The hexagonal pyramid* $\{min\}$, Fig. 188. The faces of the hexagonal pyramid will have the following symbols, and lie on the several zones [Σ] in the subjoined order:

on zone $[\Sigma_2]$ or $[1\bar{2}1]$, min nim $\bar{m}\bar{i}\bar{n}$ $\bar{n}\bar{i}\bar{m}$;
on zone $[\Sigma_3]$ or $[11\bar{2}]$, nmi mni $\bar{n}\bar{m}\bar{i}$ $\bar{m}\bar{n}\bar{i}$;
on zone $[\Sigma_1]$ or $[\bar{2}11]$, inm imn $\bar{i}\bar{n}\bar{m}$ $\bar{i}\bar{m}\bar{n}$.

In Fig. 188 is represented the hexagonal pyramid $\{51\bar{3}\}$ corresponding to the above parameters.

The symbol of the form truncating the Σ-edges of a di-scalenohedron $\{hkl, efg\}$ may be obtained either by determining the symbols of the face in which a zone containing two adjacent faces of the discalenohedron symmetrical to a plane Σ is tautohedral with the zone of which Σ is the zone-plane; for instance, the face in which $[hkl, gfe]$ and $[\bar{1}2\bar{1}]$ are tautohedral; or by the briefer method of adding corresponding indices of the two symbols hkl gfe when brought into corresponding form, as indicated in article **133**, p. 154.

A dirhombohedron and a hexagonal pyramid are precluded by the symmetry of the system from being mutually congruent: nor can a pole of the form $\{hkk, eff\}$ be equidistant with a pole of the form $\{min\}$ from the pole $\{111\}$. A pole of a form $\{min\}$ equidistant with a pole (hkk) from (111) would in fact have the symbols

$$(h + 2k + \sqrt{3}\,\overline{h-k}, \quad h + 2k, \quad h + 2k - \sqrt{3}\,\overline{h-k}),$$

the indices of which can only be rational when $h = k$, and the symbol (min) becomes (111), or when $2k+h = 0$, and the symbol (min) becomes (111); that is to say, in the cases where the hexagonal pyramid passes over into one of its limiting forms, the pinakoid or the deutero-prism.

229. *The hexagonal prisms.* The general symbol $\{pqr\}$ has been assigned to such forms as have their poles on the equatorial zone-circle C. Where such a form has three different literal indices the form is the diprism (Art. **225**). But since $p = -(q+r)$, if $q = r$ the symbol becomes $\{2\bar{1}\bar{1}\}$, a particular case of the symbol $\{hkk\}$, and thus represents a form having its poles on the proto-systematic zone-circles.

If, again, $p = -r$, then $q = 0$, and the symbol is $\{10\bar{1}\}$, a particular case of the symbol $\{min\}$; it thus represents a form having its poles on the deutero-systematic zone-circles. This symbol is in fact that of a face truncating the faces (hkk) and $(\bar{e}\bar{f}\bar{f})$, i.e. $(3h\ 3k\ 3k)$ and $(\overline{h-4k}\ \ \overline{-2h+k}\ \ \overline{-2h+k})$, namely $(2\overline{h-k}\ \ \overline{k-h}\ \ \overline{k-h})$, or $(2\bar{1}\bar{1})$; and similarly a face truncating the edge $[min\ \ \bar{n}\bar{i}\bar{m}]$ is $(m-n\ \ 0\ \ n-m)$ or $10\bar{1}$.

230. 5. *The hexagonal proto-prism* $\{2\bar{1}\bar{1}\}$, (Fig. 189). This form, which may be considered as the form of which the faces truncate the equatorial edges of the dirhombohedron, or as a limiting form

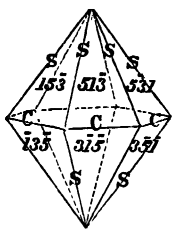


Fig. 188.

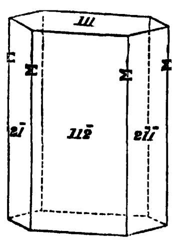


Fig. 189.

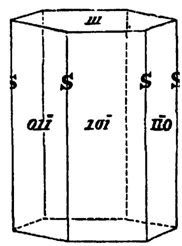


Fig. 190.

of the figures which may be produced by combining those faces of the dirhombohedron which have only h and k or e and f in their symbols, has for its poles the axial points of the axes $[SC]$; on which axes therefore its faces are ortho-symmetrical. The faces are parallel to the deutero-symmetral planes Σ,

those parallel to Σ_1 being $2\bar{1}\bar{1}$ and $\bar{2}11$,

those parallel to Σ_2 being $\bar{1}2\bar{1}$ and $1\bar{2}1$,

those parallel to Σ_3 being $\bar{1}\bar{1}2$ and $11\bar{2}$.

Its edges also lie in the planes Σ and are parallel to the hexagonal axis. It is an open form without quoins, but its faces have an ortho-symmetrical character.

231. 6. *The hexagonal deutero-prism* {10$\bar{1}$}, (Fig. 190). This form, which is identical in features with the proto-prism, has for the normals of its faces the ortho-symmetral axes [ΣC], the faces themselves being parallel to the proto-systematic planes *S*.

The faces parallel to S_1 are 01$\bar{1}$ and 0$\bar{1}$1,
" " to S_2 are 10$\bar{1}$ and $\bar{1}$01,
" " to S_3 are 1$\bar{1}$0 and $\bar{1}$10.

The edges of the form also lie in the planes *S*.

The horizontal sections of the proto- and deutero-prisms are regular hexagons, and the edges of the one form are truncated by the faces of the other form, and are bevilled by the faces of the diprism.

232. 7. *The pinakoid* {111}. This form, the two faces of which are parallel to the trito-systematic plane *C*, has the morphological axis for its normal, and is, as its symbol implies, the pyrametral form, meeting the three positive or the three negative arms of the crystallographic axes with equal intercepts.

The form is further, in holo-symmetrical crystals, dihexagonally symmetrical on the [$S\Sigma$] axis. Its two faces (111) and ($\bar{1}\bar{1}\bar{1}$) truncate the terminal quoins of the discalenohedron and the isosceles dodecahedra, and they close the dihexagonal and hexagonal prisms (Figs. 185, 189, 190) with faces that are symmetrical dodecagons in one set of cases and equiangular hexagons in the other.

Hexagonal System. B.—*Mero-symmetrical Forms.*

233. The mero-symmetrical forms that are possible in the Hexagonal system are numerous and varied in their types; the faces of the discalenohedron being capable of partition, in accordance with the principles of mero-symmetry, into correlative forms in which they possess one or other of the following kinds of alternate distribution:—

I. Twelve faces corresponding to twelve normals are extant or are absent: *holohexagonal haplohedral forms.*

II. Twelve faces corresponding to six normals are so: *hemihexagonal diplohedral forms.*

III. Six faces corresponding to six normals: *hemihexagonal haplohedral forms.*

IV. Six faces corresponding to three normals: *tetarto-hexagonal diplohedral forms.*

V. Three faces corresponding to three normals: *tetarto-hexagonal haplohedral forms.*

In the following table of the symbols of the faces of a discalenohedron $\{hkl, efg\}$, the symbols of the pair of faces belonging to each normal form a column in the upper or in the lower half of the table; and the triads of faces being for convenience in notation indicated by the letters employed for them in Articles **114**, **115**, **116** and **120**, pp. 130–138, those designated as p, $\bar{p}'$, $\bar{q}$, $\bar{q}'$ will be mutually metastrophic, and antistrophic to the remaining faces of the form.

TABLE G.

p ...	hkl	lhk	klh	hlk	khl	lkh	... p',
$\bar{p}$ ...	$\bar{h}\bar{k}\bar{l}$	$\bar{l}\bar{h}\bar{k}$	$\bar{k}\bar{l}\bar{h}$	$\bar{h}\bar{l}\bar{k}$	$\bar{k}\bar{h}\bar{l}$	$\bar{l}\bar{k}\bar{h}$	... $\bar{p}'$,
q ...	efg	gef	fge	egf	feg	gfe	... q',
$\bar{q}$ ...	$\bar{e}\bar{f}\bar{g}$	$\bar{g}\bar{e}\bar{f}$	$\bar{f}\bar{g}\bar{e}$	$\bar{e}\bar{g}\bar{f}$	$\bar{f}\bar{e}\bar{g}$	$\bar{g}\bar{f}\bar{e}$	... $\bar{q}'$.

234. In a holo-systematic haplohedral crystal the poles of a triad p, q, p' or q', cannot concur with the poles of a parallel triad $\bar{p}$, $\bar{q}$, $\bar{p}'$ or $\bar{q}'$, but they may concur with those of any other of the latter triads.

Thus, we may only have combinations of the following kinds:—

I. i. p and q with $\bar{p}'$ and $\bar{q}'$, and its correlative $\bar{p}\bar{q}p'q'$.
 ii. p and p' with $\bar{q}$ and $\bar{q}'$, " " $\bar{p}\bar{p}'qq'$.
 iii. p and q' with $\bar{p}'$ and $\bar{q}$, " " $\bar{p}\bar{q}'p'q$.
 iv. p and p' with q and q', " " $\bar{p}\bar{p}'\bar{q}\bar{q}'$.

II. And, in a hemi-systematic diplohedral crystal, any triad p, q, p' or q' must be accompanied by the corresponding parallel triad $\bar{p}$, $\bar{q}$, $\bar{p}'$ or $\bar{q}'$. So that the only possible combinations are:—

 i. p and $\bar{p}$ with p' and $\bar{p}'$, and its correlative $q\bar{q}q'\bar{q}'$.
 ii. p and $\bar{p}$ with q' and $\bar{q}'$, " " $q\bar{q}p'\bar{p}'$.
 iii. p and $\bar{p}$ with q and $\bar{q}$, " " $p'\bar{p}'q'\bar{q}'$.

III. A haplohedral form of a hemi-systematic crystal must belong to one of the tetartohedral types resulting from the concurrence of the following face-triads:—

i. p with $\bar{p}'$: and the three correlatives $q\,\bar{q}'$, $p'\bar{p}$, $q'\bar{q}$.
ii. p with $\bar{q}'$: „ „ $q\,\bar{p}'$, $p'\bar{q}$, $q'\bar{p}$.
iii. p with $\bar{q}$: „ „ $q\,\bar{p}$, $p'\bar{q}'$, $q'\bar{p}'$.
iv. p with p': „ „ $q\,q'$, $\bar{p}\,\bar{p}'$, $\bar{q}\,\bar{q}'$.
v. p with q': „ „ $q\,p'$, $\bar{p}\,\bar{q}'$, $\bar{q}\,\bar{p}'$.
vi. p with q: „ „ $p'q'$, $\bar{p}\,\bar{q}$, $\bar{p}'\bar{q}'$.

IV. And since a tetarto-systematic partition of the twelve normals into four groups of three normals is also possible, the discalenohedron may yield four diplohedral correlative forms. They are the face-triads that concur in one of the four following correlative groups:—

p with $\bar{p}$, q with $\bar{q}$, p' with $\bar{p}'$, and q' with $\bar{q}'$.

V. A tetarto-hexagonal haplohedral form would be represented by the faces p, and the discalenohedron would yield seven other forms correlative to it. These octo-forms would be hemimorphous.

235. The following table exhibits the forms which result from these mero-symmetrical modes of distributing the faces of the discalenohedron; and with them are associated such mero-symmetrical developements of the other holo-symmetrical forms as are *geometrically distinct* from the forms from which they may be derived.

I. Holo-systematic haplohedral forms; or holohexagonal haplohedra:—

i. Symmetrical to no systematic plane—

The ditrapezohedron $a\,\{h\,k\,l,\ e\,f\,g\}$ or $a\,\{h\,l\,k,\ e\,g\,f\}$.

ii. Symmetrical to the C- and S-systematic planes—

The ditrigonal (proto- or) S-pyramid $x\,\{h\,k\,l\}$ or $x\,\{e\,f\,g\}$.
The ditrigonal (proto- or) S-prism . $x\,\{p\,q\,r\}$ or $x\,\{\bar{p}\,\bar{q}\,\bar{r}\}$.
The trigonal (proto- or) S-pyramid . $x\,\{h\,k\,k\}$ or $x\,\{e\,f\,f\}$.
The trigonal (proto- or) S-prism. . $x\,\{2\bar{1}\bar{1}\}$ or $x\,\{\bar{2}11\}$.

iii. Symmetrical to the *C*- and Σ-systematic planes—

The ditrigonal (deutero- or) Σ-pyramid $\xi\{hkl, gfe\}$ or $\xi\{hlk, gef\}$.

The ditrigonal (deutero- or) Σ-prism $\xi\{pqr\}$ or $\xi\{prq\}$.

The trigonal (deutero- or) Σ-pyramid $\xi\{min\}$ or $\xi\{mni\}$.

The trigonal (deutero- or) Σ-prism $\xi\{10\bar{1}\}$ or $\xi\{1\bar{1}0\}$.

iv. Symmetrical to the *S*- and Σ-planes—

Hemimorphous hexagonal forms:—

Discalenohedral hemimorphs $\rho\{hkl, efg\}$ or $\rho\{\bar{h}\bar{k}\bar{l}, \bar{e}\bar{f}\bar{g}\}$; &c.

II. Hemi-systematic diplohedral forms; or hemihexagonal diplohedra.

i. Symmetrical ditrigonally to the *S*-systematic planes—

The ditrigonal scalenohedron $\pi\{hkl\}$ or $\pi\{efg\}$.

The rhombohedron $\pi\{hkk\}$ or $\pi\{eff\}$.

ii. Symmetrical ditrigonally to the Σ-planes—

The deutero-scalenohedron $\psi\{hkl, gfe\}$ or $\psi\{efg, lkh\}$.

The deutero-rhombohedron $\psi\{min\}$ or $\psi\{nim\}$.

iii. Symmetrical to the *C*-plane—

Gyroidal forms:—

The tritopyramid $\phi\{hkl, efg\}$ or $\phi\{lkh, gfe\}$.

The hemi-diprism $\phi\{pqr\}$ or $\phi\{rqp\}$.

III. Hemi-systematic haplohedral forms; or hemihexagonal haplohedra: each represented by four correlative forms:—

i. Normals symmetrical to the *S*-planes—

The trigonal trapezohedron $a\pi\{hkl\}$.

ii. Normals symmetrical to the Σ-planes—

The trigonal deutero-trapezohedron $a\psi\{hkl\}$.

iii. Normals symmetrical to the *C*-plane—

The skew trigonohedron $x\phi\{hkl\}$.

Hemimorphous hemihexagonal forms:—

iv. Symmetrical ditrigonally to the *S*-planes—

The hemi-protopyramid $\rho\pi\{hkl\}$.

The hemi-protorhombohedron $\rho\pi\{hkk\}$.

v. Symmetrical ditrigonally to the Σ-planes—

The hemi-deuteropyramid $\rho\psi\{h\,k\,l\}$.

The hemi-deuterorhombohedron $\rho\psi\{m\,i\,n\}$.

vi. Symmetrical to no systematic plane—

The hemi-tritopyramid $\rho\phi\{h\,k\,l\}$.

IV. Tetarto-systematic diplohedral forms; tetarto-hexagonal diplohedra.

Three extant normals.

The hemiscalenohedron or skew rhombohedron $\pi\phi\{h\,k\,l\}$.

V. Tetarto-systematic haplohedral forms.

The hemimorphous hemiscalenohedron $\rho\pi\phi\{hkl\}$; &c.

Of these combinations, the third under the holo-systematic type with the merosymmetrical prefix ξ, the second under the hemi-systematic diplohedral type distinguished by the prefix ψ, and the second under the haplohedral type of the latter division with prefix $a\psi$—have only a theoretical interest, since if any hexagonal crystal should present forms corresponding to either the ditrigonal pyramid or the scalenohedron, the systematic planes to which such forms were symmetrical would offer themselves for choice as the proto-systematic group, for it is contrary to mero-symmetrical principles that, on any crystal, forms symmetrical to only one group should be associated with such as are symmetrical to a different group of systematic planes.

The consideration of such forms has however a theoretical interest, and their recognition is so far necessary that without them a complete view of the modes of mero-symmetrical dissection of which the hexagonal discalenohedron is susceptible cannot be taken.

236. I. Holo-systematic haplohedral forms.

i. *The ditrapezohedron* $a\{hkl,\ efg\}$ or $a\{hlk,\ egf\}$, Figs. 191 (*a*), (*c*), is contained by either set of alternate faces of the discalenohedron, the other faces forming the correlative figure; all the faces of one of the semiforms are antistrophic to those of the other, and the semiforms themselves are enantiomorphous.

The symbols of the twelve faces comprised under the form-symbol $a\{hkl,\ efg\}$ are

$$\begin{array}{lll} hkl & lhk & klh \\ & & \bar{h}\bar{l}\bar{k}\quad \bar{k}\bar{h}\bar{l}\quad \bar{l}\bar{k}\bar{h} \\ efg & gef & fge \\ & & \bar{e}\bar{g}\bar{f}\quad \bar{f}\bar{e}\bar{g}\quad \bar{g}\bar{f}\bar{e}; \end{array}$$

the correlative form $a\{hlk,\ egf\}$ comprising the remaining faces of Table G, Art. **233**. Figs. 191 (*a*) and (*c*) represent the forms $a\{2\bar{1}0,\ \bar{4}52\}$ and $a\{20\bar{1},\ \bar{4}25\}$, respectively, derived from the discalenohedron $\{20\bar{1},\ \bar{4}25\}$ represented in Fig. 191 (*b*).

The faces are disposed in hexagonal (not dihexagonal) symmetry round the axis $[S\Sigma]$, so that the six culminating edges E, which unite in a quoin to form the apex of the figure at either

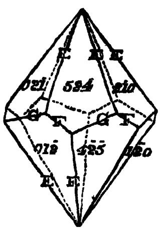


Fig. 191 (*a*).

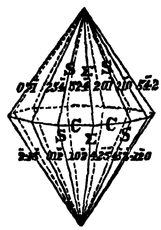


Fig. 191 (*b*).

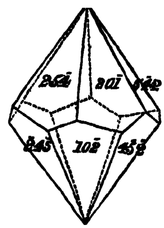


Fig. 191 (*c*).

end, are similar. But the figure has no plane or centre of symmetry. Its remaining edges F and G are of two kinds traversing the equatorial plane in zigzag. The faces are trapezoids bounded each by two edges E, an edge F, and an edge G. The twelve lateral quoins are similar, each being formed by the meeting of three dissimilar edges E, F, and G. The forms $a\{pqr\}$, $a\{hkk,\ eff\}$, $a\{min\}$, $a\{2\bar{1}\bar{1}\}$, $a\{10\bar{1}\}$ and $a\{111\}$ are geometrically identical with the holo-symmetrical forms from which they are derived. No crystal presenting this type of hemi-symmetry has yet been observed.

237. ii. and iii. To the next two subdivisions under holo-hexagonal merosymmetry belong forms which, by the condition of the division, are not centro-symmetrical. They are, however symmetrical to the trito-systematic plane, and to one or other of the triads of systematic planes S and Σ.

It has already been observed that in reality we need only take cognizance of one of these types as of actual crystal-forms; for the two cannot concur, and the occurrence of one type on a crystal would ensure that triad of systematic planes to which it was symmetrical being selected for the proto-systematic planes S.

238. ii. *The ditrigonal proto-pyramid* or *S-pyramid* $x\,\{h\,k\,l\}$ or $x\,\{e f g\}$, Figs. 192 (*a*), (*c*). In the ditrigonal proto-pyramid the symmetral character of the deutero-systematic planes is in abeyance. The dihexagonal axis of the original discalenohedron becomes a ditrigonal axis by virtue of its being the zone-axis of three similar planes of symmetry S_1, S_2, S_3. The plane C being also a plane of symmetry, the following are the constituent faces of the form $x\,\{hkl\}$,

$$h k l \quad l h k \quad k l h \qquad h l k \quad k h l \quad l k h,$$
$$\bar{e}\bar{f}\bar{g} \quad \bar{g}\bar{e}\bar{f} \quad \bar{f}\bar{g}\bar{e} \qquad \bar{e}\bar{g}\bar{f} \quad \bar{f}\bar{e}\bar{g} \quad \bar{g}\bar{f}\bar{e}.$$

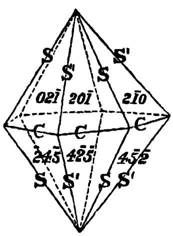


Fig. 192 (*a*).

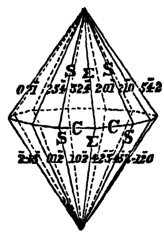


Fig. 192 (*b*).

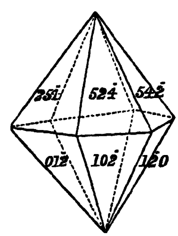


Fig. 192 (*c*).

They correspond to the groups p, p', $\bar{q}$, $\bar{q}'$ of Table G: the form $x\,\{efg\}$ comprises the remaining faces of Table G, viz. those of the groups $\bar{p}$, $\bar{p}'$, q, q'. Figs. 192 (*a*) and (*c*) represent the forms $x\,\{20\bar{1}\}$ and $x\,\{\bar{4}25\}$ respectively derived from the discalenohedron $\{20\bar{1}, \bar{4}25\}$ represented in Fig. 192 (*b*).

The faces of the discalenohedon are thus seen to be extant or to be suppressed in alternate pairs in either hemisphere, and in adjacent pairs on opposite sides of the equatorial plane. They are triangular, and, as their adjacent faces are antistrophic to each other, they are scalene. The edges of each face are therefore of three kinds; two, S and S', are culminating edges lying in the S-planes of symmetry, and, of these, six—three of each kind—meet on the $[S]$ axis in a quoin at each apex of the figure: the third edge of each face lies in the C-plane, and of these similar edges C the figure has six. Hence any section of the form perpendicular to the ditrigonal axis is a symmetrical hexagon: the aspect of the form is that of a double or crystallographic pyramid having no centre of symmetry; and its lateral quoins are of two kinds, three of them formed by two edges S and two edges C, and three, alternating with these, formed by two edges S' and two edges C; and they are ortho-symmetrical on the $[CS]$ lateral axes. The two correlative figures are identical in their aspect, as seen in opposite directions of an axis $[SC]$: and, from the antistrophic character of adjacent faces in each, the figures are tautomorphous, a revolution of either form through 60°, or any odd multiple of 60°, round the $[S]$ axis bringing it into congruence with the correlative form.

239. *The ditrigonal proto-prism* or *S-prism* $x\{pqr\}$ or $x\{\bar{p}\bar{q}\bar{r}\}$, Figs. 193 (*a*), (*c*), may be regarded as truncating the basal edges

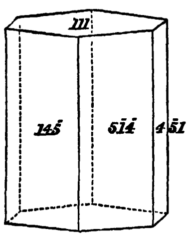


Fig. 193 (*a*).

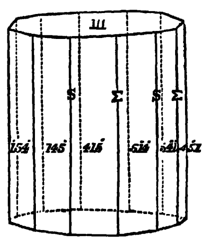


Fig. 193 (*b*).

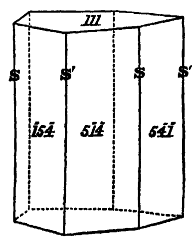


Fig. 193 (*c*).

of a ditrigonal proto-pyramid, or as a limiting case of the same figure in which the faces have become parallel to the ditrigonal axis and pairs of faces symmetrical to the equatorial plane have thus become coincident. It is an open six-sided prism without a centre of symmetry: the edges are of two kinds S and S' and lie in the S-planes of symmetry.

The faces of the form $x\{pqr\}$ have the symbols

$$pqr \quad rpq \quad qrp \quad prq \quad qpr \quad rqp,$$

the symbols of faces of the correlative form $x\{\bar p \bar q \bar r\}$ being

$$\bar p \bar q \bar r \quad \bar r \bar p \bar q \quad \bar q \bar r \bar p \quad \bar p \bar r \bar q \quad \bar q \bar p \bar r \quad \bar r \bar q \bar p.$$

Figs. 193 (*a*) and (*c*) represent the forms $x\{5\bar 1\bar 4\}$ and $x\{\bar 5 1 4\}$, respectively, derived from the diprism $\{5\bar 1\bar 4\}$ shown in Fig. 193 (*b*).

240. *The trigonal proto-pyramid* or *S-pyramid* $x\{h k k\}$ or $x\{e f f\}$, Figs. 194 (*a*), (*c*), may be regarded either as truncating the *S*- or *S'*-edges, respectively, or as being a limiting case of the ditrigonal proto-pyramid, in which pairs of faces symmetrical to the *S*-planes of symmetry have become coincident. The figure s thus a double pyramid with an equilateral triangle for base. There are six similar culminating edges *S* or *S'* lying in the *S*-planes of symmetry, and three similar basal edges *C* lying in the equatorial plane.

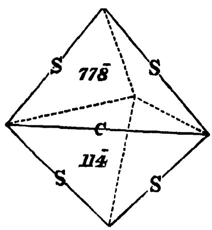


Fig. 194 (*a*).

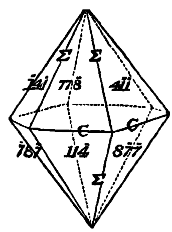


Fig. 194 (*b*)

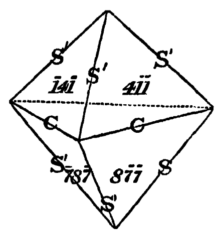


Fig. 194 (*c*).

The symbols of the faces of the form $x\{h k k\}$ are

$$h k k \quad k h k \quad k k h$$
$$\bar e \bar f \bar f \quad \bar f \bar e \bar f \quad \bar f \bar f \bar e,$$

those of the form $x\{e f f\}$ being

$$\bar h \bar k \bar k \quad \bar k \bar h \bar k \quad \bar k \bar k \bar h$$
$$e f f \quad f e f \quad f f e.$$

Figs. 194 (*a*) and (*c*) represent the forms $x\{\bar 8 7 7\}$ and $x\{4\bar 1\bar 1\}$, respectively, derived from the dirhombohedron $\{4\bar 1\bar 1, \bar 8 7 7\}$ represented in Fig. 194 (*b*).

241. *The trigonal proto-prism* or *S-prism* $x\{2\bar 1\bar 1\}$ or $x\{\bar 2 1 1\}$, Figs. 195 (*a*), (*c*), is an open prism with an equilateral triangle for base. The poles lie at the intersections of the *S*- and *C*-planes, and the three similar edges *S* or *S'* lie in the *S*-planes of symmetry.

The faces of the form $x\{2\bar{1}\bar{1}\}$ are $2\bar{1}\bar{1}$, $\bar{1}2\bar{1}$, $\bar{1}\bar{1}2$,
those of the form $x\{\bar{2}11\}$ being $\bar{2}11$, $1\bar{2}1$, $11\bar{2}$.

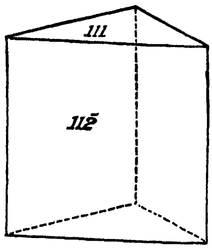


Fig. 195 (*a*.)

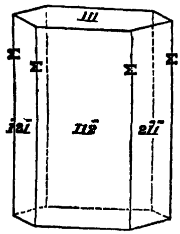


Fig. 195 (*b*).

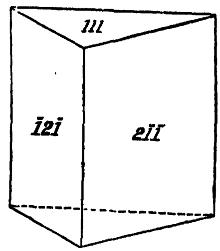


Fig. 195 (*c*).

Figs. 195 (*a*) and (*c*) represent the forms $x\{\bar{2}11\}$ and $x\{2\bar{1}\bar{1}\}$, respectively, derived from the hexagonal proto-prism $\{2\bar{1}\bar{1}\}$ represented in Fig. 195 (*b*).

The forms $x\{m\,i\,n\}$, $x\{10\bar{1}\}$, and $x\{111\}$ are geometrically identical with the holo-symmetrical forms from which they are derived.

242. iv. Hemimorphous hexagonal forms.

By the condition of this subdivision only the *S*- and Σ-systematic planes retain their symmetry. Hence the faces of any form be-

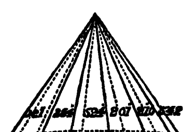


Fig. 196 (*a*).

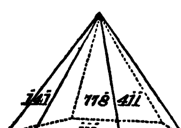


Fig. 196 (*b*).

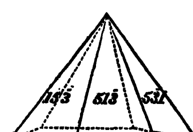


Fig. 196 (*c*).

longing to it will all lie at one side of the equatorial plane, and the figure will be represented by the upper or lower half of the corresponding holo-symmetrical form. The only forms which are geometrically distinct from those presenting holo-symmetry are

$\rho\{hkl, efg\}$, $\rho\{hkk, eff\}$, $\rho\{min\}$, and $\rho\{111\}$, with their correlatives.

Figs. 196 (*a*), (*b*), (*c*) represent respectively the forms

$$\rho\{20\bar{1}, \bar{4}25\}, \ \rho\{4\bar{1}\bar{1}, \bar{8}77\}, \text{ and } \rho\{51\bar{3}\},$$

closed by the form $\rho\{\bar{1}\bar{1}\bar{1}\}$.

243. II. Hemi-systematic diplohedral forms.

The characteristic distinction of the mero-symmetrical forms of the Hexagonal as contrasted with those of the Tetragonal system is, that in the Tetragonal system centro-symmetry and symmetry to the equatorial plane are concurrent conditions, whereas in the Hexagonal system these two kinds of mero-symmetry are quite independent of each other; so much so indeed that a centro-symmetrical semiform in the latter system can only be symmetrical to the equatorial plane in the case of a form which is symmetrical to no other plane than this.

Of diplohedral semiforms therefore we have two kinds. In the one kind, the normals are symmetrical to one or other but not to both of the systematic plane-triads, and each carries its two faces; in the other kind, alternate normals, symmetrically distributed round the axis of form as a hexagonal (not dihexagonal) axis, are diplohedral. In the former kind the discalenohedron is dissected into two similar scalenohedra, in the latter kind the partition is into two correlative forms of which each is a double pyramid, with a geometrically equilateral hexagonal base.

The scalenohedra can in theory be of two kinds; proto-scalenohedra symmetrical to the proto-systematic planes *S*, and deutero-scalenohedra symmetrical to the deutero-planes Σ. The three planes, however, to which the scalenohedral forms are symmetrical being those which naturally offer themselves as the proto-systematic planes, we have only to deal practically with the scalenohedron as a proto-systematic diplohedral form.

In accordance with the notation adopted for such a form its symbol will be $\pi\{hkl\}$ or $\pi\{efg\}$; that of a deutero-scalenohedron would be $\psi\{hkl, gfe\}$ or $\psi\{hlk, egf\}$.

244. i. *The ditrigonal scalenohedron*, $\pi\{hkl\}$ or $\pi\{efg\}$, Figs. 197 (*a*), (*c*). The symbols of the twelve faces of the scalenohedron

$\pi\{h\,k\,l\}$ symmetrical to the proto-systematic planes S correspond to the groups $p, p', \bar{p}, \bar{p}'$ in Table G (Art. **233**) and are

$$h\,k\,l \quad l\,h\,k \quad k\,l\,h \qquad h\,l\,k \quad k\,h\,l \quad l\,k\,h,$$

$$\bar{h}\,\bar{k}\,\bar{l} \quad \bar{l}\,\bar{h}\,\bar{k} \quad \bar{k}\,\bar{l}\,\bar{h} \qquad \bar{h}\,\bar{l}\,\bar{k} \quad \bar{k}\,\bar{h}\,\bar{l} \quad \bar{l}\,\bar{k}\,\bar{h};$$

those of the form $\pi\{efg\}$ lie in the two lower rows of Table G, and correspond to the groups $q, q', \bar{q}, \bar{q}'$. They are,

$$e\,f\,g \quad g\,e\,f \quad f\,g\,e \qquad e\,g\,f \quad f\,e\,g \quad g\,f\,e,$$

$$\bar{e}\,\bar{f}\,\bar{g} \quad \bar{g}\,\bar{e}\,\bar{f} \quad \bar{f}\,\bar{g}\,\bar{e} \qquad \bar{e}\,\bar{g}\,\bar{f} \quad \bar{f}\,\bar{e}\,\bar{g} \quad \bar{g}\,\bar{f}\,\bar{e}.$$

Each face of the form is a scalene triangle, of which two of the edges S and S', culminating in a ditrigonal quoin on the axis $[S]$, lie in the planes S, while the third edge lies obliquely to the plane C.

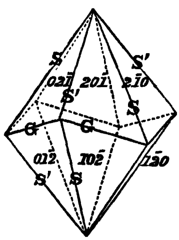


Fig. 197 (*a*).

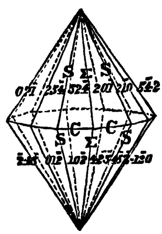


Fig. 197 (*b*).

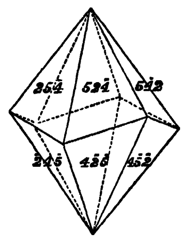


Fig. 197 (*c*).

The figure has therefore two ditrigonal quoins forming its vertices, and constituted by three edges S alternating with three edges S'.

It has moreover six lateral edges G, which are similar but lie in zigzag athwart the plane C, and meet in six quoins similar but alternately inverted in position, in each of which an edge S and an edge S' meet two of the zigzag edges G symmetrically as regards a plane S. The two correlative scalenohedra are tautomorphous, since adjacent faces on each form are antistrophic to each other. And a revolution of either figure through an angle that is an odd multiple of 60° brings it into congruence with its correlative form.

Figs. 197 (*a*) and (*c*) represent respectively the forms $\pi\{20\bar{1}\}$ and $\pi\{\bar{4}25\}$ derived from the discalenohedron $\{20\bar{1}, \bar{4}25\}$.

245. *The ditrigonal prism,* $\pi\{pqr\}$. The zone-circles passing through the pole 111 and successive poles of a scalenohedron will intersect the zone-circle [S] in the poles of a prism the faces of which are parallel to the axis of the zone [S].

From a geometrical point of view the figure, considered as an open form, is identical with the diprism $\{p\,q\,r\}$. Crystallographically however it is not so, as is evident when a ditrigonal form is combined with the prism. The ditrigonal prism is in fact the limiting form of the two correlative scalenohedra, in the case in which their faces become parallel to the morphological axis and their poles fall on the equatorial zone-circle.

246. *The rhombohedron,* $\pi\,\{h\,k\,k\}$ or $\pi\,\{e\,f\,f\}$, Figs. 198 (*a*), (*c*), 199 (*a*), (*c*). The relation of the rhombohedron to the scalenohedron is analogous to that of the dirhombohedron to the discalenohedron; and, as the faces of two correlative scalenohedra have, in the one, symbols into which only the literal indices $h\,k\,l$, and

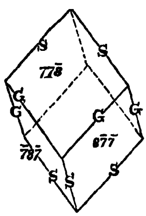


Fig. 198 (*a*).

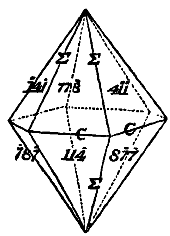


Fig. 198 (*b*).

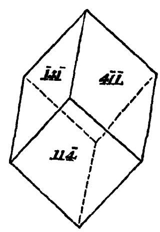


Fig. 198 (*c*).

in the other, symbols into which only the literal indices $e\,f\,g$ enter, so the faces of two correlative rhombohedra have in their symbols the indices $h\,k\,k$ and $e\,f\,f$ respectively.

If the normal-angle of two adjacent faces of the scalenohedron be supposed to become zero, the faces come to lie in a plane, and each pair of antistrophic faces becomes a single quadrilateral face, euthysymmetrical to the line originally representing their edge, which is the trace on the face of a plane of symmetry S: the opposite angles are equal in pairs, and the form of the face is geometrically a rhomb. Crystallographically however it is not so, since its adjacent culminating edges which represent the S'-edges of the scalenohedron are not similar to those which represent the lateral or G-edges of the latter figure, though they have become parallel to them.

The twelve faces of the scalenohedron being represented by six faces in the rhombohedron the figure is a rhombic hexahedron or six-faced rhombohedron, and may be acute or obtuse according as the three-edged quoins which are the vertices on the morphological axis are formed by edges which meet in three acute or three obtuse angles.

Figs. 198 (*a*) and (*c*) represent the correlative acute rhombohedra $\pi\,\{\bar{8}77\}$ and $\pi\,\{4\bar{1}\bar{1}\}$ derived from the dirhombohedron $\{4\bar{1}\bar{1}, \bar{8}77\}$ of Fig. 198 (*b*), while Figs. 199 (*a*) and (*c*) represent the correlative obtuse rhombohedra $\pi\,\{\bar{1}22\}$ and $\pi\,\{100\}$ derived from the dirhombohedron $\{100, \bar{1}22\}$ of Fig. 199 (*b*).

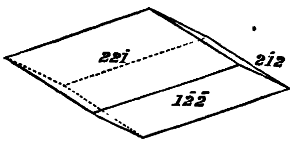


Fig. 199 (*a*).

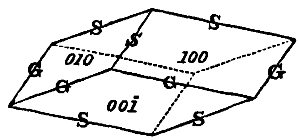


Fig. 199 (*c*).

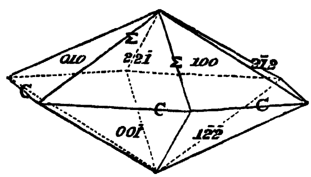


Fig. 199 (*b*).

The summit-quoins are symmetrical ditrigonally on the [*S*] axis. The six other quoins are symmetrical to no axis, but only to the *S*-planes of symmetry, and are each formed by the meeting of a culminating edge *S* with two of the six lateral or zigzag edges *G*. In each lateral quoin of an acute rhombohedron two lateral edges make an acute angle with each other and an obtuse angle with a summit-edge. In each lateral quoin of an obtuse rhombohedron two lateral edges make an obtuse angle with each other and meet a summit-edge in two acute angles. The alternate quoins are directly congruent; those intermediate to them are also congruent with them, but are inverted in their orientation relative to the morphological axis.

The limiting form dividing the acute from the obtuse varieties of the rhombohedron is the cube, the faces of which are geometrically squares. The Cubic system is in fact a developement of the Rhombohedral system resulting from the condition that, when the faces of the rhombohedron become squares, the zigzag edges become crystallographically, as well as geometrically, similar to the culminating edges.

247. *The trigonal dihexahedron*, $\pi\{m\,i\,n\}$. Should the poles of a scalenohedron fall on the great circles corresponding to the deutero-systematic planes, the resulting twelve-sided figure would evidently be a limiting form between the two series of correlative scalenohedra. Such a form is geometrically identical with the hexagonal dihexahedron already considered, but its adjacent faces are crystallographically antistrophic.

The forms $\pi\{2\bar{1}\bar{1}\}$, $\pi\{10\bar{1}\}$ and $\pi\{111\}$ will likewise be geometrically identical with the corresponding holo-symmetrical forms.

248. The zigzag edges of a scalenohedron, and therefore also those of a rhombohedron, are truncated by the faces of the hexagonal deutero-prism $\{10\bar{1}\}$. For if the poles of two adjacent faces meeting in one of these lateral edges be considered, the poles $h\,k\,l$ and $\bar{l}\,\bar{k}\,\bar{h}$ for instance, in the one form, or the poles $h\,k\,k$ $\bar{k}\,\bar{k}\,\bar{h}$ in the other, the sum of their indices gives the symbol $(10\bar{1})$ for the face truncating the edge; and similarly for the other edges of the form. And the lateral quoins in both figures will be replaced symmetrically by the faces of the hexagonal proto-prism $\{1\bar{2}1\}$, which are perpendicular to the proto-systematic planes to which these quoins are symmetrical. The summit-edges of the scalenohedron will be truncated by the faces of rhombohedra, direct or inverse according as the poles of the scalenohedron faces forming these edges are disposed symmetrically in respect to such quadrants of the systematic great circles *S* as contain the poles of the form $\{100\}$, or to the remaining quadrants of those great circles. In fact, the edge of the faces $h\,k\,l$ and $h\,l\,k$ is truncated by the face $(2h\ \ k+l\ \ k+l)$, the symbol of which is obtained by adding the indices of the two symbols, and has the form $h'k'k'$, while the edge of the faces $l\,h\,k$ and $l\,k\,h$ is truncated by the face $(2l\ \ h+k\ \ h+k)$, the symbol of which has the form $e'f'f'$.

249. The indices of a face will all have positive values where the pole of the face lies within the spherical triangle of which the angular points are at the poles 100, 010, and 001, and they will be all negative when the pole lies in the triangle with its angular points at the poles $\bar{1}00$, $00\bar{1}$, $0\bar{1}0$. The values of one or of two of the indices will be negative in every other symbol.

250. Forms consisting of two correlative scalenohedra or rhombohedra associated together but differing in their physical characters are of very frequent occurrence; so frequent in fact that the existence of truly holo-symmetrical hexagonal crystals has been disputed: Professor Miller indeed only recognised a Rhombohedral system, holosymmetrical in itself, and referred no crystals to a Hexagonal system. That there exists, inherent in the laws which regulate the coordination of the crystal-molecules, a cause for the great prevalence of trigonal diplohedral symmetry in crystals belonging to this type may be accepted as a result of experience; but that it is a cause which precludes the recognition of a Hexagonal system, such as is necessary to the theoretical completeness of the principles of geometrical mero-symmetry, is hardly to be admitted. It would not be so were the evidence against the existence among known crystals of holo-symmetrical hexagonal forms complete; but in fact this evidence is not complete, and there are crystals, of which those of beryl may be taken as an example, which in their cleavage and in the physical as well as geometrical characters of their forms offer no sufficient grounds for referring them to a trigonal rather than a hexagonal type of symmetry.

251. ii. *The deutero-scalenohedron*, $\psi\,\{hkl\}$ and *deutero-rhombohedron*, $\psi\,\{min\}$ need only to be noticed as forms of which some cognizance must be taken in a complete review of the various mero-symmetrical partitions of which the faces of a discalenohedron and a hexagonal pyramid are theoretically susceptible.

Detailed descriptions of these deutero-symmetrical forms are unnecessary, inasmuch as in everything but in the form of their symbols and the particular planes to which they would be symmetrical they are identical in character with the forms considered in the foregoing articles.

252. iii. *Gyroidal forms.* There remains a group of centrosymmetrical forms belonging to the hemi-systematic division of the Hexagonal system, which present no symmetry to the proto- or deutero-triad of systematic planes, but which are symmetrical to the trito-systematic plane *C*. The morphological axis is an axis of simply hexagonal symmetry for such forms, in which therefore of the faces of the discalenohedron which lie on the same side of the equatorial plane only alternate ones can be extant. It is only the discalenohedron and the diprism that will present geometrically distinct forms under this type, and from their symmetry to the plane *C* they have been termed the trito-pyramid and trito-prism or hemi-diprism.

253. *The trito-pyramid,* $\phi\,\{hkl,\ efg\}$ or $\phi\,\{lkh,\ gfe\}$, Figs. 200 (*a*), (*c*), differs from the ditrapezohedron in that, while in the

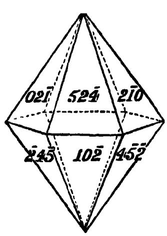


Fig. 200 (*a*).

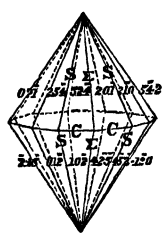


Fig. 200 (*b*).

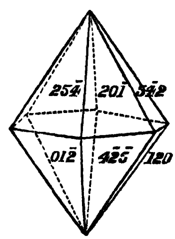


Fig. 200 (*c*).

latter figure the whole of the faces of a semiform were metastrophic, only those on one side of the plane *C* are so in the trito-pyramid, those on the other side being symmetrical, and therefore antistrophic, to the former. The correlative forms are thus tautomorphous, being capable of being brought into congruence by a rotation round the morphological axis, after interchange in the positions of the vertices of one of them.

The figure is formed by two hexagonal pyramids, the base common to which is a regular hexagon. Its two summit-quoins are formed each by six similar edges, and each of its six lateral quoins by the meeting of two of these edges with two of the basal edges, which are all similar and lie in the equatorial plane. It is evident however from the gyroidal position of its faces in respect to the vertical axis, that none of its summit-edges can undergo truncation, since no two planes of the system can be equally inclined on them.

The symbols of the faces of the form $\phi\{hkl, efg\}$ are contained in the left-hand half of the Table G (Art. **233**). They are,

$$\begin{array}{lll} h\,k\,l & l\,h\,k & k\,l\,h \\ \bar{h}\,\bar{k}\,\bar{l} & \bar{l}\,\bar{h}\,\bar{k} & \bar{k}\,\bar{l}\,\bar{h}, \\ e\,f\,g & g\,e\,f & f\,g\,e \\ \bar{e}\,\bar{f}\,\bar{g} & \bar{g}\,\bar{e}\,\bar{f} & \bar{f}\,\bar{g}\,\bar{e}. \end{array}$$

Those of the form $\phi\{lkh, gfe\}$ are contained in the right-hand half of that Table.

Figs. 200 (*a*) and (*c*) represent respectively the forms $\phi\{20\bar{1}, \bar{4}25\}$ and $\phi\{\bar{1}02, 52\bar{4}\}$ derived from the discalenohedron $\{20\bar{1}, \bar{4}25\}$ represented in Fig. 200 (*b*).

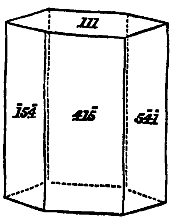


Fig. 201 (*a*).

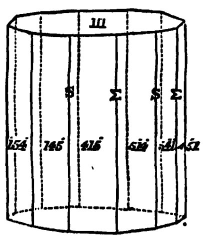


Fig. 201 (*b*).

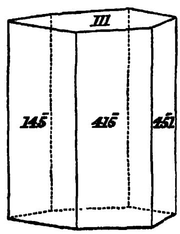


Fig. 201 (*c*).

254. The *hemi-diprism* or *trito-prism*, $\phi\{pqr\}$ or $\phi\{prq\}$, Figs. 201 (*a*), (*c*), has the geometrical character of a regular hexagonal prism, its edges having normal-angles of 60°. But its poles do not lie at angles of the systematic triangles, and the faces are crystallographically symmetrical only to the equatorial plane. Figs. 201 (*a*) and (*c*) represent respectively the forms $\phi\{5\bar{4}\bar{1}\}$ and $\phi\{5\bar{1}\bar{4}\}$ derived from the diprism $\{5\bar{1}\bar{4}\}$ shown in Fig. 201 (*b*). The forms $\phi\{hkk, eff\}$, $\phi\{min\}$, $\phi\{2\bar{1}\bar{1}\}$, $\phi\{10\bar{1}\}$, and $\phi\{111\}$

are geometrically identical with the corresponding holo-symmetrical forms.

255. III. and IV. Tetarto-symmetry.

The distinct types of tetartohedra that may occur in the Hexagonal system, apart from those resulting from hemimorphism, are four in number.

Three of these types are haplohedral, and in them each of the six alternate normals of the discalenohedron is represented by a single face; they correspond, in fact, to the six different hemi-systematic forms from which they are the haplohedral derivatives. The fourth is a unique case of tetarto-symmetry in which three normals only are represented, but by both their faces: it is therefore of diplohedral tetarto-systematic type (Art. **265**).

Four correlative tetartohedra admit of being combined in three different ways, and so can only reproduce three distinct pairs of correlative hemi-symmetrical forms. It will thus be seen that four different tetrads are needed to build up six distinct hemi-symmetrical derivatives from the discalenohedron; for one pair of hemi-symmetrical forms must be common to any two of the groups that are built up by separate tetrads of tetarto-symmetrical forms.

All four of the tetarto-symmetrical forms alluded to must therefore take part in the construction of the six hemi-symmetrical forms which are not hemimorphous. The projections in plates IV to VII exhibit the relations of these forms to one another.

The tetartohedral forms of whatever kind may be treated on a principle analogous to that followed for the hemi-symmetrical forms. In the latter, the symmetrical distribution of the *poles* of a form in respect to the different systematic planes was taken for this purpose. In the tetarto-symmetrical forms it is, further, the distribution of those *normals* which are conceived as extant that determines the nature of the resulting quarter-forms.

Thus the normals which are represented each by a pole may be symmetrical to the proto-, to the deutero-, or to the trito-systematic planes; or, symmetrical to none of these, they may be gyroidally grouped round the axis of form. A form of the last kind is however necessarily hemimorphous in its character.

256. III. Hemi-systematic haplohedral forms.

If the six normals are to be symmetrical in respect to the proto-systematic planes, they will be the same normals which in a ditrigonal scalenohedron $\pi\{hkl\}$ are diplohedral; or which in the trapezohedron $a\{hkl,\ efg\}$ and ditrigonal Σ-pyramid $\xi\{hkl,\ gfe\}$ are haplohedral. Now these normals may carry their six faces all on one side of the plane C in a simply hemimorphous way (Art. **261**); or, the faces may be distributed partly above and partly below that plane, and this can only occur in one way, that namely indicated in the projections in Plate V.

257. i. *The trigonal trapezohedron*, $a\pi\{hkl\}$, Figs. 202 (*a*), (*c*), (*d*), (*f*). The figure corresponding to any one of these four projections has trapezoids for its faces, which meet in two trigonal quoins on the $[S]$ axis, and have in a crystallographic sense three sorts of edges. The faces of each tetartohedron are mutually metastrophic. The symbols of the faces of the four quarter-forms are as follows:—

$$(1)\quad a\pi\{hkl\}\quad \begin{cases} hkl \quad lhk \quad klh \quad \bar{h}\bar{l}\bar{k} \quad \bar{k}\bar{h}\bar{l} \quad \bar{l}\bar{k}\bar{h} \quad \text{or} \\ \qquad\quad p \qquad\qquad\qquad\qquad \bar{p}' \end{cases}$$

$$(2)\quad a\pi\{hlk\}\quad \begin{cases} \bar{h}\bar{k}\bar{l} \quad \bar{l}\bar{h}\bar{k} \quad \bar{k}\bar{l}\bar{h} \quad hlk \quad khl \quad lkh \quad \text{or} \\ \qquad\quad \bar{p} \qquad\qquad\qquad\qquad p' \end{cases}$$

$$(3)\quad a\pi\{efg\}\quad \begin{cases} efg \quad gef \quad fge \quad \bar{e}\bar{g}\bar{f} \quad \bar{f}\bar{e}\bar{g} \quad \bar{g}\bar{f}\bar{e} \quad \text{or} \\ \qquad\quad q \qquad\qquad\qquad\qquad \bar{q}' \end{cases}$$

$$(4)\quad a\pi\{egf\}\quad \begin{cases} \bar{e}\bar{f}\bar{g} \quad \bar{g}\bar{e}\bar{f} \quad \bar{f}\bar{g}\bar{e} \quad egf \quad feg \quad gfe \quad \text{or} \\ \qquad\quad \bar{q} \qquad\qquad\qquad\qquad q' \end{cases}$$

Of these forms the first and third, and the second and fourth, are tautomorphous, but, taken otherwise in pairs, the forms are enantiomorphous. In Figs. 202 (*a*) and (*c*) are represented the forms $a\pi\{20\bar{1}\}$ and $a\pi\{2\bar{1}0\}$, which together compose the form $\pi\{20\bar{1}\}$ shown in Fig. 202 (*b*): the remaining correlative tetartohedral forms $a\pi\{\bar{4}25\}$ and $a\pi\{\bar{4}52\}$ shown in Figs. 202 (*d*) and (*f*) together compose the form $\pi\{\bar{4}25\}$ represented in Fig. 202 (*e*).

The two enantiomorphous groups $a\pi\{hkl\}$ and $a\pi\{hlk\}$ are of great interest from the occurrence of several forms of one or other of these particular groups on crystals of certain minerals, of which quartz is a conspicuous example; the particular group, when it occurs on the crystal, indicating by its presence the nature of a rotatory influence of the crystal on a ray of polarised light traversing it in the direction of the optic axis, which is the morphological axis $[S\Sigma]$. In a crystal of quartz the forms $a\pi\{hkl\}$ are indicative of lævo-rotatory, the forms $a\pi\{hlk\}$ of dextro-rotatory action.

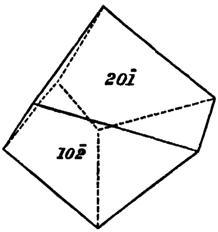


Fig. 202 (*a*).

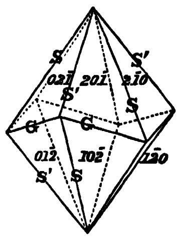


Fig. 202 (*b*).

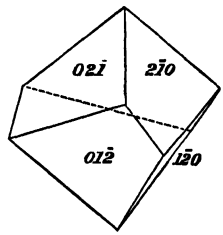


Fig. 202 (*c*).

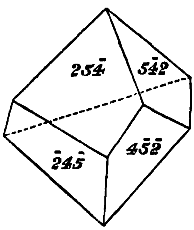


Fig. 202 (*d*).

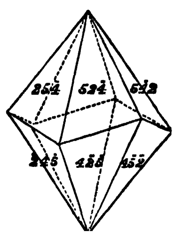


Fig. 202 (*e*).

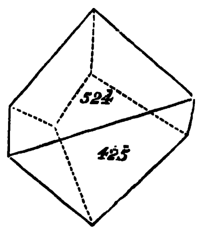


Fig. 202 (*f*).

The form $a\pi\{hkk\}$ is geometrically identical with the form $\pi\{hkk\}$, and the forms $a\pi\{2\bar{1}\bar{1}\}$, $a\pi\{111\}$ with the corresponding holo-symmetrical forms; the remaining forms $a\pi\{pqr\}$, $a\pi\{min\}$, and $a\pi\{10\bar{1}\}$ are geometrically identical with the ditrigonal forms $\xi\{pqr\}$, $\xi\{min\}$, and $\xi\{10\bar{1}\}$, and only differ from the forms represented in Figs. 193, 194, and 195 in that they are symmetrical to the Σ-planes instead of the S-planes.

258. ii. *The trigonal deutero-trapezohedron*, $a\psi\{hkl\}$. The six normals to which its extant faces belong will be symmetrical to the deutero-systematic planes, the faces of any two symmetrically placed normals being in opposite hemispheres. The symbols $a\psi\{hkl\}$, $a\psi\{hlk\}$, $a\psi\{efg\}$, $a\psi\{egf\}$ would represent the tetrad of forms. *Mutatis mutandis*, the figure is quite similar to that of the trigonal trapezohedron described in the last article.

If the extant poles of the pairs of symmetrical normals lay on the same hemisphere, the form would be the hemimorphous tetartohedron $\rho\psi\{hkl\}$, (Art. **263**).

259. Where the six extant normals are in pairs symmetrical to the trito-systematic plane, the faces will either all lie on one side of that plane and form a gyroidal hemimorphous form $\rho\phi\{hkl\}$, (Art. **264**), or they will be distributed symmetrically in respect to the plane C and form a pyramidal figure in which two trigonal pyramids (of which the faces, though geometrically isosceles, are crystallographically scalene) stand base to base and are trigonally symmetrical on the morphological axis of the system.

260. iii. *The skew trigonohedron*, $x\phi\{hkl\}$, Figs. 203 (*a*), (*c*), (*d*), (*f*). The faces of the form which carries the face hkl are those which are common to the forms $\phi\{hkl\}$, $x\{hkl\}$, and $\xi\{hkl\}$, their symbols being

$$hkl \quad lhk \quad klh$$
$$\bar{e}\bar{f}\bar{g} \quad \bar{g}\bar{e}\bar{f} \quad \bar{f}\bar{g}\bar{e}.$$

The summit-quoins are three-faced, formed by trigonally similar culminating edges metastrophic to each other, which can only be replaced by planes which do not truncate them; so that two of these edges belonging to a face are not alike in their relations to that face. The three lateral quoins are four-faced, two culminating edges alternating with two lateral edges, but they are symmetrical to no axis. In Figs. 203 (*a*) and (*c*) are represented the forms $x\phi\{20\bar{1}\}$ and $x\phi\{\bar{4}25\}$, which together compose the form $\phi\{20\bar{1}, \bar{4}25\}$ shown in Fig. 203 (*b*); the remaining correlative tetartohedral forms $x\phi\{\bar{4}52\}$ and $x\phi\{2\bar{1}0\}$ shown in Figs. 203 (*d*) and (*f*) together compose the form $\phi\{2\bar{1}0, \bar{4}52\}$ represented in Fig. 203 (*e*).

The form $x\phi\{pqr\}$ would be a skew three-sided prism with an equilateral section, and would consist of the three faces pqr, rpq, qrp: the forms $x\phi\{hkk\}$ and $x\phi\{2\bar{1}\bar{1}\}$ would be geometrically identical with $x\{hkk\}$ and $x\{2\bar{1}\bar{1}\}$, and $x\phi\{min\}$, $x\phi\{10\bar{1}\}$ with $\xi\{min\}$, $\xi\{10\bar{1}\}$: the form $x\phi\{111\}$ would carry both its faces.

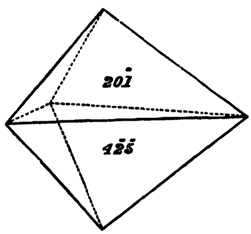


Fig. 203 (*a*).

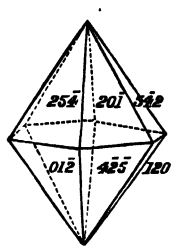


Fig. 203 (*b*).

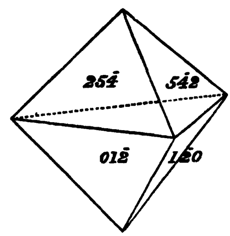


Fig. 203 (*c*).

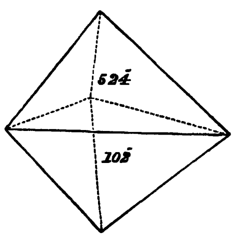


Fig. 203 (*d*).

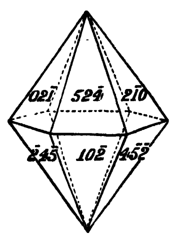


Fig. 203 (*e*).

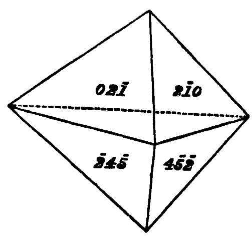


Fig. 203 (*f*).

No crystal is as yet known which carries the faces of a skew trigonohedron.

261. iv. v. vi. *Hemimorphous tetartohedral forms.* The hemimorphous derivatives from hemi-symmetrical forms will be represented by three tetartohedral modes of dissecting the general hemimorphous form $\rho\{hkl, efg\}$, or $\rho\{\bar{h}\bar{k}\bar{l}, \bar{e}\bar{f}\bar{g}\}$, and the forms $\rho\{hkk, eff\}$, $\rho\{min\}$, $\rho\{pqr\}$, &c.

Each tetrad of correlative tetartohedral forms in this division will therefore reproduce the two correlative hemihedral forms with the above symbols, and also two corresponding correlatives of the six remaining hemi-symmetrical forms.

Their symbols, taking that one of the four forms which contains a pole hkl for example, are,

iv. the hemi-protopyramid $\rho\pi\{hkl\}$,

v. the hemi-deuteropyramid $\rho\psi\{hkl\}$,

vi. the hemi-tritopyramid $\rho\phi\{hkl\}$.

262. iv. *The hemi-protopyramid*, or, more completely expressed, *the hemimorphous ditrigonal protopyramid*, as represented by the form $\rho\pi\{hkl\}$, is the form the normals of which are symmetrical to the *S*-planes (Fig. 204): its faces therefore are those of the upper half

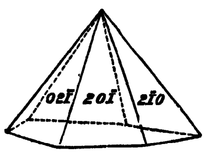


Fig. 204.

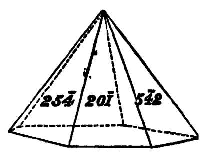


Fig. 205.

of a scalenohedron $\pi\{hkl\}$, or of a ditrigonal *S*-pyramid $x\{hkl\}$, as well as of $\rho\{hkl\}$. The symbols of the tetrad are

The hemimorphous ditrigonal prism $\rho\pi\{pqr\}$ is a ditrigonal prism with six faces, forming a symmetrical hexagon, of which the alternate edges are similar and adjacent edges dissimilar, as are the forms which close its opposite ends.

The hemimorphous rhombohedron and *hemimorphous trigonal S-pyramid* represent in the forms $\rho\pi\{hkk\}$ and $\rho x\{hkk\}$ the upper halves of those figures; geometrically the same figures, they do not differ crystallographically. Their symbols, taking $\rho\pi$ for the designating letters, are

$$\rho\pi\{hkk\}, \quad \rho\pi\{eff\}, \qquad \rho\pi\{\bar{h}\bar{k}\bar{k}\}, \quad \rho\pi\{\bar{e}\bar{f}\bar{f}\}.$$

A hemimorphous dihexahedron $\rho\pi\{min\}$ would be identical with the form $\rho\{min\}$, excepting in its crystallographic association with ditrigonal forms.

263. v. *The hemi-deuteropyramid* $\rho\psi\{h\,k\,l\}$, *hemi-deuteroprism* $\rho\psi\{p\,q\,r\}$, and *hemi-deuterorhombohedron* $\rho\psi\{m\,i\,n\}$, would be analogous forms to those just described if the deutero-systematic series of forms were to be recognised as actual forms; but they have only a theoretical interest, for the same reason as that which removes the ditrigonal deutero-scalenohedron and the deutero-rhombohedron from among the forms belonging to practical crystallography.

264. vi. The normals represented in *the hemi-tritopyramid* are symmetrical to no systematic plane, but are hemihexagonally disposed round the axis of form. In the form taken as example, they are the normals which are common to the forms $\rho\{h\,k\,l,\ e\,f\,g\}$, $a\,\{hkl\}$, and $\phi\,\{h\,k\,l\}$, (Fig. 205). The tetrad is composed of the forms

$$\rho\phi\{h\,k\,l,\ e\,f\,g\},\qquad \rho\phi\{h\,l\,k,\ e\,g\,f\},$$
$$\rho\phi\{\bar{h}\,\bar{k}\,\bar{l},\ \bar{e}\,\bar{f}\,\bar{g}\},\qquad \rho\phi\{\bar{h}\,\bar{l}\,\bar{k},\ \bar{e}\,\bar{g}\,\bar{f}\}.$$

The hemimorphous tritoprism $\rho\phi\{pqr\}$ differs crystallographically from the hemi-diprism $\phi\{pqr\}$ in the different character of the forms closing its opposite extremities, and, so far only, is also geometrically different.

265. IV. Tetarto-systematic diplohedral forms.

The hemiscalenohedron or *skew rhombohedron* $\pi\phi\{hkl\}$, Figs. 206 (*a*), (*c*), (*d*), (*f*), belongs to the remaining tetarto-symmetrical type, namely, that in which each of the three extant normals carries both its faces.

The symbols of the faces of the form $\pi\phi\{h\,k\,l\}$ are

$$h\,k\,l\quad l\,h\,k\quad k\,l\,h$$
$$\bar{h}\,\bar{k}\,\bar{l}\quad \bar{l}\,\bar{h}\,\bar{k}\quad \bar{k}\,\bar{l}\,\bar{h}.$$

The figure is geometrically a rhombohedron: it differs from the crystallographic rhombohedron of Art. **246** in that its edges do not lie in systematic planes, and are thus incapable of truncation or bevilment: further, the two three-faced quoins on the morphological axis are trigonally, not ditrigonally, symmetrical.

In Figs. 206 (*a*) and (*c*) are represented the forms $\pi\phi\{20\bar{1}\}$ and $\pi\phi\{2\bar{1}0\}$, respectively, which together compose the scalenohedron $\pi\{20\bar{1}\}$ shown in Fig. 206 (*b*), and in Figs. 206 (*d*) and (*f*) are represented the remaining correlative forms $\pi\phi\,\{\bar{4}52\}$ and $\pi\phi\,\{\bar{4}25\}$, which together compose the correlative scalenohedron $\pi\,\{\bar{4}25\}$ shown in Fig. 206 (*e*).

The forms $x\phi\,\{2\bar{1}\bar{1}\}$, $x\phi\,\{10\bar{1}\}$, and $x\phi\,\{111\}$ are geometrically identical with the holo-symmetrical forms $\{2\bar{1}\bar{1}\}$, $\{10\bar{1}\}$, and $\{111\}$, and the forms $x\phi\,\{p\,q\,r\}$, $x\phi\,\{h\,k\,k\}$, $x\phi\,\{m\,i\,n\}$ with the hemi-symmetrical forms $\phi\,\{p\,q\,r\}$, $\pi\,\{h\,k\,k\}$, and $\psi\,\{m\,i\,n\}$ respectively.

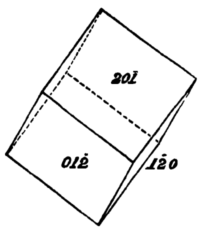


Fig. 206 (*a*).

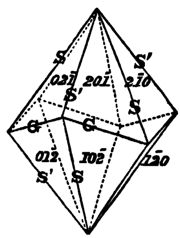


Fig. 206 (*b*).

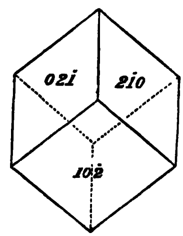


Fig. 206 (*c*).

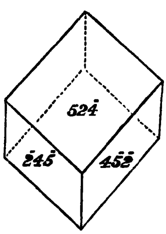


Fig. 206 (*d*).

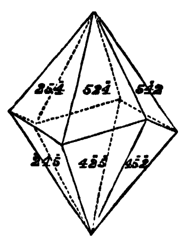


Fig. 206 (*e*).

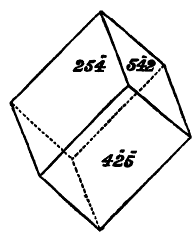


Fig. 206 (*f*).

266. V. Tetarto-systematic haplohedral forms.

The remaining type of mero-symmetry is one in which only three normals are present, each carrying a single face. The symbols of the faces of the form $\rho\pi\phi\,\{h\,k\,l\}$ are

$$h\,k\,l \quad l\,h\,k \quad k\,l\,h.$$

The form may be regarded as a hemimorphous developement of the hemiscalenohedron. This type of mero-symmetry has been observed on crystals of sodium periodate (Art. **276**).

Hexagonal System. C.—*Combinations of Forms.*

267. The holo-symmetrical type of the Hexagonal system is represented by so limited a number of examples, in the known range of crystals, as to have induced Professor Miller to treat all crystals presenting hexagonal and trigonal symmetry as being wholly of trigonal symmetry, and, in fact, as belonging exclusively to a Rhombohedral system (Art. **250**).

It is possible to do this by considering a discalenohedron or a dirhombohedron as resulting from the combination of two distinct and independent transverse scalenohedral or rhombohedral forms respectively; or, where the poles of the latter forms lie on a single triad of systematic planes, by viewing them as hexagonal pyramids with symbols of the form $\{m i n\}$.

Notwithstanding this high authority for such a treatment of the Hexagonal system, it is more in accord with theoretical considerations to regard the Hexagonal system as, potentially at least, a natural one; and it would still be so, even if the system were not represented in its holo-symmetrical completeness by known crystals. There are crystals, however, of which those of beryl, nepheline, pyrrhotine, and zinc oxide (both artificial and as the mineral spartalite) are instances, that occur with holo-symmetry as hexagonal forms, with cleavages parallel to a hexagonal prism and the pinakoid, and offer no evidence of a mero-symmetrical structure. Even had not this been the case, the actual occurrence of crystals exemplifying any of the types of hemi-symmetry, other than the rhombohedral type, derivable from a Hexagonal system would necessitate the recognition of the actuality of the holo-symmetrical system from which such types must, in common with the rhombohedral type, be derived. And crystals, though their species are few in number, are not wanting to establish the existence of such hemi-symmetry (Art. **271**).

268. (*a*) *Holo-symmetrical forms.* The combinations of hexagonal forms are as remarkable for symmetrical and equipoised character as are the combinations of forms in the Tetragonal system. This character is conspicuous in beryl, of which Fig. 207 is representative, and in connellite, which is illustrated in Fig. 208, the beryl exhibiting discalenohedral and dirhombohedral forms in subordination to the pinakoid- and prism-faces, while in connellite the converse is the case: the faces *r z* in Fig. 208 belong to the dirhombohedron $\{100, \bar{1}22\}$, the faces *o ω* to the discalenohedron $\{46\bar{7}, 9\bar{4}\bar{2}\}$, and the faces *b a* to the proto- and deutero-prisms respectively.

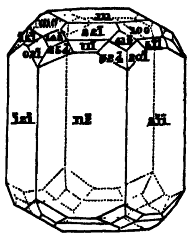


Fig. 207.

Fig. 208.

269. (*b*) *Hemi-symmetrical forms.* Of the varieties of hemi-symmetry of which the Hexagonal system is susceptible, that diplohedral type in which only the proto-systematic planes are symmetral, often designated as the Rhombohedral system, is the most frequent, and, indeed, the habitual type to which crystals in the Hexagonal system belong.

In rhombohedral crystals the trigonal grouping round the morphological axis is quite as marked a feature as the hexagonal grouping in the holo-symmetrical crystals, or the tetragonal grouping in the crystals of the system last considered.

Where prism-forms are extant, the crystals assume a hexagonal aspect; for the orthogonal projection of the side- or zigzag-edges of a rhombohedron on a plane perpendicular to the trigonal axis is a regular hexagon, the sides of which are the traces on the plane of projection of the faces of the hexagonal deutero-prism $\{10\bar{1}\}$.

Thus the aspect of crystals in which prism-forms are present, even if they be not predominant, differs considerably from that of crystals in which only rhombohedral or scalenohedral forms are extant. Figures 209 (*a*), 210, 211 (*a*), 212, represent in orthogonal projection on the equatorial plane the relations of a direct

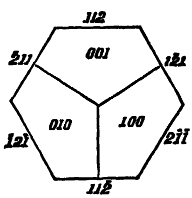


Fig. 209 (*a*).

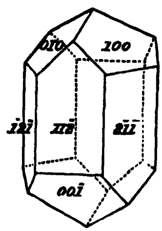


Fig. 209 (*b*).

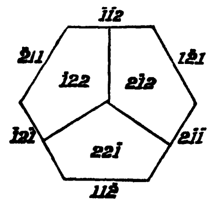


Fig. 210.

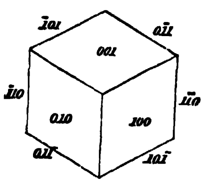


Fig. 211 (*a*).

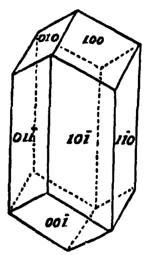


Fig. 211 (*b*).

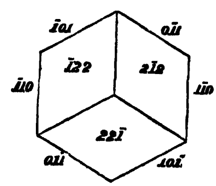


Fig. 212.

and an inverse rhombohedron to the proto-prism and deutero-prism; Figs. 209 (*b*), 211 (*b*) represent combinations of forms corresponding respectively to those of Figs. 209 (*a*), 211 (*a*), but projected on a plane distinct from the equatorial plane.

270. Rhombohedra and scalenohedra are not unfrequently combined in tautozonal series, the faces of one rhombohedron truncating the edges of another more acute. In the following table any symbol in either of the inner columns is that of a rhombohedron, of which the faces truncate the edges of the rhombohedron having its symbol in the other inner column and on the line next below: a symbol in either of the outer columns is that of a rhombohedron transverse to (i. e. correlative with) the rhombohedron of which the symbol stands on the same line with it and in the inner column:—

	Inverse Forms.		Direct Forms.	
	$\{14\,17\,17\}$		$\{655\}$	
obtuser ↑		$\{233\}$		$\{10\,7\,7\}$
	$\{255\}$		$\{211\}$	
		$\{011\}$		$\{411\}$
	$\{\bar{1}22\}$		$\{100\}$	
		$\{\bar{1}11\}$		$\{5\bar{1}\bar{1}\}$
acuter ↓	$\{\bar{7}55\}$		$\{3\bar{1}\bar{1}\}$	
		$\{\bar{5}33\}$		$\{17\,\bar{7}\,\bar{7}\}$
	$\{\bar{3}1\,17\,17\}$		$11\,\bar{5}\,\bar{5}$	

Thus the form $\{\bar{1}22\}$ is an inverse rhombohedron transverse to the direct rhombohedron $\{100\}$, the faces of which truncate the edges of the inverse rhombohedron $\{\bar{1}11\}$, and the edges of which are themselves truncated by the faces of the inverse rhombohedron $\{011\}$.

Figs. 213–218 represent combinations of forms occurring in calcite.

271. The other variety of hemi-systematic diplohedral forms exemplified on known crystals is the gyroidal type of which apatite, a phosphate and chloride or fluoride of calcium, offers the most conspicuous, if not the only illustration: a corresponding mero-symmetrical character has been developed by an etching process on faces of pyromorphite, an analogous compound in which lead takes the place of calcium. The poles of a gyroidal form $\phi \{h k l, e f g\}$ are those of the hexagonal discalenohedron in the symbols of which the indices all follow either the direct order $h k l$, $e f g$, or else the inverse order $l k h$, $g f e$; in either hemisphere the distribution of the poles is thus in asymmetric hexagonal (not dihexagonal) symmetry, and at the same time the extant faces of the form lie symmetrically with respect to the equatorial plane. Fig. 219 illustrates such a form, namely $\phi \{5 2 \bar{4}, \bar{1} 0 2\}$, in association with other forms having a holo-symmetrical hexagonal

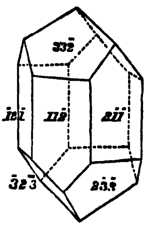


Fig. 213.

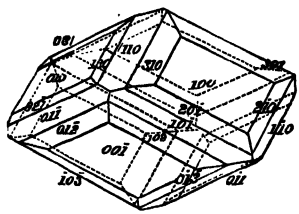


Fig. 214.

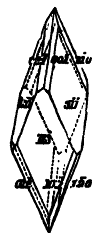


Fig. 215.

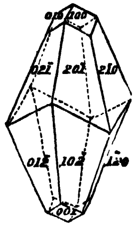


Fig. 216.

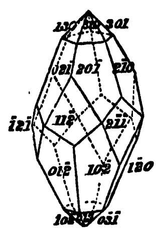


Fig. 217.

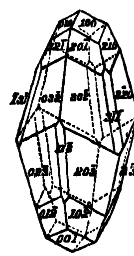


Fig. 218.

aspect. In fact crystals of apatite usually present a considerable number of forms quite hexagonal in their symmetry, while only a few forms exhibit, in the defalcation of their alternate faces, the gyroidal hemi-symmetry, which, however, must be held really to dominate the structure of the entire crystal.

272. Hemimorphous forms are not rare in the Hexagonal system, but they occur most often as hemimorphs, of hemi-symmetrical types of crystal, and are then tetartohedral in their symmetry. But as representing the hemimorphous type of a crystal otherwise holosymmetrical, greenockite, the cadmium sulphide, occurs in crystals carrying generally hexagonal pyramids (or dirhombohedra) arranged in series tautozonal with the pinakoid and the hexagonal deutero- (or proto-) prism; but the faces of some of these forms exist only on one side of the equatorial plane, and the crystals often exhibit on the other side of that plane only a single

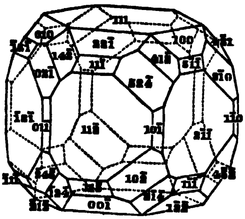


Fig. 219.

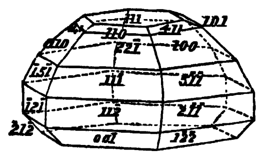


Fig. 220.

pinakoid-face (Fig. 220). Artificial crystals of cadmium sulphide have the parameters of greenockite, but present both rhombohedral and scalenohedral forms. Perhaps with greenockite may be associated the hexagonal zinc sulphide, wurtzite.

273. (*c*) *Tetarto-symmetrical forms.* Of the four types of tetarto-symmetry that, exclusive of hemimorphs of hemi-symmetrical forms, are possible in this system, two are illustrated in known crystals: they may both be derived from the rhombohedral type of hemi-symmetry, the one as a haplohedral, the other as a diplohedral derivative. The *trapezohedron* is the general tetartohedron of the former type; of the latter, or tetarto-hexagonal type, the general form is the *hemiscalenohedron* or *skew rhombohedron.*

Of the trapezohedral tetartohedron, quartz and sodium periodate afford examples in their crystals: of the hemiscalenohedron, instances are met with on certain crystals of phenakite, dioptase, and ilmenite.

274. As already mentioned in Art. **257**, the trapezohedra that occur on quartz belong to two correlative groups associated, the one with lævo-rotatory, the other with dextro-rotatory polarisation: and the forms of the two groups, if conceived as occurring simultaneously, would be symmetrical with respect to the protosystematic planes.

When examining with a nicol-prism divergent light originally plane-polarised that has traversed a quartz-plate of a certain thickness with its faces cut perpendicularly to the axis, it will be found that on turning the analysing prism to the right (as the hands of a watch move) the isochromatic rings *contract* and their centre passes through tints in the order *blue*, *plum-colour* (the 'sensitive' tint), *orange*, *red*, when the crystal is '*left hand;*' the rings *dilate* and the tints follow the order *red*, *orange*, *plum-colour*, *blue*, when the crystal is '*right hand:*' the phenomena being reversed in either case if the analyser be turned towards the left.

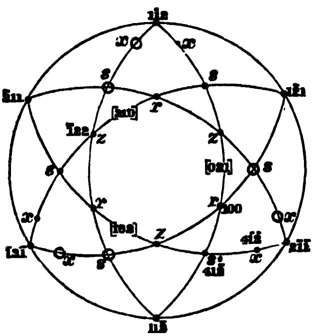


Fig. 221 (λ).

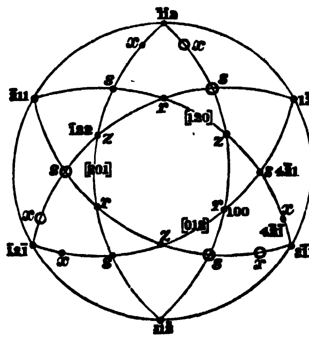


Fig. 221 (ρ).

In quartz the predominating faces of the lævo-rotatory forms $a\pi\{hkl\}$ lie in zones [102], [210], [021] on the upper or positive hemisphere, and in zones [120], [012], [201] on the negative hemisphere; those of the dextro-rotatory are in the latter zones on the positive, in the former zones on the negative hemisphere. These zones are tautohedral in faces of the rhombohedron $a\pi\{100\}$, the *S*-prism $a\pi\{11\bar{2}\}$, and the hexagonal pyramid $a\pi\{41\bar{2}\}$, (Figs. 221 λ and ρ).

In the case of quartz the faces of the form $a\pi\{41\bar{2}\}$ or $a\pi\{4\bar{2}1\}$ are small and rhomboidal in shape: they are usually striated parallel to their intersections with faces of the forms $a\pi\{100\}$, $a\pi\{11\bar{2}\}$, (see plate V). Edges of the predominating trapezohedra (the so-called plagihedral forms) are parallel to those edges of the form $a\pi\{41\bar{2}\}$ which run athwart the striation.

A crystal of quartz, when so placed that the faces 100 010 001 are at its upper end, will, if right-handed, present on three alternate quoins at each end of the prism plagihedral faces arranged in the form of a right-handed screw. Furthermore the two plagihedral faces, belonging to a single form, which are associated with any of the three prism-faces $2\bar{1}\bar{1}$ $\bar{1}2\bar{1}$ $\bar{1}\bar{1}2$ will be to the right of the observer as he looks at the prism-face (Fig. 221 *r*). On a left-handed crystal the plagihedral faces are arranged in the form of a left-handed screw and the two associated with any of the above prism-faces are seen to the left of the observer (Fig. 221 *l*). The faces of the form $a\pi\{\bar{1}22\}$ denoted in the figures by the letter *z* are generally smaller and more dull than the faces of the form $a\pi\{100\}$ denoted by the letter *r*: in Figs. 221 (*l*) and 221 (*r*) the *s* faces are the rhomb-faces of the form $a\pi\{41\bar{2}\}$, and the *x* faces are those of the plagihedral forms $a\pi\{4\bar{2}\bar{1}\}$ and $a\pi\{4\bar{1}\bar{2}\}$ respectively.

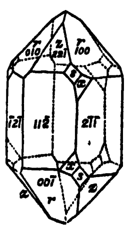


Fig. 221 (*l*).

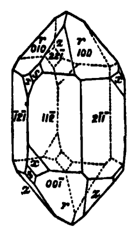


Fig. 221 (*r*).

275. A group of dithionates (or hyposulphates) offers examples of tetartohedrism of the type under consideration; the lead salt in particular ($Pb\,S_2\,O_6 . 4\,H_2\,O$) exhibits trapezohedral forms in conjunction with rotatory action on plane polarised light. These forms are, however, so distributed that of the two prevalent correlative rhombohedra the more developed would have to be taken as the form $a\pi\{\bar{1}22\}$ transverse to the axial rhombohedron $a\pi\{100\}$ in order that the situation of the plagihedral faces relative to the latter form might correspond with that presented by crystals of quartz, that is to say, that their faces might be gyroidally to the left of the primary rhombohedral faces in lævo-rotatory, and to the right in dextro-rotatory crystals. The more developed rhombohedron, however, is usually taken as $a\pi\{100\}$, so that the gyroidal distribution of the plagihedral forms becomes inverse to that obtaining in quartz.

276. Sodium periodate crystallised with three molecules of water also has a rotatory action on plane-polarised light, and carries forms of trapezohedral type indicative of the direction of the rotatory action; the crystals exhibit the additional peculiarity that they are hemimorphous in developement. Fig. 224 represents one end of a crystal of sodium periodate projected on the equatorial plane: at the other end only the pinakoid plane is present.

These crystals will be further considered under the subject of twin-crystals belonging to this type of mero-symmetry.

277. The hemiscalenohedron, when present, is subordinate to rhombohedral forms, as in Figs. 222 and 223. The first of these represents a crystal of dioptase in which the hemiscalenohedron

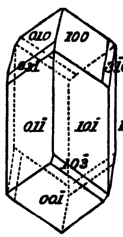


Fig. 222.

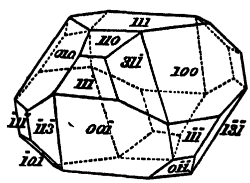


Fig. 223.

$\pi\phi\,\{3\bar{1}0\}$ is combined with the rhombohedron $\pi\phi\,\{100\}$ and the deutero-prism $\pi\phi\,\{10\bar{1}\}$: in Fig. 223 is illustrated a crystal of ilmenite presenting the hemiscalenohedron $\pi\phi\,\{31\bar{1}\}$ in combination with the rhombohedra $\pi\phi\,\{100\}$, $\pi\phi\,\{110\}$, and $\pi\phi\,\{11\bar{1}\}$.

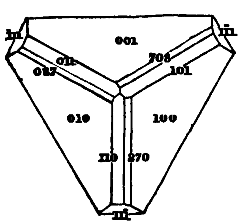


Fig. 224.

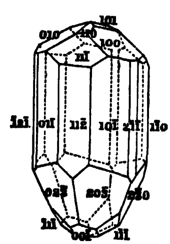


Fig. 225.

278. Of the hemimorphous developement of crystals of rhombohedral type tourmaline is a conspicuous example, different forms being present on the opposite sides of the equatorial plane. A not infrequent accompaniment of hemimorphism, the pyroelectric character, is strongly illustrated in this remarkable mineral. Fig. 225 represents a crystal of tourmaline in which the lower extremity of the morphological axis is the so-called *analogous* pole, and the upper the *antilogous* pole, being respectively positive and negative in electrification when the temperature is rising.

Hexagonal System. D.—*Twinned Forms.*

279. I. *Twins of holo-symmetrical crystals.* Simple holo-symmetrical crystals are of such rare occurrence in this system that few examples of twin-structure of holo-symmetrical types are to be expected, and in fact the only one on record is a minute crystal of zinc oxide, obtained accidentally as a furnace product, which has been described by vom Rath*. The twin-plane was very near to (100), although the measurements obtained indicated a face of more complex symbol. It seems quite possible that there is in this case only an accidental simulation of twin-structure.

280. II. *Twins of hemi-symmetrical crystals.* Hemi-symmetrical crystals afford abundant examples of twin-structure: their great

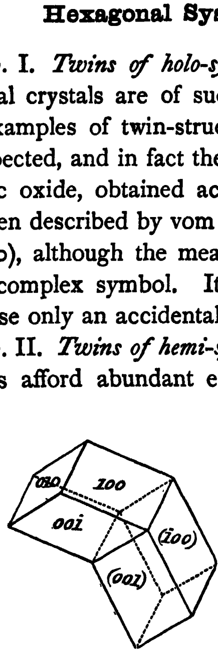


Fig. 226.

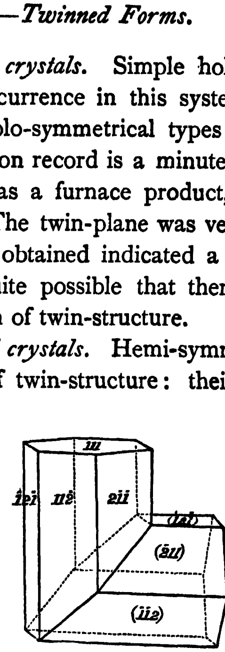


Fig. 227.

variety in aspect is however due rather to the diversities exhibited by individual crystals, even of the same substance, than to any great variety in the twin-laws under which crystals belonging to

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this system are united. It is in the numerous crystals of rhombohedral type that we must look for examples of such unions; and

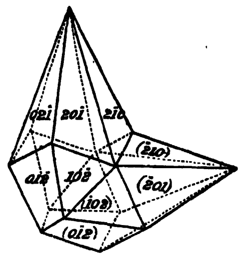

Fig. 228.

of these examples the multiform mineral calcite presents the most conspicuous and varied illustrations.

281. The twin-laws governing the union of rhombohedral crystals are the following:—

1. Twin-plane a face of a direct rhombohedron.

(*a*) *Twin-plane a face of the form* $\{100\}$.

Figures 226, 227, 228 represent respectively the rhombohedron $\pi\{100\}$, the hexagonal proto-prism $\pi\{2\bar{1}\bar{1}\}$, and the scalenohedron $\pi\{20\bar{1}\}$ of calcite twinned upon a face $(0\bar{1}0)$ of the form $\pi\{100\}$, the twin-plane being likewise the face of union.

282. (*b*) *Twin-plane a face of the form* $\pi\{211\}$.

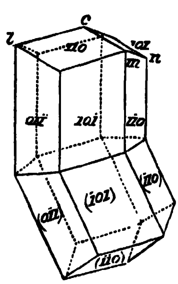

Fig. 229.

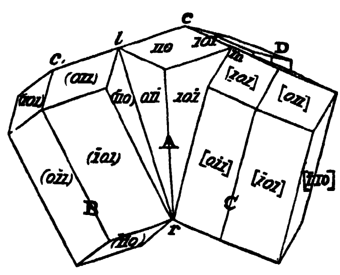

Fig. 230.

Fig. 229 illustrates a crystal of pyrargyrite twinned about the plane $(\bar{1}2\bar{1})$, which is parallel to the plane truncating the edge *cl* formed by the faces (110) (011): in this case the twin-plane is the face of union of the two individuals. Sometimes the twin-plane is at right angles to the face of union and three individuals are in twin-position relative to a central one, as shown in Fig. 230. If the line *lr* be at right angles to the edge *cl* it is the twin-axis for the individuals *A B*: hence the edges *lc* lc_1 of the two crystals fall in the same straight line and the faces 110 (011) have the same direction; the faces (110) 011 on the other side of the edges *lc* lc_1 are likewise co-planar. The planes truncating the edges *cm cn* are respectively twin-planes for the pair *AC* and the pair *AD*.

If the above be the true expression of the law of this twin-structure, the face of union, having its pole in a proto-systematic plane and being at right angles to a plane of the form $\pi\{211\}$, cannot be a plane with rational indices.

283. 2. Twin-plane a face of an inverse rhombohedron.

(*a*) *Twin-plane a face of the form* {011}.

Fig. 231 represents a simple rhombohedron $\pi\{100\}$ of calcite twinned on the plane ($\bar{1}\bar{1}0$) which truncates the edge between the faces $\bar{1}00$ $0\bar{1}0$: the twin-plane is also the face of union. Four cleavage-planes of the twin-structure, namely 010, 100, ($\bar{1}00$) and ($0\bar{1}0$), form a prism, the angle between the planes 100 010 and also between the planes ($\bar{1}00$) ($0\bar{1}0$) being 105° 5′, and the angle between the planes 100 ($\bar{1}00$) and also between 010 ($0\bar{1}0$) being 74° 55′: of the remaining cleavage-planes $00\bar{1}$ and (001) form a re-entrant angle of 38° 8′, while 001 and ($00\bar{1}$) form a salient angle of the same magnitude.

In scrutinising a transparent block of the calcite from Iceland, known as Iceland spar, there is frequently to be found a very narrow twin-lamina or twin-plate of calcite intercalated in the mass of the crystal in accordance with this law. And in other cases of its occurrence, as at the Rathhaus-berg near Gastein, and at Auerbach, Hesse-Darmstadt, the intercalated laminæ are so thin and so numerous as to give quite a lamellar character to the crystal-aggregate.

Crystals of the metal bismuth also occur twinned in obedience to this law.

284. (*b*) *Twin-plane a face of the form* $\{\bar{1}11\}$.

In Fig. 232 is illustrated a scalenohedron $\pi\{20\bar{1}\}$ of calcite twinned on a face $\bar{1}1\bar{1}$ of the inverse rhombohedron $\{\bar{1}11\}$ the edges of which would be truncated by the faces of the primary rhombohedron $\pi\{100\}$. The face $1\bar{1}1$ itself truncates the edge *cm*

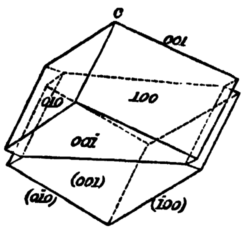

Fig. 231.

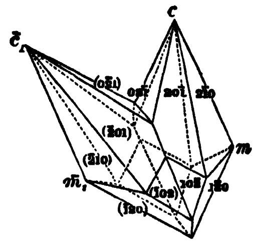

Fig. 232.

between the faces $2\bar{1}0 . 0\bar{1}2$. In the twin-crystal the edges *cm* $\bar{c}_1\bar{m}_1$ are parallel to each other and to the twin-plane which is likewise the face of union.

285. 3. Twin-plane a face of the pinakoid $\{111\}$.

(*a*) *Face of union coincident with the twin-plane.*

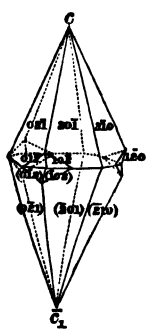

Fig. 233.

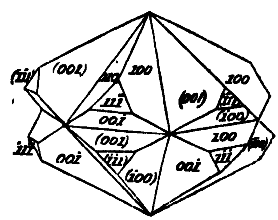

Fig. 234.

Twinned scalenohedra of calcite from Derbyshire, similar to Fig. 233, are familiar illustrations of this twin-structure. The aspect of the twin is similar to that which would result if the scalenohedron $\pi\{20\bar{1}\}$ were cut through its centre by a plane parallel to the faces of the pinakoid $\pi\{111\}$ and then one-half of it turned round the morphological axis through 60°, or any odd multiple of that angle. The twin-plane is the face of union: the edges which lie in this plane and are formed by the meeting of the two individuals are salient and re-entrant in alternate pairs.

(*b*) *Face of union perpendicular to the twin-plane.*

The crystal of hæmatite illustrated in Fig. 62, page 178, is an example of this kind of twin-growth.

(*c*) *Interpenetrating crystals.*

Rhombohedra of dolomite, and notably of chabazite, twinned about the morphological axis and mutually interpenetrating afford illustrations of this variety. Fig. 234 represents a twin-growth of the latter mineral: the former mineral is now regarded as being tetarto-symmetrical in its structure.

286. III. *Twins of tetarto-symmetrical crystals.*

In considering the twin-growths of tetarto-symmetrical crystals we shall be almost entirely limited to those of quartz, the crystals of which besides being the most common of mineral products are seldom entirely simple in their structure.

A remarkable class of regular growths of quartz, simulating the aspect of simple crystals, was first explained by Rose. As already stated in Art. **274**, a right-handed crystal of this mineral presents on three alternate quoins at each end of the prism plagihedral faces arranged in the form of a right-handed screw, and if the crystal is so placed that the faces 100 010 001 are at the upper end, the two plagihedral faces belonging to a single form, which are associated with any of the three prism-faces $2\bar{1}\bar{1}$ $\bar{1}2\bar{1}$ and $\bar{1}\bar{1}2$, are to the right of the observer as he looks at the prism-face (Fig. 235 *a*). In the illustrative figures the faces of the rhombohedron $\alpha\pi\{100\}$ are indicated by the letter r and those of the rhombohedron $\alpha\pi\{\bar{1}22\}$ by the letter z; the plagihedral faces are denoted by the letter x: suffixes to these letters refer to the position of the simple crystal to which the corresponding faces belong. The crystal represented in Fig. 235 (*a*), after half a revolution round the morphological axis takes the position represented in Fig. 235 (*c*): the faces r_1 and z_1 are parallel respectively to the faces z and r of the first position, and the prism-faces $(\bar{2}11)$ $(\bar{1}\bar{1}2)$ $(1\bar{2}1)$

of Fig. 235 (*c*) have positions which correspond to those of the prism-faces $2\bar{1}\bar{1}$ $11\bar{2}$ $\bar{1}2\bar{1}$ of Fig. 235 (*a*); it will be seen that in this position of the crystal the two plagihedral faces associated with a face corresponding to any of the above three prism-faces $2\bar{1}\bar{1}$ $\bar{1}2\bar{1}$ $\bar{1}\bar{1}2$ of the crystal in its first position are now to the left of the observer.

An intergrowth of two crystals with a common morphological axis and the relative positions shown in Figs. 235 (*a*), 235 (*c*) may thus have for result an apparently simple crystal without re-entrant angles. But in such a compound crystal the plagihedral faces, instead of presenting themselves systematically on alternate quoins, may be either present or absent on any of the quoins according as the quoin belongs to a crystal with the first or second position.

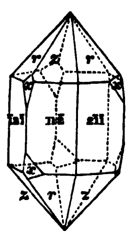

Fig. 235 (*a*).

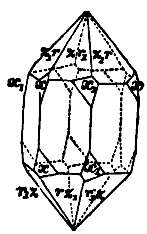

Fig. 235 (*b*).

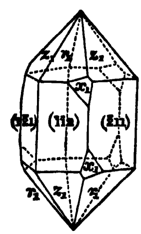

Fig. 235 (*c*).

And in the same way, any one of the six faces forming the pyramidal termination of the compound crystal may belong wholly to the form $a\pi$ {100} or wholly to the form $a\pi$ {$\bar{1}22$} according as it is part of a crystal with the first or second position, and on the other hand it may be a compound face, one portion of it belonging to the form $a\pi$ {100} of one position and the remainder to the form $a\pi$ {$\bar{1}22$} of the other: as the *r* faces are generally brighter than the *z* faces, this compound nature is usually apparent. Fig. 235 (*b*) represents a possible form of a compound crystal resulting from the intergrowth of simple crystals having the positions shown in Figs. 235 (*a*) and (*c*).

287. Similar growths of two left-handed crystals are met with. Fig. 236 (*c*) represents the position of a left-handed crystal after half a revolution round the morphological axis from the position shown in Fig. 236 (*a*): Fig. 236 (*b*) illustrates a possible form of a compound crystal resulting from the intergrowth of crystals having these positions.

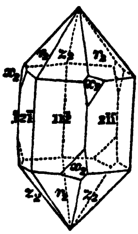

Fig. 236 (*a*).

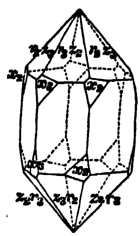

Fig. 236 (*b*).

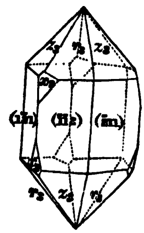

Fig. 236 (*c*).

288. Another kind of intergrowth, described by Rose as a twin-growth, does not come within the definition of a twin given in Art. **154**, but yet is so similar in aspect to those just explained that it may be here most conveniently mentioned. It consists of a left-handed and a right-handed crystal with the relative positions shown in Figs. 237 (*a*) and (*c*) respectively, the similar rhombohedral planes having corresponding positions in the two individuals: the crystals may be described as having *homologous* positions; as

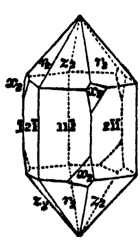

Fig. 237 (*a*).

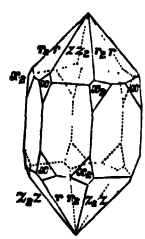

Fig. 237 (*b*).

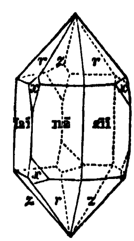

Fig. 237 (*c*).

already pointed out, crystals presenting incongruent forms cannot be parallel (Art. **197**). Adjacent quoins at either end of such a compound crystal, if modified at all, present plagihedral faces of opposite kinds: the pyramidal terminations are in their physical characters not to be distinguished from those of simple crystals. Fig. 237 (*b*) illustrates a compound crystal of this kind resulting from the intergrowth of crystals having the positions shown in Figs. 237 (*a*) and (*c*). That crystals, apparently simple but yet presenting both right and left plagihedral faces, are really composite, as suggested by Rose, was proved by Groth, who found that a slice from such a compound crystal rotates the plane of polarisation oppositely in different parts and in accordance with the right or left position of the adjacent plagihedral faces.

289. The surface of junction of two right-handed or left-handed individuals or of a right-handed and a left-handed individual is generally a very irregular one: the difference in the physical characters on opposite sides of the junction often renders it possible to trace the boundary on the faces of the compound crystal; in Fig. 238 is illustrated a crystal belonging to the British

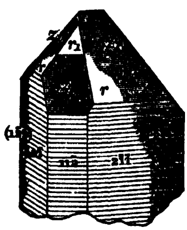

Fig. 238.

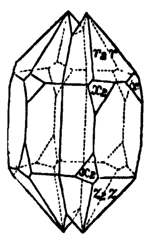

Fig. 239.

Museum. Growths of right and left crystals similar to the one shown in Fig. 237 (*b*) are sometimes twinned about the morphological axis as if they were simple crystals: Fig. 239 illustrates a growth of this character presenting re-entrant angles.

290. Since the time of Rose, composite crystals of right and left quartz, with the general aspect of simple crystals but with a higher symmetry, have been described in which the *r* faces of the one individual are co-planar with the *z* faces of the other, a relative position of the two crystals which may be regarded, from a purely geometrical point of view, as due to a half-revolution of one individual round the morphological axis from a position in which it was homologously disposed to the other. Figs. 237 (*a*) and (*c*) represent a left-handed and a right-handed crystal in homologous positions: Figs. 240 (*a*) and (*c*) represent the relative positions of the two individuals after the left-handed crystal shown in 237 (*a*) has been turned through half a revolution round the morphological axis.

In a composite crystal resulting from the intergrowth of two crystals with these relative positions three alternate quoins at either end of the prism can present no plagihedral faces at all, while each of the remaining three quoins may present either right or left plagihedral faces or both, according as the quoin is simple and belongs to the right-handed or to the left-handed crystal, or is

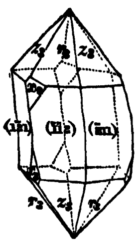

Fig. 240 (*a*).

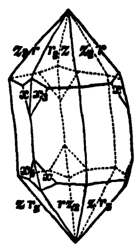

Fig. 240 (*b*).

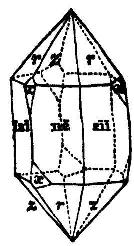

Fig. 240 (*c*).

composite and belongs partly to the one and partly to the other. Fig. 240 (*b*) illustrates a possible disposition of the plagihedral faces on a growth of this kind.

291. Growths of a right and a left crystal with the above relative positions but presenting re-entrant angles and an evident composite character have likewise been described. As both the individuals did not present plagihedral faces, the real nature of the growth was misunderstood until an optical investigation revealed the right and left character of the crystals.

292. Another class of regular growths, one of which was first described by Weiss in 1829, is presented by certain rare specimens of quartz from La Gardette, Traversella, Munzig, and Japan. These growths are of two kinds, which are so far similar that in both of them the morphological axes of the two individuals are in a Σ-plane and have a mutual inclination of 84° 34′: but whereas, relatively to the plane bisecting the above angle between the morphological axes and at right angles to their plane, the *r* faces of one individual are symmetrically disposed to the *r* faces of the other in the first kind, they are symmetrically disposed to the *z* faces of the other in the second kind.

293. The relative positions of the two individuals may also be imagined in the following way. In Fig. 241 (*a*) the line ot_1 is in a Σ-plane and inclined at an angle of 47° 43′ to the morphological axis; it is therefore at right angles to a terminal edge *ca*

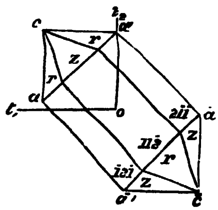

Fig. 241 (*a*).

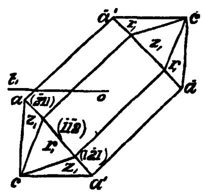

Fig. 241 (*b*).

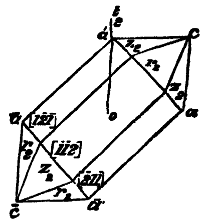

Fig. 241 (*c*).

of the ordinary six-sided pyramid *rz* and also to the plane of the form $a\pi$ $\{52\bar{1}\}$ which would truncate that edge: the line ot_2 is in the same Σ-plane and perpendicular to the line ot_1: it is therefore parallel to the edge *ca* and makes an angle of 42° 17′ with the morphological axis.

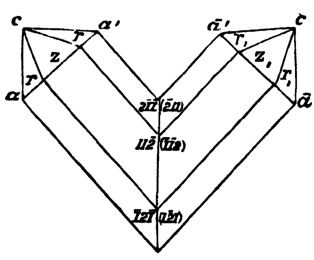

Fig. 242 (*a*).

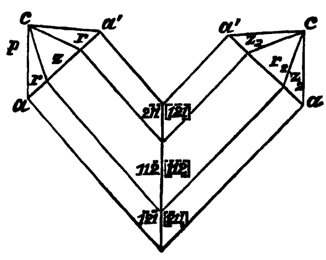

Fig. 242 (*b*).

A half revolution of the crystal represented in Fig. 241 (*a*) round the line ot_1 will bring it into the position shown in Fig. 241 (*b*): the morphological axes of two crystals with these relative positions will be inclined to each other at an angle of 84° 34′ and will lie in a Σ-plane; further, the *r* faces of the two individuals will be symmetrically disposed relative to a plane bisecting the angle 84° 34′ between the morphological axes, the prism-faces $11\bar{2}.(\bar{1}\bar{1}2)$ will be co-planar, and the edge *ca* of one individual will be parallel to the edge $\bar{c}\bar{a}$ of the other. A growth of two individuals with these relative positions will thus be one of the first kind (Fig. 242 *a*).

294. Again, a half-revolution round the line ot_2 of the crystal represented in Fig. 241 (*a*) will bring the crystal into the position represented in Fig. 241 (*c*): as before, the morphological axes will be inclined to each other at an angle of 84° 34′ and will lie in a Σ-plane; but now the *r* faces and the *z* faces will be symmetrically disposed relative to a plane bisecting the angle 84° 34′: the prism-faces $11\bar{2}.[\bar{1}\bar{1}2]$ will be co-planar, and the edge *ca* of one individual will be parallel to the edge *ca* of the other. A growth of two individuals with these relative positions will thus be one of the second kind.

A crystallographically identical result would be obtained by a half-revolution, round the morphological axis, of one of the two individuals of a growth of the first kind (Fig. 242 *a*).

295. The positions of the plagihedral faces of a crystal of quartz being defined by those of the *r* and *z* faces and the right or left character of the crystal (Art. **274**), the above growths may be classed as twins, always falling under the definition given in Art. **154**, whether the two individuals be both right-handed or both left-handed, and may still be geometrically explained by a reference to a single half-revolution even when one individual is right-handed and the other left-handed, if the pair be first placed in homologous positions, as explained in Art. **288**.

In the majority of the few known examples of this class of regular growths, it is practically difficult to decide, owing to the absence of plagihedral faces, as to the right or left character of both individuals: still specimens have been described in which the individuals are regarded as being both right-handed or both left-handed, or one right-handed and the other left-handed in character.

296. In no case, however, has the simple character of the individuals themselves been satisfactorily demonstrated: we know, indeed, that the observations of Des Cloizeaux have led him to infer that apparently simple crystals of quartz from La Gardette are not exclusively composed of either the right or the left variety, and that a homogeneous simple crystal of quartz is one of the greatest of mineralogical rarities: further, vom Rath, in his examination of the regular growth from Japan, though he failed to find any evidence as to the right or left character of the individuals, found abundant proof of their compound structure. It is thus quite possible that we have here essentially a single kind of twin-growth, geometrically explained by a half-revolution, round the normal to a plane of the form $a\pi\{52\bar{1}\}$, of one individual from a position in which it is parallel or homologously disposed to the other, and that the second kind of growth is merely a result of the

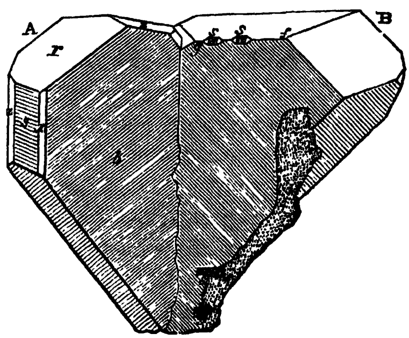

Fig. 243.

fact that one or both the individuals is itself a twin-growth of the kind described in Arts. **286–290**.

297. An idea of the actual developement of the individuals and of the difficulty of deciding as to their true character may be obtained from an examination of Fig. 243, which represents a very fine British Museum specimen of a twin-growth of this class from La Gardette. The individual *A* carries largely-developed plagihedral faces characteristic of a simple left-handed crystal. On the edge *fg* of the individual *B* are small notches *ss* of which the sides are parallel to those faces of the rhombohedron and prism which are connected by the edge, and at the bottom of which is a plane of the form *s* or $a\pi\{41\bar{2}\}$: the plane at the bottom of one of the larger notches carries striations indicative of a right-handed crystal.

On the lower part of the same individual there are small faces *s s x* likewise characteristic of a right-handed crystal. It will be seen, however, from Art. **286** that the disposition of these faces on the individual *B* is that of the faces of a composite crystal twinned about the morphological axis. As the limits of the individuals of which *B* is composed cannot be traced, and it is difficult in this specimen to distinguish the *r* from the *z* faces by means of their physical characters, we cannot be certain as to which of the rhombohedral faces of *B* belong to the forms $a\pi\{100\}$ and $a\pi\{\bar{1}22\}$ respectively.

Some of the rhombohedral faces of both *A* and *B* show dull patches, such as are usual in composite crystals twinned about the morphological axis.

The boundary of the individuals *A* and *B* is of a zigzag shape, as is seen from its trace on the co-planar prism-planes.

298. Of other tetarto-symmetrical minerals than quartz, phenakite is remarkable for its twin-growths. The individuals are twinned about the morphological axis, and are mutually interpenetrant: owing to the subordinate character of the planes which are tetarto-symmetrically developed the twin-growths have an aspect somewhat similar to that of chabazite twins (Fig. 234), but further exhibit the planes of the hexagonal prism truncating the zigzag edges of the rhombohedron.

299. It has already been remarked that crystals of sodium

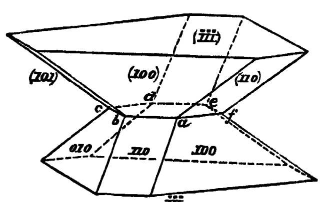

Fig. 244.

periodate carry forms which may be regarded as a hemimorphous development of a tetarto-symmetrical type, and that the crystals, like those of quartz, are right-handed and left-handed in their rotation of the plane of polarisation. Twin-growths of this substance have been described by Groth. The individuals of a composite crystal are respectively right-handed and left-handed in optical character, and are united in a face parallel to the pinakoid; they are not symmetrical with respect to this plane, but would become so if one of the individuals were turned through 60°, or any odd multiple of 60°, round the morphological axis (Fig. 244). It will be seen from the figure that a half-revolution of one of the individuals round the line *ad* would bring it into such a position that the faces of the forms present would coincide with those of the other individual, and that the two individuals would then be in homologous positions. The line *ad* is parallel to the line *bc* in which a rhombohedron-face meets a plane parallel to the pinakoid; it is therefore perpendicular to a face of the hexagonal deutero-prism.

SECTION IV.—The Ortho-rhombic or Ortho-symmetric System.

A.—*Holo-symmetrical Forms.*

300. The characteristics of this system are that its holo-symmetrical forms are symmetrical to three perpendicular planes of unconformable symmetry intersecting in three perpendicular mutually incongruent axes of orthogonal symmetry.

These planes are, the proto-systematic plane S, the deutero-systematic plane Σ, and the trito-systematic plane C which is taken as horizontal in position.

The systematic planes being taken for the axial planes, the three systematic axes, which are also their normals, become the crystallographic axes; the vertical axis $[S\Sigma]$ being taken for the axis of Z, and the zone-lines $[\Sigma C]$ and $[SC]$ for the axes of X and Y respectively.

The elements of the crystal are

$$\xi = \eta = \zeta = 90^\circ, \qquad a : b : c,$$

the parametral ratios having different values. The systematic planes being mutually incongruent, and equally important in their relation to the symmetry of the crystal, any one of them may be selected as the plane Σ, S, or C. It would have been more satisfactory if in the description of crystals it had been adopted as a convention that the intercepts made on the axes OX, OY and OZ by the parametral plane should be in descending order of magnitude. Very often, however, that axis has been selected as the direction OZ which presents itself as the axis of a well-developed prismatic zone, or as the normal of a large pinakoid face, independently of the magnitude of its parametral intercept; the greater of the two remaining intercepts being then taken for the axis OX: or again, the axes have been so selected as to

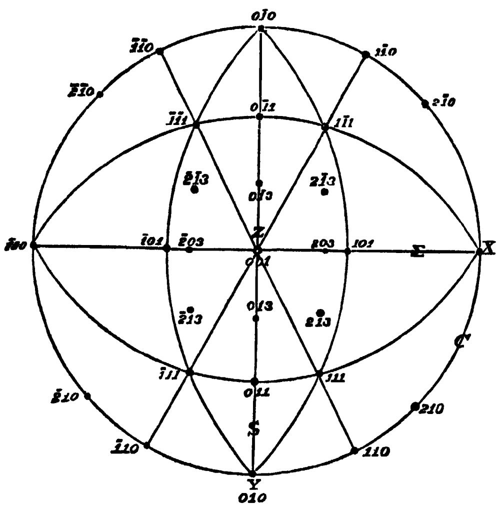

Fig. 245.

suggest that the crystals are similar in developement to others having an analogous chemical composition.

The systematic triangle is quadrantal and all its angles are right angles, each such triangle being commensurate with one of the octants of the system.

The general scalenohedron $\{h\,k\,l\}$ of the system is the scalene octahedron; and the remaining six forms, representing those of which the poles lie on corresponding arcs or at the several angles of the systematic triangle, are

The proto-dome	$\{0kl\}$.
The deutero-dome	$\{h0l\}$.
The prism	$\{hk0\}$.
The proto-pinakoid	$\{100\}$.
The deutero-pinakoid	$\{010\}$.
The trito-pinakoid	$\{001\}$.

The positions of the poles of such forms on the sphere of projection are illustrated in Fig. 245 for the case where

$$a : b : c = 1{\cdot}894 : 1 : 1{\cdot}797,$$

as in topaz; the axis of the well-developed prism of this mineral being generally taken for the axis *OZ*, although the parametral intercept on that axis is greater than the intercept on one of the other axes.

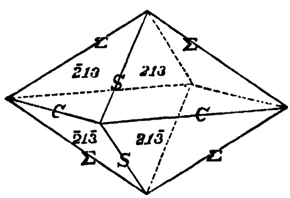

Fig. 246.

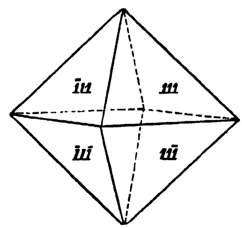

Fig. 247.

301. *The scalene octahedron* $\{hkl\}$, Figs. 246–7, is the general independent form of the system. Its faces are scalene triangles, their symbols being

$$h k l \quad \bar{h} k l \quad \bar{h} \bar{k} l \quad h \bar{k} l,$$
$$\bar{h} \bar{k} \bar{l} \quad h \bar{k} \bar{l} \quad h k \bar{l} \quad \bar{h} k \bar{l}.$$

The three systematic planes pass through the twelve edges of the form, the four similar edges in which the form is intersected by each plane giving a rhombic section. The three rhombic sections however in the different systematic planes are obviously not congruent, the edges in one section being dissimilar from those in a different section.

The six quoins are four-faced and ortho-symmetrical on the respective axes, but the pair on one axis is not congruent with that on a different axis. In Fig. 246 is represented the form $\{213\}$ corresponding to the above elements.

The parametral form $\{111\}$ is a scalene octahedron (Fig. 247).

302. *Prismatid forms.* The designations of these open forms, the horizontal prismatid forms as domes, the vertical one as a prism, have already been given in article **109**. They are the forms of which the faces truncate the edges of a scalene octahedron, each therefore consisting of a rhombic based prism, the rhombic sections of which have their diagonals parallel to two of the axes of the system. These diagonals are termed the proto- and deutero-diagonals of the prism, and either of them associated with the vertical axis $[S\Sigma]$ forms the pair of diagonals of one of the dome forms.

i. *The proto-dome* $\{0kl\}$, Figs. 248–9, has its four edges parallel to the proto-diagonal $[\Sigma C]$, the axis X; the symbols of its faces are

$$0kl \quad 0\bar{k}l \quad 0\bar{k}\bar{l} \quad 0k\bar{l};$$

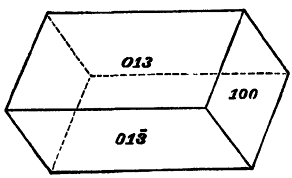

Fig. 248.

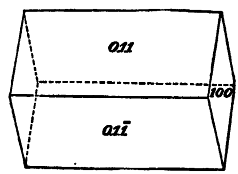

Fig. 249.

they are intersected perpendicularly to their edges by the proto-systematic plane S, which therefore divides them symmetrically, the traces of the section being rhombs the diagonals of which are the vertical axis and the deutero-diagonal. Fig. 248 represents the proto-dome {013}, closed by the faces of the form {100}. In the parametral dome {011} (Fig. 249) these diagonals are in the ratio $\frac{c}{b}$.

ii. The *deutero-dome* $\{h0l\}$, Figs. 250–1, has its four edges parallel to the deutero-diagonal $[SC]$, the axis Y; the symbols of its faces are

$$h0l \quad \bar{h}0l \quad \bar{h}0\bar{l} \quad h0\bar{l};$$

they are euthy-symmetrical to the intersections with them of the deutero-systematic plane Σ, the diagonals of the rhombic section of the dome being the vertical axis and the proto-diagonal. Fig. 250

represents the deutero-dome {203} closed by the faces of the form {010}.

The parametral deutero-dome {101} (Fig. 251) represents in its diagonals the ratio $\frac{c}{a}$.

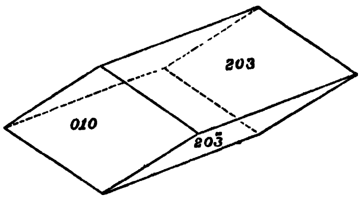

Fig. 250.

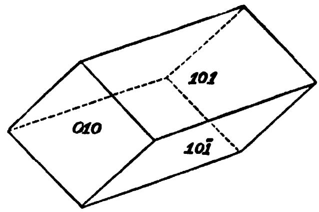

Fig. 251.

iii. The four faces of the *rhombic prism* $\{hk0\}$, Figs. 252–3, have for their symbols

$$hk0 \quad \bar{h}k0 \quad \bar{h}\bar{k}0 \quad h\bar{k}0;$$

they are euthy-symmetrically intersected by the plane C, which is the plane of their zone, as S and Σ are of those of the proto- and deutero-domes. Fig. 252 represents the rhombic prism {210}

$\bar{2}10$ $\quad$ $2\bar{1}0$ $\qquad\qquad$ $\bar{1}10$

Fig. 252. $\qquad$ Fig. 253.

corresponding to the above elements, closed by the faces of the form {001}.

The parametral prism {110} (Fig. 253) gives in its diagonals the ratio $\frac{a}{b}$.

303. The faces parallel to each systematic plane constitute a distinct *pinakoid* form.

Those of the *proto-pinakoid* {100} are parallel to the proto-systematic plane *S*. They are 100, $\bar{1}00$ (Figs. 248, 249).

The faces of the *deutero-pinakoid* {010}, namely 010 and $0\bar{1}0$, are parallel to the deutero-systematic plane Σ (Figs. 250, 251): and the faces of the *trito-pinakoid* or *basal pinakoid* {001}, namely 001 and $00\bar{1}$, are parallel to the horizontal plane of symmetry *C* (Figs. 252, 253).

The poles of the pinakoids form the angular points of the systematic triangles; those of the prismatid forms lie on the arcs of those eight triangles.

Ortho-rhombic System. B.—*Mero-symmetrical Forms.*

304. In the Tetragonal and Hexagonal systems the mero-symmetrical forms admissible by the conditions of the systems were remarkable for their variety, and except in the case of the rhombohedral division hardly less so for the paucity and in some cases the absence of known crystal species that exemplify them.

In the Ortho-rhombic system the simpler character of the system is illustrated as well in the few and simple kinds of hemi-symmetry it offers, as in the greater number of crystals by which these are represented.

In the *holo-systematic* section of this system there are four normals to the general form, the scalene octahedron $\{h\,k\,l\}$; and in the case of hemi-symmetry a face will be extant for each of these normals.

There are two ways in which the suppression of the four absent faces may occur.

305. I. *The rhombic sphenoid* $a\,\{hkl\}$ or $a\,\{\bar{h}kl\}$, Figs. 254 (*a*), (*c*). In the first, the asymmetrical case, no systematic plane is symmetral; but each of the three ortho-symmetral axes becomes an axis of diagonal symmetry. Hence the alternate faces of the holohedral form $\{h\,k\,l\}$ are absent or extant. The resulting form is tetrahedrid in character, its four faces being similar scalene triangles. It is therefore a scalene sphenoid. Its six edges are similar in pairs only, a pair of similar edges being oppositely placed on the crystal and having their directions symmetrically inclined in respect to one of the systematic planes. The four quoins are three-faced and similar, but present no symmetry. Figs. 254 (*a*) and 254 (*c*) represent respectively the forms $a\{213\}$ and $a\{\bar{2}13\}$, which together compose the scalene octahedron $\{213\}$ of Fig. 254 (*b*).

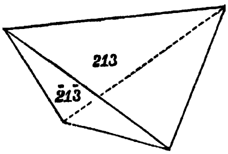

Fig. 254 (*a*).

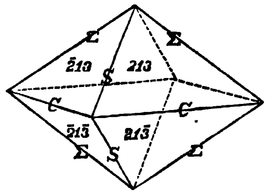

Fig. 254 (*b*).

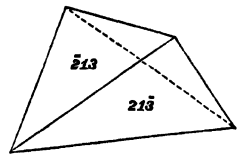

Fig. 254 (*c*).

Sphenoidal hemi-symmetry of an ortho-symmetrical crystal can only be represented in a distinct form by faces of the rhombic sphenoid. Any other form of the system than the octahedrid scalenohedron can only exhibit sphenoidal characters in the mero-symmetrical features it acquires by association with sphenoidal forms.

II. The second division of the holo-systematic section of this system is one in which, two of the systematic planes being planes of symmetry, the third fails of being so; conditions resulting in three kinds of hemimorphous forms.

i. *The trito-hemioctahedron* $\rho\{hkl\}$ or $\rho\{hk\bar{l}\}$. Where the proto- and deutero-systematic planes are symmetral, the form is hemimorphous in respect to the plane *C*, and the form is a trito-hemioctahedron $\rho\{hkl\}$, the correlative form being $\rho\{hk\bar{l}\}$, the former comprising the faces

$$hkl \quad \bar{h}kl \quad h\bar{k}l \quad \bar{h}\bar{k}l,$$

the latter the faces

$$hk\bar{l} \quad \bar{h}k\bar{l} \quad h\bar{k}\bar{l} \quad \bar{h}\bar{k}\bar{l}.$$

ii. *The proto-hemioctahedron* $\xi\{hkl\}$ or $\xi\{\bar{h}kl\}$. The form hemimorphous to the plane *S* is symmetrical to the planes Σ and *C*. Its faces are

for the form $\xi\{hkl\}$,

$$hkl \quad h\bar{k}l \quad hk\bar{l} \quad h\bar{k}\bar{l},$$

for the form $\xi\{\bar{h}kl\}$,

$$\bar{h}kl \quad \bar{h}\bar{k}l \quad \bar{h}k\bar{l} \quad \bar{h}\bar{k}\bar{l}.$$

Those of the *proto-pinakoid* {100} are parallel to the proto-systematic plane *S*. They are 100, $\bar{1}00$ (Figs. 248, 249).

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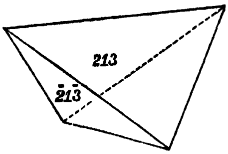


Fig. 254 (*a*).

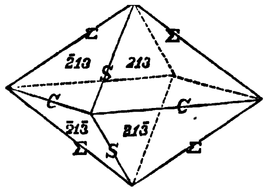
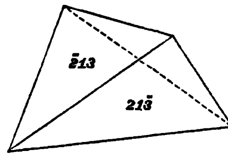


Fig. 254 (*b*).



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$$hkl \quad \bar{h}kl \quad h\bar{k}l \quad \bar{h}\bar{k}l,$$

the latter the faces

$$hk\bar{l} \quad \bar{h}k\bar{l} \quad h\bar{k}\bar{l} \quad \bar{h}\bar{k}\bar{l}.$$

ii. *The proto-hemioctahedron* $\xi\{hkl\}$ or $\xi\{\bar{h}kl\}$. The form hemimorphous to the plane *S* is symmetrical to the planes Σ and *C*. Its faces are

for the form $\xi\{hkl\}$,

$$hkl \quad h\bar{k}l \quad hk\bar{l} \quad h\bar{k}\bar{l},$$

for the form $\xi\{\bar{h}kl\}$,

$$\bar{h}kl \quad \bar{h}\bar{k}l \quad \bar{h}k\bar{l} \quad \bar{h}\bar{k}\bar{l}.$$

iii. *The deutero-hemioctahedron* $x\{hkl\}$ or $x\{h\bar{k}l\}$ is symmetrical to the planes S and C. The faces of the form

$x\{hkl\}$ are $hkl\ \bar{h}kl\ hk\bar{l}\ \bar{h}k\bar{l}$,

and of $x\{h\bar{k}l\}$ are $h\bar{k}l\ \bar{h}\bar{k}l\ h\bar{k}\bar{l}\ \bar{h}\bar{k}\bar{l}$.

Since the forms are hemimorphous on the plane Σ, in each of them the sign of the k index is the same for every face.

In a hemimorphous developement of the scalene octahedron $\{213\}$ of Fig. 254 (*b*) the upper and lower halves would be $\rho\{213\}$ and $\rho\{21\bar{3}\}$, the right and left halves $\xi\{213\}$ and $\xi\{\bar{2}13\}$, and the front and back halves $x\{213\}$ and $x\{2\bar{1}3\}$ respectively.

It is evident that all the forms may be hemimorphous to a systematic plane with the exception of such as have their faces perpendicular to the systematic plane in respect to which the crystal is hemisymmetrical. Thus in the first case, where the horizontal systematic plane C has its symmetral character in abeyance, besides the forms $\rho\{hkl\}$ and $\rho\{hk\bar{l}\}$, there may be a hemi-protodome $\rho\{0kl\}$ or $\rho\{0k\bar{l}\}$, a hemi-deuterodome $\rho\{h0l\}$ and $\rho\{h0\bar{l}\}$, and a hemi-tritopinakoid $\rho(001)$ or $\rho(00\bar{1})$; but the proto- and deutero-pinakoids and the rhombic prism will only betray a hemimorphous character in their association with the other hemimorphous forms: and similarly for the forms $\xi\{0kl\}$, $\xi\{010\}$, and $\xi\{001\}$, $x\{h0l\}$, $x\{100\}$, and $x\{001\}$ in the remaining groups of hemimorphous forms.

306. The question as to the existence of a *hemi-systematic* section of the merohedral forms belonging to this system presents a certain ambiguity; for the hemi-symmetrical and tetarto-symmetrical forms which might seem to accord with the law of mero-symmetry will be found to present the symmetrical characters of the holo-symmetrical and hemi-symmetrical forms respectively of the Clino-rhombic system. So that it is very questionable whether such forms could, in accordance with mero-symmetrical principles, hold a place in the system under consideration, as not being impressed with any especial characteristics of that system. Thus, if we consider the hemi-systematic diplohedral form, it evidently presents four faces belonging to two normals.

The four planes thus concurring will belong to the zone which contains the two normals, and in the plane of which lies the zone-axis to which these are diagonally symmetrical; the systematic plane which is perpendicular to this axis will also belong to the zone, and its trace on the zone-circle will, with the axis normal to it, ortho-symmetrically divide the zone.

The form would be a rhombic prism, and, if a form of the Ortho-rhombic system, would be represented by one or other of the symbols $\pi\{h k l\}$ or $\pi\{h \bar{k} l\}$, $\psi\{h k l\}$ or $\psi\{\bar{h} k l\}$, $\phi\{h k l\}$ or $\phi\{\bar{h} k l\}$.

Since no symmetral axes in the Ortho-rhombic system are similar, each axis, as also each systematic plane, is independent of the others in respect to the mero-symmetry it may thus present. But this condition of a plane of symmetry having an axis of diagonal symmetry for its normal is precisely that which characterises the holo-symmetrical forms of the Clino-rhombic system. A hemi-systematic ortho-rhombic form would thus differ from a holo-systematic clino-rhombic form, not in the type of its symmetry, but only in the ortho-symmetrical character of the one systematic zone-circle to which it was symmetrical.

All the crystals to which this clino-rhombic habit in their hemi-systematic forms had been attributed have been shown, by reference to their optical and other physical characters, to belong either to the Clino-rhombic or to the Anorthic system.

Ortho-rhombic System. C.—*Combinations of Forms.*

307. (*a*) *Holo-symmetrical Forms.* Crystals belonging to the Ortho-rhombic system present great diversity of aspect in their combinations; sometimes the predominance of an octahedrid or pinakoid form imparts to the crystal a pyramidal or a tabular character; or again, a more or less prismatid aspect results from the presence of a conspicuous zone of prism- or dome-forms; while frequently the different kinds of form are so balanced that the crystal cannot be classed under either of these three types.

In Figs. 255–9 are illustrated various combinations of the forms $\{111\}$, $\{110\}$, $\{011\}$, $\{101\}$, $\{100\}$, $\{010\}$, $\{001\}$, corresponding to the elements given in Art. **300**. In Fig. 255 the parametral octahedron {111} is predominant and the combination is pyramidal in aspect: four of the edges of the octahedron are truncated by the faces of the proto-dome {011} and two of its quoins by the faces of the proto-pinakoid {100}; the parallelism of the edges formed by the faces (100), (111), (011), ($\bar{1}$11), ($\bar{1}$00) points to the tautozonality of these faces. The combination shown in Fig. 256

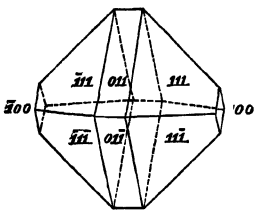

Fig. 255.

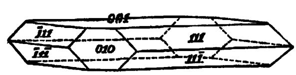

Fig. 256.

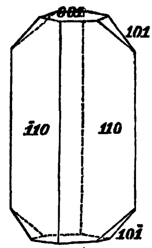

Fig. 257.

is tabular, owing to the size of the faces of the basal pinakoid {001}: and that of Fig. 257 is prismatid through the predominance of the prism {110}, which is closed by smaller faces of the basal pinakoid {001} and the deutero-dome {101}. In the combinations shown in Figs. 258, 259 more than one form is largely developed: in Fig. 258 two edges of the dome {101} are trun-

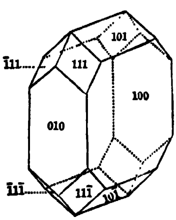

Fig. 258.

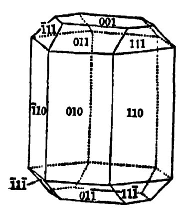

Fig. 259.

cated by two large faces of the proto-pinakoid {100} perpendicular to which are the faces of the deutero-pinakoid {010}, while the edges of intersection of the forms {010} {101} are replaced by the faces of the parametral octahedron {111}; on the other hand, the crystal figured in Fig. 259 is prismatid in developement; the prism {110} has two of its edges truncated by the faces of the deutero-pinakoid {010}, and is closed by those of the basal pinakoid {001}; the edges of intersection of the latter form with {010} {110} respectively being replaced by faces of the forms {011} {111}.

308. Fig. 260 represents a frequent combination exhibited by crystals of native sulphur. The parametral octahedron {111} is predominant, and imparts a distinctly octahedrid aspect to the crystal. Four of its edges lying in the Σ-symmetral plane are truncated by faces of the dome {101}, and its summit-quoins by the basal pinakoid {001}; the edges formed by its faces with this pinakoid are replaced by the faces of the octahedron {113}. Fig. 261 represents a crystal of witherite, and illustrates the pseudo-

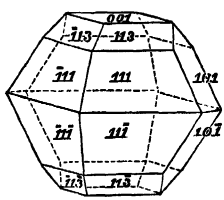

Fig. 260.

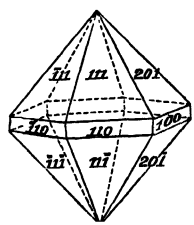

Fig. 261.

hexagonal aspect of many crystals in this system: the angles $100 \cdot 110$ and $110 \cdot \bar{1}10$ are $59^\circ 15'$ and $61^\circ 30'$ respectively, so that the combination of the pinakoid {100} with the parametral prism {110} approximates very nearly in angles and in aspect to the regular hexagonal prism with normal-angles of 60°.

Fig. 262 is that of a crystal of topaz (in the British Museum). The crystals of this mineral are frequently embedded at one extremity in the rock, the faces on the other extremity being often numerous, and those belonging to different forms very distinctly recognisable by their physical features: the aspect is usually prismatid. In the figure the prisms {110}, {210} and {320} are associated with the proto-domes {011} and {013}, the deutero-domes {101} and {201}, the octahedrid forms {113}, {112}, and (almost microscopically developed) {111}.

809. Had the prism {210}, the angles of the adjacent faces of which are 86° 52′ and 93° 8′, been adopted as the parametral prism, topaz would have been a more evident illustration of the approximation to tetragonal type seen in a large number of the crystals that belong to the Ortho-rhombic system. Thus andalusite, a mineral closely allied to topaz in chemical type, has its parametral ratio $\frac{a}{b} = 0{\cdot}9856$. Calco-uranite, a complex hydrated phosphate of uranium and calcium, has the ratio $\frac{a}{b} = 0{\cdot}9876$, while the analogous minerals in which copper takes the place of calcium or arsenic takes the place of phosphorus in the former compounds are actually tetragonal. In the enstatite

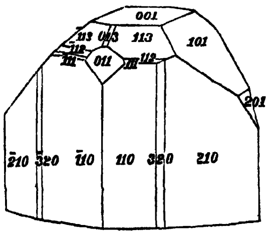

Fig. 262.

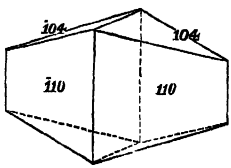

Fig. 263.

group of silicates the ratio $\frac{a}{b} = 1{\cdot}03$, and a similar approximation to equality in parameters is seen both in the mono-symmetric or augite group and the anorthic or pajsbergite group of minerals of similar chemical type with enstatite. The natrolite group of silicates offers analogous illustrations of this coincidence. The libethenite group of phosphates and arsenates also presents parametral ratios in the prism that differ little from unity; and in other crystals this character is met with in one or other of the domes. In the group of crystallised substances isomorphous with hepta-hydrated magnesium sulphate the ratio $\frac{a}{b}$ varies from 0·9804 to 0·9901.

810. Fig. 263 illustrates the combination of a dome {104} with a parametral prism {110} that is not rare in crystals of mispickel.

In Fig. 264 a crystal of göthite of prismatid developement due to the presence of the forms {100}, {110} and {120} is terminated by the forms {111} and {101}. Fig. 265 shows a crystal of aragonite on which the predominant forms are the proto-pinakoid {100}, the parametral prism {110}, and the deutero-dome {101}, the remaining forms, {001}, {111} and {211}, though subordinate, being still well-developed.

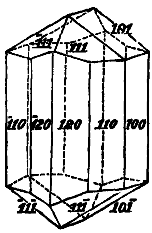

Fig. 264.

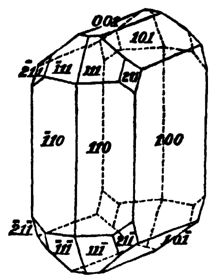

Fig. 265.

And in Figs. 266 and 267 representations of barytes and bournonite respectively illustrate tabular crystals of this system facetted by a large number of forms.

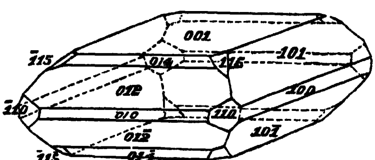

Fig. 266.

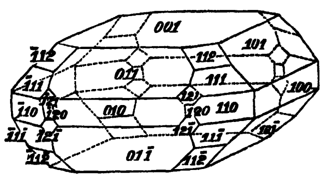

Fig. 267.

311. (*b*) *Mero-symmetrical Forms.* The hemi-symmetrical developements of crystals in this system are numerous: illustrations of crystals exhibiting combinations of hemi-symmetrical forms are shown in Figs. 268 (*l*), 268 (*r*), 269, and 270.

Figs. 268 (*l*) and (*r*) represent crystals of sodium ammonium lævo- and dextro-tartrate respectively, in which the forms {111} and {211} are sphenoidally developed and are combined with the three pinakoids, the parametral domes and prism, as also the prism {210}. If the mixed crystals deposited from a solution of the corresponding racemate be separated into two sets according as they present the developement shown in Fig. 268 (*l*) or that of Fig. 268 (*r*), their solutions will rotate a plane polarised ray to the left and right respectively, although the crystals are themselves

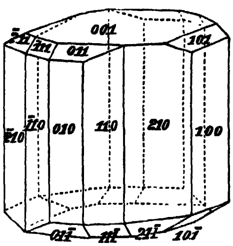

Fig. 268 (*l*).

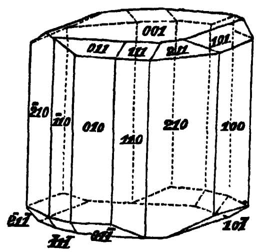

Fig. 268 (*r*).

devoid of this property. A series of corresponding tartrates also exhibits the same characters.

312. In Fig. 269 sphenoidal hemi-symmetry is seen in a crystal of hepta-hydrated magnesium sulphate ($Mg\,SO_4$, $7H_2O$) in which the sphenoid *a* {111} is combined with the prism *a* {110}, the latter having all the faces of the holo-symmetrical prism {110}.

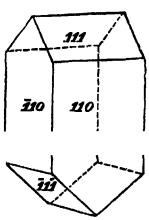

Fig. 269.

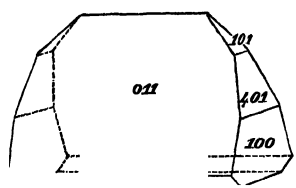

Fig. 270.

The correlative sphenoid *a* {$\overline{111}$} is frequently associated with *a* {111} on the crystals of this substance.

313. Fig. 270 of struvite and Fig. 271 of hemimorphite present characteristic examples of hemimorphism in ortho-rhombic crystals. The latter of these minerals is a conspicuous illustration of the significant concurrence of hemimorphism with pyro-electric properties already alluded to in Article **278** in the case of tourmaline.

314. The only variety of tetarto-symmetry of which an ortho-rhombic crystal is susceptible is illustrated in crystals of milk-sugar: Fig. 272 represents a crystal of this substance presenting the form $\rho a\ \{1 1 \bar{1}\}$ associated with other forms holo-symmetrically developed. These are sphenoidal in their type, but are at the same

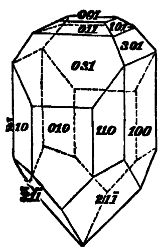

Fig. 271.

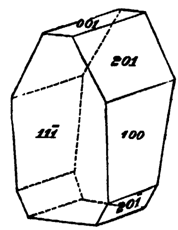

Fig. 272.

time hemimorphous in their developement. The solution of milk-sugar rotates a polarised ray to the right, but the crystals themselves possess no gyratory action: in this respect they present some analogy with the series of tartrates already mentioned.

Ortho-rhombic System. D.—*Twinned Forms.*

315. The three ortho-symmetral axes in this system are by their nature precluded from being twin-axes, except in the case where the axis has been deprived of all symmetral character by mero-symmetrical suppression. This is not the case with a sphenoidal form, but is so where the crystal is hemimorphous in respect to either of the systematic planes passing through the axis. Any other face-normal of a holo-symmetrical crystal may, however, be a twin-axis: hence either a dome- or prism-face, or again, a face of an octahedrid form may be a twin-face: a pinakoid-face can only be a twin-face when the crystal is hemimorphous in respect to one of the remaining pinakoid-faces.

I. *Twin-plane a face of a scalene octahedron.*

316. This mode of twinning is rare, and it is singular that, in the two or three crystallised substances known to present it, the

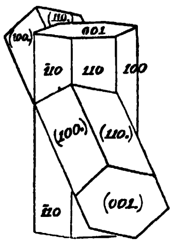

Fig. 273.

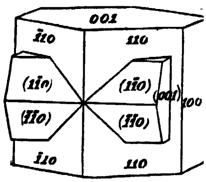

Fig. 274.

twin-plane is parallel to the face of a form not actually met with so far on the crystals. Fig. 273 represents an interpenetrant twin-crystal of staurolite twinned round the normal of a face with the symbol (322) never observed on this mineral. The normal-angle between a face of this form {322} and the adjacent pinakoid-face of the form {001} is 60° 37′, so that the pinakoid-faces with identical symbol 001 in the two individuals are inclined in the twin at a normal-angle of 121° 14′; the zone-plane of these faces is inclined on the zone-plane [010] at an angle of 54° 37′. The twin-axis is inclined on the normal of (100) or the zone-axis [001·010] at an angle of 59° 42′, whence it follows that the normal-angle of two pinakoid-faces with identical symbol 100 in the two individuals will be 119° 24′.

Copper glance, and stromeyerite isomorphous with that mineral, also present twins of which the twin-plane is a face of an as yet unobserved octahedrid form {112}.

II (*a*). *Twin-plane a face of a dome.*

317. Examples of this kind of twinning are afforded by staurolite, the twin-plane being a face of the unobserved form {302} (Fig. 274); by anhydrite, with a twin-face belonging to the form {102}; and by copper glance, twinned on a face of the form

{403}. In the staurolite twin, with twin-plane (302) (Fig. 274), the angle between the Z-axes is near to 90° (91° 36′), the normal-angle (302·001) being 45° 48′.

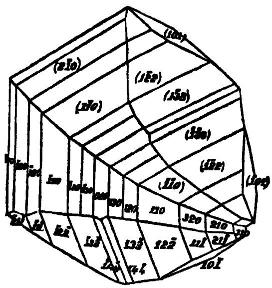

Fig. 275.

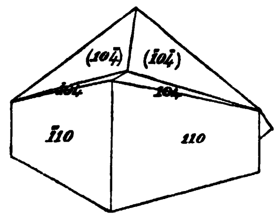

Fig. 276.

Fig. 275 represents a twin of manganite. The twin-plane is (101): the normal-angle of the faces (001·101) being 28° 35′, the Z-axes of the two crystals are inclined to each other at an angle of 57° 10′ (near to 60°) and lie in the zone-plane [010].

Grouped twins of alexandrite, with faces of the form {301} for twin-planes, have been alluded to in Article **165** as illustrating a pseudo-hexagonal type of combination resulting from repeated twinning on a dome-face.

Mispickel is sometimes twinned on a face of the parametral proto-dome {011}, so that the Z-axes of the crystals lie in the zone-plane [100] and are inclined to each other at an angle of 120° 48′ (near to 120°), the angle between the faces $0\bar{1}1$·001 being 60° 24′. In Fig. 276 the twin-plane is ($0\bar{1}1$): a simple crystal is shown as Fig. 263.

Twins of humite occur, twinned in two ways; the one on faces of a form {307}, the other on those of a form {107}, neither of which forms has been actually observed on crystals of the substance: these twin-planes are inclined to the basal pinakoid at normal-angles of nearly 60° and 30° respectively.

II (*b*). *Twin-plane a face of a prism.*

318. Aragonite offers a conspicuous example of the great variety capable of being presented by the twinned crystals of this system.

The twins of aragonite are frequently pseudo-hexagonal in their character as a result of the prism-angle being not very remote from 60°, and the twinning being repeated on the faces of the form {110}. Usually the prism is associated with the pinakoid {100} and the dome {101}, and forms belonging to the zone [100·101] are of frequent concurrence with these. The octahedron {111} is also a frequently prevalent form (Fig. 265). The normal-angle of the faces $110 \cdot \bar{1}10$ of the prism {110} is $63^\circ 44'$, and the normal-angle 100·110 is therefore $58^\circ 8'$.

If now the crystal is twinned by the half-revolution of a parallel individual round the normal of the face $1\bar{1}0$, and this plane be likewise the plane of junction of the two crystals, the re-entrant normal-angle between the face 100 of the first and the face ($\bar{1}00$) of the second individual is ($121^\circ 52' - 58^\circ 8'$ or) $63^\circ 44'$, and the

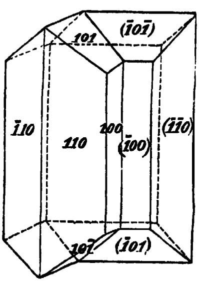

Fig. 277.

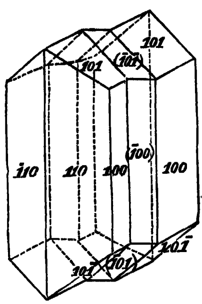

Fig. 278.

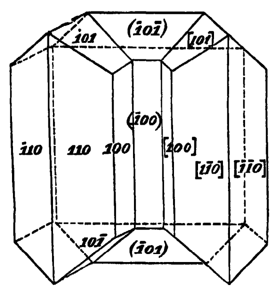

Fig. 279.

salient angle between the face 110 of the first and the face ($\bar{1}\bar{1}0$) of the second individual is ($116^\circ 16' - 63^\circ 44'$ or) $52^\circ 32'$.

319. The repetition of the twinning on the successive prism-faces of the different individuals produces complex crystal-structures, that vary in their aspect and angular relations according to the faces on which the twinning occurs.

In Fig. 278 there are three individuals; while the front crystal retains its position, the second crystal is twinned on the normal of the face $1\bar{1}0$ of the first, and the third on the normal of the face ($1\bar{1}0$) of this intercalated second crystal. In Fig. 279 the three individuals are successively twinned, the second as before on the

$1\bar{1}0$ face of the first and left-hand individual, but the third on the $(\bar{1}\bar{1}0)$ face of the second crystal.

320. At Herrengrund in Hungary and Girgenti in Sicily are found very fine specimens of aragonite compounded of several individuals, but which at first sight may be taken for simple hexagonal prisms terminated by a basal pinakoid. It is seen, however, that the striations on the basal pinakoid are in diverse directions, and that obtuse re-entrant edges traverse some of the faces of the pseudo-hexagonal prism in a direction parallel to its edges. The structure of these crystals is illustrated by Figs. 280–2.

Fig. 280 is a projection of three individuals on the basal pinakoid (001), each being a combination of the proto-pinakoid $\{100\}$, the parametral prism $\{110\}$, and the basal pinakoid $\{001\}$, of

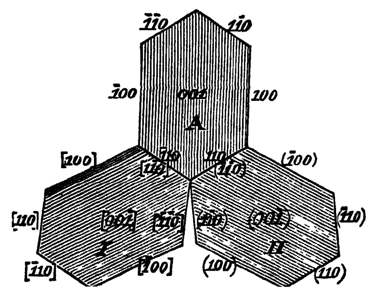

Fig. 280.

which the latter is grooved with striations parallel to its intersections with the faces of the proto-pinakoid $\{100\}$. The crystals I and II are twinned respectively on the planes $\bar{1}10$ and 110 of the crystal A. The normal-angle $110 \cdot \bar{1}10$ being $63^\circ 44'$, the face $[\bar{1}\bar{1}0]$ of crystal I makes a small re-entrant angle of $11^\circ 12'$ with the face $(1\bar{1}0)$ of crystal II: the basal pinakoids 001, $[00\bar{1}]$, $(00\bar{1})$ are coincident in direction: the faces $100 \cdot (\bar{1}00)$ and the faces $\bar{1}00 \cdot [100]$ form re-entrant normal-angles of $63^\circ 44'$, and the faces $[\bar{1}00]$ (100) a re-entrant normal-angle of $52^\circ 32'$: hence the striations of I and II make an angle of $116^\circ 16'$ with those of *A*, and an angle of $127^\circ 28'$ with each other.

321. If the crystals of Fig. 280 are interpenetrant, the resulting compound crystal may present the aspect illustrated in Fig. 281. The faces $1\bar{1}0$, $[1\bar{1}0]$ and their parallels $\bar{1}10$, $[\bar{1}10]$, of crystals A and I are coincident in direction: similarly for the faces $\bar{1}\bar{1}0$, $(\bar{1}\bar{1}0)$ and their parallels 110, (110), of crystals A and II: while the

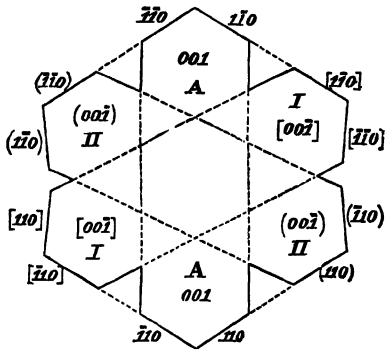

Fig. 281.

faces $(1\bar{1}0)$, $[110]$, and their parallels $(\bar{1}10)$, $[\bar{1}\bar{1}0]$, of crystals I and II make re-entrant normal-angles of 11° 12′.

322. If the crystals grow until the faces of the proto-pinakoid disappear, the structure presents the aspect of a pseudo-hexagonal prism, of which each edge has a normal-angle of 63° 44′, and of

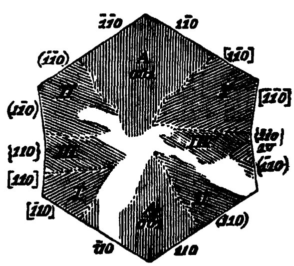

Fig. 282.

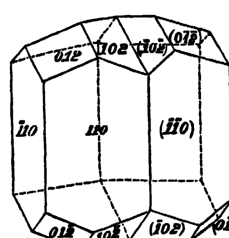

Fig. 283.

which two opposite faces are traversed by an obtuse re-entrant edge of 11° 12′. In Fig. 282 are shown the striations to be observed on the basal pinakoid of a compound crystal from Girgenti, now in the British Museum, very similar in structure to the compound crystal of Fig. 281: corresponding parts of the two figures are distinguished by the same letters and numbers. In the building up of this complex structure two additional individuals III and IV have a part: the individual III is a twin of the individual I about the plane [110], so that the faces [110], {110}, as also their parallels [$\bar{1}\bar{1}0$], {$\bar{1}\bar{1}0$}, are co-planar; the minute individual IV is similarly a twin of the individual II about the plane ($\bar{1}\bar{1}0$).

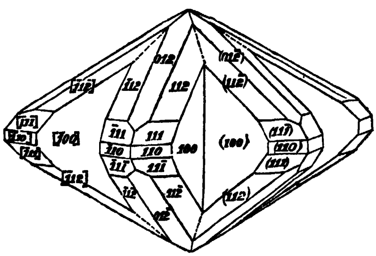

Fig. 284.

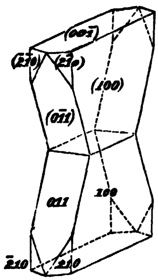

Fig. 285.

323. Figs. 283 and 284 represent twinned crystals of cerussite, the former being a simple twin on the face $1\bar{1}0$, and the latter a triple combination with faces of the form {110} for twin-planes.

Copper glance and stephanite again afford examples quite analogous to the above in the character of the twins they form, the twin-planes belonging in each case to the form {110}, and the pseudo-hexagonal character being due to the approximation of the prism-angles to 60°: in the case of copper glance it is so near that value as 59° 48′.

Crystals of cerussite twinned about a plane of the prism {310}, the faces of which make an angle of 92° 45′ (nearly 90°) with those of the parametral prism {110}, have been observed.

III. *Twin-plane a face of a pinakoid.*

324. It has already been observed that a systematic axis can only become a twin-axis in the event of its symmetral attributes being in abeyance. This can only occur when the crystal is hemimorphous with respect to the systematic plane in which the zone-axis in question lies. Fig. 285 represents a crystal of struvite twinned about the pinakoid (100), and having for plane of junction the perpendicular face 001 with respect to which the crystal is hemimorphous: crystals of hemimorphite and seignette salt (sodium potassium tartrate with four equivalents of water) also afford examples of this variety of twin.

Section V.—**The Mono-symmetric or Clino-rhombic System.**

A.—*Holo-symmetrical Forms.*

325. The Mono-symmetric system is characterised by symmetry to a single plane. The zone of which this is the zone-plane is accordingly (see Art. **91**, p. 107) not symmetrical in respect to any of its own planes, so that no faces in the zone can be permanently perpendicular to each other or inclined successively at any crystallometric angle.

Since the sphere of projection is divided into hemispheres by a single symmetral plane, there is no systematic triangle. The diplohedral form is however symmetrical to the zone-axis of the systematic plane as an axis of diagonal symmetry, the general form $\{h\,k\,l\}$ presenting four faces symmetrical to the systematic plane and to a plane perpendicular to it. The systematic plane and two origin-planes perpendicular to it are taken as axial planes: the lines in which they intersect each other are the crystallographic axes, the one which is normal to the systematic plane being taken as the axis Y; the axes X and Z lie in the systematic plane and are perpendicular to the axis Y. The obtuse angle η formed by these two axes will be taken for the positive angle XOZ, so that OC, OA the normals of the planes YOX, YOZ are contained within it.

The poles 100, $\bar{1}00$ and 001, $00\bar{1}$ of the axial planes will therefore lie on the arcs XZ and $\overline{X}\overline{Z}$; the arc η' between the poles 100 001 or $\bar{1}00$ $00\bar{1}$ being the supplement of the arc η. In stereographic projections the plane of symmetry is generally taken as the plane of the drawing: the axis of Z being placed in a vertical position *OA* the normal to it in the systematic plane is horizontal. The elements of a crystal in this system are

where the relative magnitudes of the parameters are not indicated by the alphabetical order of the letters.

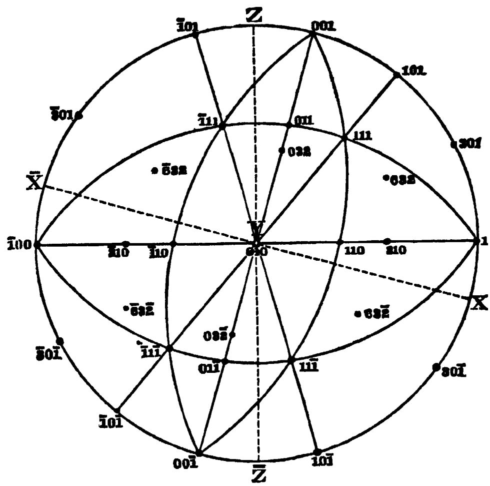

Fig. 286.

326. The varieties in the forms of the Mono-symmetric system are:

I. Such as have their poles neither on the axis nor on the circumference of the zone-circle [010];

viz. (*a*) the general form, either a *positive* (*prismatid form or*) *prismatid* $\{h\,k\,l\}$, or a *negative prismatid* $\{\bar{h}\,k\,l\}$;

(*b*) particular cases of this form in which

i. the poles lie on a zone-circle passing through (100), viz. the *ortho-prism* $\{h\,k\,0\}$, including the *parametral prism* $\{110\}$;

ii. the poles lie on a zone-circle passing through (001), viz. the *ortho-dome* $\{0\,k\,l\}$, including the *parametral dome* $\{011\}$.

II. Such as have their poles on the zone-circle [010],
the *positive* and *negative hemi-domes* $\{h0l\}$ and $\{\bar{h}0l\}$,
including the *parametral hemi-domes* {101} and $\{\bar{1}01\}$;
the *ortho-pinakoid* {100};
and the *basal pinakoid* {001}.

III. The *systematic pinakoid* {010}.

In this nomenclature the term *dome* is employed not in contra-distinction to the term *prism* or *prismatid*, but, like the latter term, conventionally and merely to distinguish these forms from one another: the dome, by analogy with the Orthorhombic system, being a form with its poles on the zone-circles [010 001] and [100 001].

Fig. 286 represents in stereographic projection the positions of the poles of these kinds of forms for the case where

$$\eta = 106^\circ\ 1' \text{ and } a : b : c = 0{\cdot}541 : 1 : 0{\cdot}913,$$

as in diopside.

327. I. That the general mono-symmetric form $\{hkl\}$ will be a rhombic prism is evident from its having four poles lying on an orthosymmetrical zone-circle passing through the pole of the systematic plane (010). If one of its poles lie in the octant formed by the great circles passing through the poles 100, 010, 001, the form is termed a *positive prismatid* and its four faces are

$$hkl \quad h\bar{k}l \quad \bar{h}k\bar{l} \quad \bar{h}\bar{k}\bar{l};$$

those four of the octants formed by the great circles in question which contain the poles of the positive form being similar. And the remaining four octants will similarly contain the poles of the *negative prismatids*, viz.

$$\bar{h}kl \quad \bar{h}\bar{k}l \quad hk\bar{l} \quad h\bar{k}\bar{l}.$$

The symbol of the zone containing the four faces of the form $\{hkl\}$ being $[hkl,\ 010]$ or $[\bar{l}0h]$, the first and last index in the symbol of each of the faces must have the same sign: and similarly a face of a form $\{\bar{h}kl\}$ must have different signs in its first and third indices. Figs. 287 (*a*) and (*b*) represent respectively for the above elements the positive prismatid {632} and the negative prismatid $\{\bar{6}32\}$ closed by the faces of the form $\{100\}$.

Where the indices have the same magnitudes in each form, a positive and a negative prismatid combine to produce a quasi-

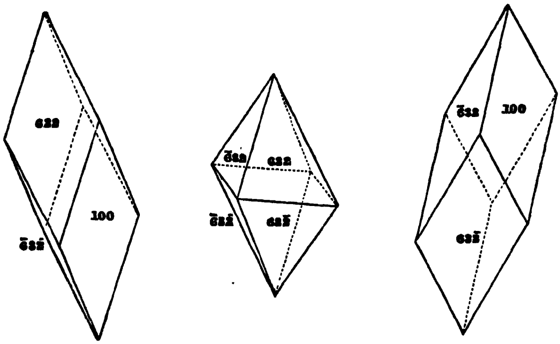

Fig. 287 (*a*). Fig. 287 (*b*). Fig. 287 (*c*).

octahedral form, as in Fig. 287 (*b*). The faces however of the positive and negative component prismatids are not similar. They are scalene triangles of two kinds corresponding to the different forms; and the edge in which two adjacent faces of these forms meet is incapable of undergoing truncation or bevilment.

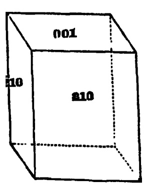

Fig. 288.

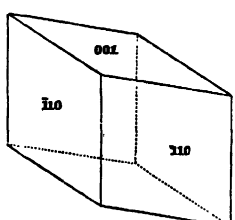

Fig. 289.

328. Among the varieties of prismatids, of which the poles always lie in a zone perpendicular to the zone-circle of symmetry [010], two are especially noticeable.

One of these varieties includes the *vertical* or *ortho-prism* $\{hk0\}$, usually distinguished as the *prism-form*, the faces of which lie in the zone [100, 010]; in Fig. 288 is illustrated the ortho-prism {210} closed by the form {001}. The *parametral prism* {110} belongs to this type and is generally one of the most frequent and important forms on crystals belonging to this system (Fig. 289); in fact, the zone selected as that of the ortho-prism is usually chosen on account of the pre-eminence of these faces.

The other especially important variety consists in the suite of dome-forms with the general symbol {0*kl*} (illustrated in Fig. 290 by the form {032} closed by the faces of the form {100}) lying in the zone [010, 001]; one of them is the parametral dome {011} (Fig. 291). This variety of dome is termed the *ortho-dome*,

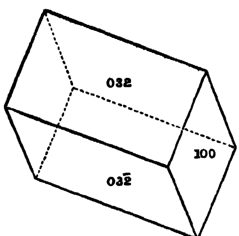

Fig. 290.

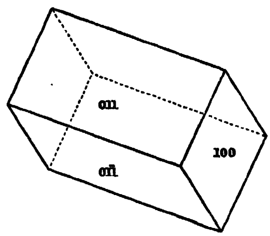

Fig. 291.

because, like the ortho-prisms, it has an ortho-symmetrical character, the zone containing it being symmetrical to the axis [010] and also to the normal of the face (001), perpendicular to that axis.

Another important zone perpendicular to the plane of symmetry is that passing through the parametral plane 111. Its symbol is evidently [10$\bar{1}$], so that for all planes belonging to it the first and last index are equal and have the same sign. This zone is tautohedral with the zone [010] in the two faces 101 and $\bar{1}0\bar{1}$ of the form {101}. The quasi-octahedrid form constituted by the union of the positive prismatid {111} and the negative prismatid {$\bar{1}$11} would correspond to the parametral octahedron of a rectangular-axed crystal: Figs. 292 (*a*) and (*c*) represent the parametral positive and negative prismatids {111} and {$\bar{1}$11} respectively, each closed by the faces of the form {100}; in Fig. 292 (*b*) is illustrated the quasi-octahedrid form produced by their union. The two faces of the form {$\bar{1}$01} are in the zone [010, $\bar{1}$11].

329. II. With regard to the forms the poles of which lie in the zone [010], which is symmetrical only to its centre, their faces will of course consist in each case of a single parallel pair; the various special forms comprised under the general symbol $\{h0l\}$ being in fact a series of pinakoids. And of this series the forms will be positive when their poles lie on the arcs joining the poles

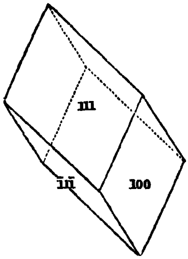

Fig. 292 (*a*).

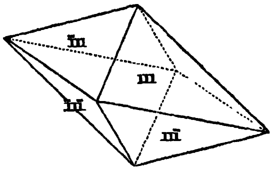

Fig. 292 (*b*).

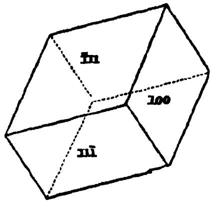

Fig. 292 (*c*).

001, 100 and $00\bar{1}$, $\bar{1}00$; negative when they lie on the arcs joining 001, $\bar{1}00$ or $00\bar{1}$, 100.

And if the indices in a positive and negative form irrespective of sign are the same, the two pinakoids, though not of necessity concurrent, may be conceived as uniting to produce a prismatid

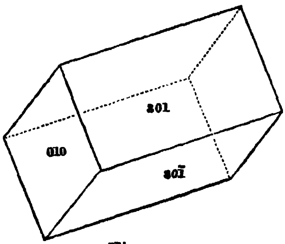

Fig. 293.

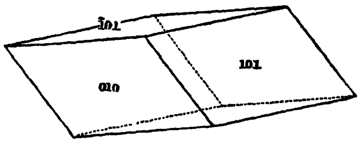

Fig. 294.

or quasi-dome-form: in Fig. 293 is shown a combination of the pinakoid $\{301\}$ with the pinakoid $\{\bar{3}01\}$, closed by the faces of the form $\{010\}$; Fig. 294 represents the positive and negative parametral pinakoids $\{101\}$ and $\{\bar{1}01\}$, closed by the faces of the form $\{010\}$. Such a double form has been termed a *clino-dome;* and it will be convenient to retain the terms *positive* and *negative*

hemidome, in order to distinguish the forms that actually compose it, the symbols of which are $\{h\,0\,l\}$ and $\{h\,0\,l\}$ respectively, from the particular cases where one of the indices h or l in the symbol $\{h0l\}$ is zero; to which the term *pinakoid* will be restricted. The *positive hemidome* $\{h0l\}$ comprises the faces $h\,0\,l$ and $\bar{h}\,0\,\bar{l}$; the *parametral hemidome* being $\{101\}$; the *negative hemidome* $\{\bar{h}0l\}$ comprises the faces $\bar{h}\,0\,l$, $h\,0\,\bar{l}$, and includes the *parametral negative hemidome* $\{\bar{1}01\}$.

The *ortho-pinakoid* $\{100\}$ is the form the faces of which are parallel to the axial-plane YZ. Its faces are 100, $\bar{1}00$, and are tautozonal with those of the prism (the ortho-prism) (Figs. 287, 290–2).

The *basal* or *clino-pinakoid* is the form $\{001\}$, the faces of which, namely 001 and $00\bar{1}$, are parallel to the axial plane YX and are tautozonal with the faces of the ortho-dome (Figs. 288, 289).

III. Finally, of the *systematic pinakoid* $\{010\}$ the two faces 010 and $0\bar{1}0$ are parallel to the plane of symmetry (Figs. 293, 294).

Mono-symmetric System. B.—*Mero-symmetrical Forms.*

330. The general form in the Mono-symmetric system, inclusive of the ortho-prism and the ortho-dome, has four poles carried by two normals and is symmetrical to the systematic plane S. Its holo-systematic mero-symmetrical forms will therefore be of two kinds.

I. One of these forms presents two poles symmetrical on the plane S; the symbol of the correlative forms being therefore $s\{hkl\}$ and $s\{\bar{h}\bar{k}\bar{l}\}$ for a positive prismatid, and $s\{\bar{h}kl\}$ and $s\{h\bar{k}\bar{l}\}$ for a negative prismatid. Thus the two extant faces of the form

$$s\{hkl\} \text{ are } hkl \text{ and } h\bar{k}l,$$

those of the form

$$s\{\bar{h}kl\} \text{ are } \bar{h}kl \text{ and } \bar{h}\bar{k}l.$$

Of the prism-form $\{hk0\}$, the correlative forms are

$$s\{hk0\} \text{ with the faces } hk0 \text{ and } h\bar{k}0,$$

$$\text{and}\quad s\{\bar{h}\bar{k}0\} \text{ with the faces } \bar{h}\bar{k}0 \text{ and } \bar{h}k0.$$

And while the ortho-dome is resolved into the correlative forms

$$s\{0\,k\,l\} \text{ with the faces } 0\,k\,l,\ 0\,\bar{k}\,l,$$

$$\text{and } s\{0\,\bar{k}\,\bar{l}\} \text{ with the faces } 0\,k\,l,\ 0\,k\,\bar{l},$$

a form $\{h\,0\,l\}$, the faces of which fall into the zone [010], will be capable of exhibiting the haplohedral hemi-symmetry in question by one only of its two faces being extant, viz. $h\,0\,l$ or $\bar{h}\,0\,\bar{l}$. And so of the negative hemi-dome either $\bar{h}\,0\,l$ or $h\,0\,\bar{l}$ and of the pinakoids 100 or $\bar{1}00$, 001 or $00\bar{1}$ will be the only extant faces, while the systematic pinakoid {010} will have both its faces extant or absent but without symmetry of form.

331. II. The other hemi-symmetrical variety of the holo-systematic forms is that in which two extant poles corresponding to two normals of the system lie on the same side of the systematic plane; so that the form is hemimorphous on that plane. In accordance with the symbolical notation we have adopted, since the form is symmetrical to the diagonal axis of symmetry of the system, though to no plane of it, the symbol for such a form would be

$$a\{hkl\} \text{ with the faces } hkl \text{ and } \bar{h}k\bar{l},$$

$$\text{or } a\{\bar{h}\bar{k}\bar{l}\} \text{ with the faces } \bar{h}\bar{k}\bar{l} \text{ and } h\bar{k}l,$$

in the case of a positive form $\{h\,k\,l\}$; $a\{\bar{h}kl\}$ or $a\{h\bar{k}\bar{l}\}$ being the symbols of the correlative semiforms of the negative form $\{\bar{h}\,k\,l\}$. The pinakoid {010} may present this hemimorphism by the absence of one of its planes: but the forms the poles of which lie in the zone-circle [010] can only exhibit hemi-symmetry of this kind in the distribution of their physical features and in their association with the mero-symmetrical forms of this type.

332. The possibility of a hemi-systematic form, in which a single normal is represented by both the faces belonging to it, involves a question similar to that already considered under the Ortho-rhombic system, Art. **306**. For such a semiform would be represented in the case for instance of a positive prismatid form by two faces, viz. hkl and $\bar{h}\bar{k}\bar{l}$, and the correlative form would present the faces $h\bar{k}l$, $\bar{h}k\bar{l}$. But such a semiform would differ from a holo-symmetrical form of the Anorthic system only in its faces belonging to an ortho-symmetrical zone or one potentially ortho-symmetrical. But there is no sufficiently characteristic distinction between such a form and an anorthic holo-symmetrical form to satisfy the principle of mero-symmetry in the Mono-symmetric system.

Mono-symmetric System. C.—*Combinations of Forms.*

333. (*a*). *Holo-symmetrical forms.* The mono-symmetric character of a crystal in this, the more symmetrical of the two oblique systems, is generally recognisable by reason of the different developement and diversity in the features of the forms it carries. The antistrophic symmetry of the two halves of the crystal as seen divided by an ideal plane of symmetry when it is looked at in the direction of that plane, and the diagonal disposition of its homologous faces from any other point of view, but especially as seen in the direction of its one axis of symmetry, are characters in general easy of recognition. Where however the axial angle η approximates to a right angle, or where the more prominent prism-forms

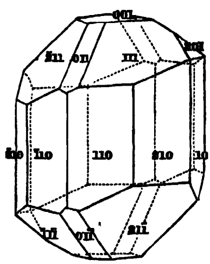

Fig. 295.

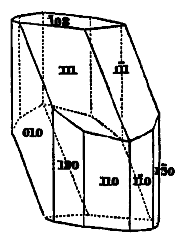

Fig. 296.

have normal-angles approximating to 45° or to 60°, and in particular where the former is united with one or other of the latter specialities, the crystal may occasionally wear the aspect of an ortho-rhombic or even of a hexagonal crystal. In such case goniometrical measurements must be had recourse to. In the not infrequent case of one or more distinct cleavages being apparent, these will, by the presence or absence of symmetrical repetition, generally guide the judgment.

334. The habit of a mono-symmetric crystal is very often distinctly prismatid; the faces of a prism $\{110\}$ or $\{hk0\}$ being then predominant: often, however, the edges $hk0$, $h\bar{k}0$ and $\bar{h}k0$, $\bar{h}\bar{k}0$ are truncated by the faces of the ortho-pinakoid {100}, or the other edges may be truncated by the faces of the systematic pinakoid {010}. The crystal of datolite in Fig. 295 exhibits the two prism-forms {110} and {210}, the edges of the latter form being truncated by the ortho-pinakoid {100}, while in Fig. 296 a crystal of gypsum is represented with prisms {110} and {120} in which the edges of the latter form are truncated by the systematic pinakoid {010}.

The crystals of augite (Fig. 297) and of hornblende (Fig. 299) exhibit prism-forms with both pairs of edges truncated by the faces of the two pinakoids {100} and {010}. The comparison of the

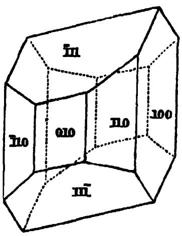

Fig. 297.

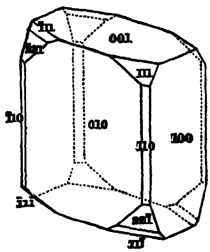

Fig. 298.

crystal of augite in Fig. 297 with that of diopside in Fig. 298, belonging to the same mineral group with it, will serve to show the different aspect assumed by a crystal according as the prismatid forms preponderate over the prism- and pinakoid-forms, or as one or both of the latter are the most developed.

335. In hornblende (Fig. 299) the normal-angles of $001 \cdot \bar{1}11$ and $001 \cdot \bar{1}\bar{1}1$ are $34^\circ 25'$, while that of $\bar{1}11 \cdot \bar{1}\bar{1}1$ is $31^\circ 32'$: further, the prism-angle $110 \cdot 1\bar{1}0$ is $55^\circ 30'$, and the angle $110 \cdot 010$ is $62^\circ 15'$. It will be seen, then, that a crystal of this mineral carrying the above forms might readily be mistaken for a rhombohedral crystal, the more so as the angle $100 \cdot 001$ is $75^\circ 2'$, while the angle $\bar{1}00 \cdot \bar{1}01$ is $73^\circ 58'$.

In some of its crystals, augite, on the other hand, presents features, in a nearly square prism {110} of which the angle is

$87^\circ\ 5'$, and in the presence of a face $\bar{1}02$ inclined to 100 at an angle of $89^\circ\ 20'$, that bring it near in aspect to an ortho-rhombic crystal.

336. The crystal of sphene represented as Fig. 300 exhibits a very oblique aspect due to a considerable prolongation of the

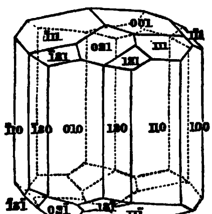

Fig. 299.

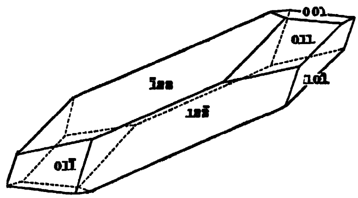

Fig. 300.

crystal in the direction of a negative prismatid $\{\bar{1}23\}$, and in epidote as shown in Fig. 301 the preponderant forms are the ortho-pinakoid {100} and basal pinakoid {001} with ortho-domes and negative prismatids, which latter impart to epidote, by the character of their developement and distribution, a characteristic

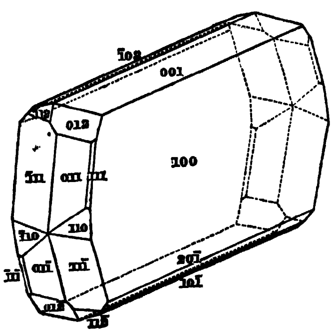

Fig. 301.

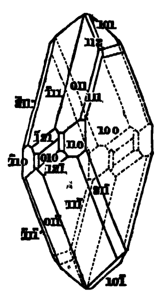

Fig. 302.

oblique appearance when looked at in the direction of the normal to the 001 face.

Fig. 302 represents a crystal of the rare mineral turnerite in the collection of the British Museum: its locality is Cornwall; it serves to give an illustration of a mono-symmetric crystal with very symmetrical developement of a large number of forms.

337. (*b*). *Mero-symmetrical forms.* The known mero-symmetrical crystals belonging to this system occur exclusively among the products of the laboratory or of the organic world. In tartaric acid an illustration is afforded of the mode in which correlative forms occur on the crystals of a substance which assumes two characters corresponding to a species of enantiomorphism in its crystals. Racemic acid has the same percentage composition and most of the characters of tartaric acid, but in solution has no action on polarised light: if from the two varieties of sodium ammonium racemate (lævo- and dextro-tartrate) described in Art. **311** the corresponding calcium salts be obtained by the addition of chloride of calcium to their respective solutions, and the calcium salts be then decomposed by sulphuric acid, solutions are obtained which have opposite rotatory actions on plane polarised light: one

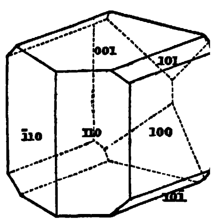

Fig. 303 (*l*).

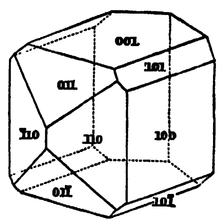

Fig. 303 (*r*).

of these solutions on evaporation yields crystals of ordinary or dextro-tartaric acid, while the other yields crystals of the same chemical composition which are termed lævo-tartaric acid: the latter crystals have the form shown in Fig. 303 (*l*), the former that of Fig. 303 (*r*). The crystals themselves do not possess this rotatory character. It will be seen by the symbols of the faces present or absent in either case that the form {011} is hemi-symmetrically developed and presents faces with positive indices for the *Y*-axis in the one case, and corresponding faces with negative indices in the other. The crystals are thus hemimorphous in developement.

338. The organic substance fichtelite or scheererite, a mineral which occurs in fossil pine-wood in the peat of the Fichtelgebirge, has been described as illustrating the variety of hemi-symmetry in which four normals of a form $\{hkl\}$ are represented by two poles symmetrical to the systematic plane.

Mono-symmetric System. D.—*Twinned Forms.*

339. The twinned forms in this system are numerous, and present considerable variation in the laws they follow even for the same substance. The analogy which is presented, at once in chemical and in crystallographic type, by many minerals is not confined to those which crystallise in one and the same system. For instance, the ortho-rhombic minerals bronzite and enstatite are homotypic in their composition with the minerals of the augite group in the Mono-symmetric system, and in their crystallographic constants present many points of close resemblance to the latter. So the felspars, which form an important and extensive group of minerals that crystallographically are anorthic, have at least one representative mineral belonging to the Mono-symmetric system. And this felspar, orthoclase, has many crystallographic features in common with the numerous minerals that are grouped under the general name of felspars in the Anorthic system. Among these characters none are more important for comparison than the methods of twinning exhibited by orthoclase, as compared with the twin-structures of the anorthic felspars. In the discussion of clino-rhombic twins this analogy has to be borne in mind.

340. In this system the twinned forms can always be explained by taking for the twin-plane a face of the crystal: but in order to keep in sight the analogy just alluded to, it is well to point out that while a mono-symmetric twin can be most simply explained as being twinned on a face-normal, it may happen that, by analogy with a corresponding anorthic twin that cannot be so simply explained, the mono-symmetric twin may also be described as twinned round a zone-axis perpendicular to the face-normal.

341. The twin-plane for crystals in this system may be a face of a form belonging to the zone [010], and therefore perpendicular to the plane of symmetry, or it may be a face oblique to this plane. In holo-symmetrical crystals it cannot be the systematic plane itself, as the normal of this plane is an axis of diagonal symmetry. The following are the various twin-laws known in this system:—

A. Twin-plane perpendicular to the plane of symmetry.

342. In the known twins falling under this law the twin-plane is either a pinakoid-face or a face of the parametral hemidome {101}.

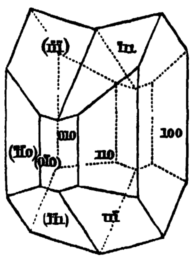

Fig. 304.

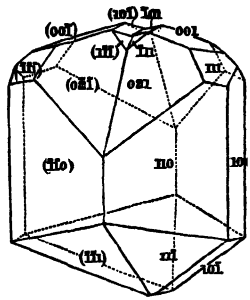

Fig. 305.

1. *Twin-plane a face of the ortho-pinakoid* {100}. Cases of this kind of twin are exemplified in augite (Figs. 304, 305) and hornblende (Fig. 306). In these twinned forms the combination-plane is parallel to the twin-plane, and the crystals are in apposition.

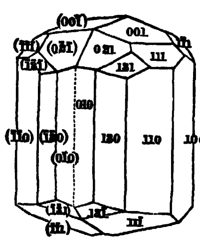

Fig. 306.

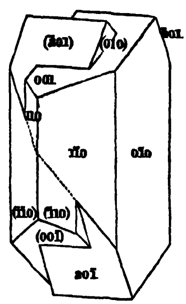

Fig. 307.

The figure of hornblende (Fig. 306) illustrates the singularly symmetrical aspect which a crystal of that mineral assumes, in certain points of view, when thus twinned. The re-entrant angles on the augite crystals (Figs. 304, 305) immediately betray the twinned nature of their structure.

Fig. 58 (*a*) (p. 175) represents a twin of orthoclase from Carlsbad, the kind designated as the Carlsbad twin: in this case the combination-plane of the juxtaposed and slightly interpenetrant individuals is perpendicular to the twin-face, and is in fact parallel to the plane of symmetry. The Carlsbad twin may equally be represented as the result of twinning round the zone-axis [001]. It is in this way described by Des Cloizeaux as indicating an analogy between the felspars belonging to the two oblique systems: the indices of the planes when the zone-axis [001] is assumed to be the twin-axis are shown in Fig. 307.

343. 2. *Twin-plane a face of the basal pinakoid* {001}. This mode of twinning is also known in orthoclase, and is well shown in crystals from Silesia: the crystals are in juxtaposition with the

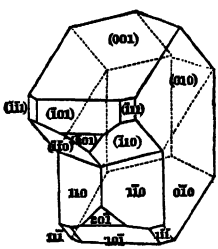

Fig. 308.

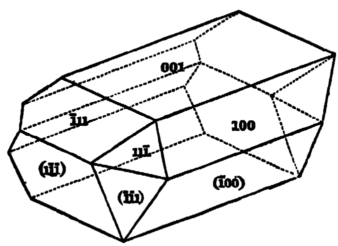

Fig. 309.

twin-plane as face of junction: Fig. 308 represents a specimen in the British Museum. An analogous twin has been recognised in the anorthic felspar, albite, but its occurrence is extremely rare.

The twin crystal of epidote represented in Fig. 309 is also an example of this law.

In Fig. 310 is illustrated a twin-crystal of sphene in which the basal pinakoid (001) is both the twin-plane and the face of junction: often the individuals are interpenetrant and have a second plane of junction normal to the first, as in Fig. 311.

344. Harmotome, long considered to be an ortho-rhombic mineral by reason of the supposed perpendicularity of its axial angles, has been shown by Des Cloizeaux to be mono-symmetric, and the twinned forms, in which alone the mineral occurs, are recognised as falling, in part at least, under this law.

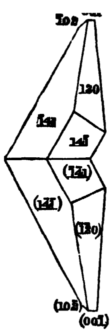

Fig. 310

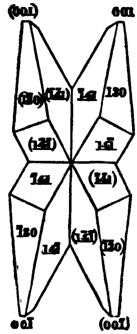

Fig. 311.

The crossed twin Fig. 314 represents a characteristic and striking variety of this mineral from Andreasberg. It is best explained as the result of a double twinning: the first twinning is round the normal to the face (001), which is likewise a face of union, as in Fig. 312: such simple twins are not met with in nature, the individuals being always so intergrown that they cross over to opposite sides of the first face of union and have a second plane of union

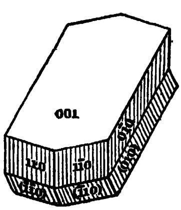

Fig. 312.

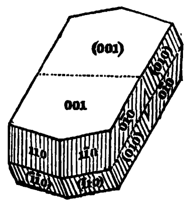

Fig. 313.

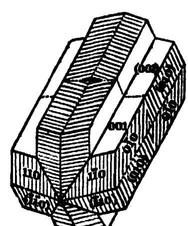

Fig. 314.

perpendicular both to the first and to the plane of symmetry (Fig. 313): the second plane of union is very nearly parallel to a face of the hemidome {101}, the angle 001 . 101 being almost exactly 90°. Such a twin has therefore three perpendicular planes of symmetry, and may be mistaken for a simple ortho-rhombic crystal. The second twinning is about a plane of the ortho-dome {011}, inclined to the basal pinakoid at an angle of 45° 18′: hence the angles between the adjacent twinned pinakoids, which form the re-entrant edges of this complex growth, are 90° 36′ and 89° 24′ alternately.

345. 3. *Twin-plane a face of the parametral hemidome* {101}.

Gypsum affords an example of a twin following this law (Fig. 315): the twin-plane is also the face of junction. This twin is

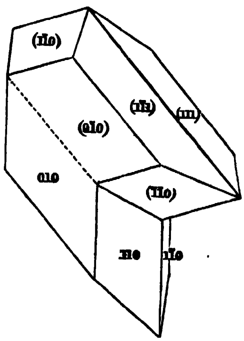

Fig. 315.

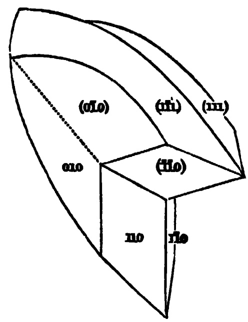

Fig. 316.

of frequent occurrence, but the faces of the extant forms of the crystals from Montmartre, near Paris, are usually much rounded, as in Fig. 316.

B. Twin-plane oblique to the plane of symmetry.

346. In its second law, where the twin-plane is a face of the ortho-dome {011}, harmotome has already supplied an illustration

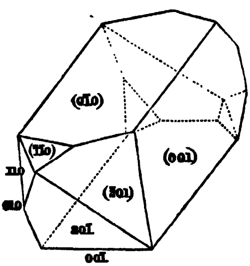

Fig. 317.

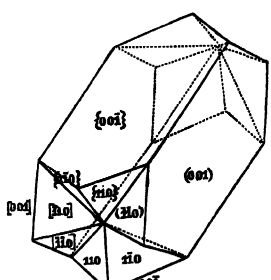

Fig. 318.

of this mode of twin-growth. The (orthoclase) felspar twin known as the Baveno twin (from one of the localities in which this form of felspar occurs) has for its twin-plane a face of the ortho-dome {021}, which is also the plane of union. An analogous twin occurs, but is very rare, in the anorthic felspar albite. Figs. 317 and 319 (*a*) represent simple twins according to this law: the systematic and basal pinakoids of the two individuals form a prism of which two normal-angles, namely $00\bar{1}$. 010 and (001) . $(0\bar{1}0)$, are exactly right angles, and the remaining two, namely 010 . $(0\bar{1}0)$ and $00\bar{1}$. (001), are 90° 6′ and 89° 54′ respectively. Twins of orthoclase, especially of the variety called adularia, that follow the Baveno law, are often interpenetrant, and exhibit repetitions of the twinning resulting in very composite structures: thus, in Fig. 319 (*b*) is illustrated one end of a complex growth in which there is

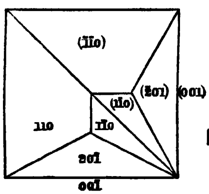

Fig. 319 (*a*).

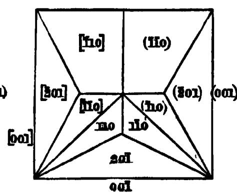

Fig. 319 (*b*).

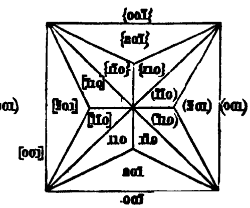

Fig. 319 (*c*).

twinning about the faces (021) and $(0\bar{2}1)$ of the central individual; Fig. 319 (*c*) shows one end of a similar growth in which a fourth individual takes a part, being twinned on a face of the form [021] of the left-hand individual; in Fig. 318 is shown the latter growth obliquely projected, the form $\{20\bar{1}\}$ being omitted for the sake of simplicity.

As in the case of harmotome, complicated growths according to two distinct twin-laws are met with in orthoclase, faces of the basal pinakoid {001} and of the ortho-dome {021} being simultaneously present as planes of twinning.

Rare pseudomorphous forms after crystals of orthoclase twinned about a face of the ortho-dome {051} have been described.

347. Fig. 320 is the representation of a twin of wolfram of which the plane of twinning as well as of union is the face $(02\bar{3})$. Like harmotome, wolfram has been removed from the Ortho-rhombic system, under which it had been previously classed in a hemi-symmetrical category really incompatible with the symmetry of that system.

348. Twinning about a face of a prismatid is very rare. In Fig. 321 is shown an interpenetrant twin of orthoclase, belonging

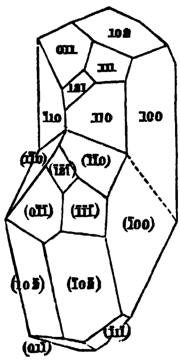

Fig. 320.

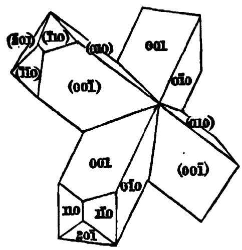

Fig. 321.

to the British Museum, in which the face $(\bar{1}11)$ is the twin-plane. Rare pseudomorphs after orthoclase crystals twinned about a face $(\bar{4}54)$ have also been described.

SECTION VI.—**The Anorthic System.**

A.—*Holo-symmetrical Forms.*

349. A form in the Anorthic system is constituted by two parallel faces, the only symmetry which characterises the system being that to a centre.

No zone on such a crystal can permanently present ortho-symmetry, nor can it have three tautozonal faces inclined consecutively at crystallometric angles. The three edges to which the axes are parallel will of necessity be inclined to each other at angles of which no two can be right angles, and the elements of the crystal are unfettered by any condition. Where more than one of the axial angles may approximate to a right angle, it is conceivable that changes of temperature inducing changes in the axial elements might momentarily bring two or all the axes into a position of perpendicularity; but it will be seen in a future chapter that this is a condition transient with the temperature, and is not a sufficient condition for the system to which the crystal belongs to assume a different type of symmetry.

In the general case then the axes are to be assumed as oblique to each other and the parameters as unequal; and in order to assign a conventional orientation to such an axial system it will be necessary to determine the ways in which a uniform principle may be applicable to it.

Adopting the alphabetical order of the letters a, b, c as that of the descending magnitudes of the parameters assigned to the axes X, Y, Z, respectively, we might further determine that in all cases the axial angle η or ZX should be an obtuse angle, as in the Mono-symmetric system. Now it will be found that under these restrictions it is always possible to make the positive octant such that the three axes which contain it shall be inclined either all at obtuse angles, or so that two of the axial angles are obtuse and one, for which ζ may be taken, is acute. The axial system would then be represented by the expression

$$\xi > 90^\circ, \quad \eta > 90^\circ, \quad \zeta \gtrless 90^\circ,$$
$$a > b > c.$$

An orientation of this kind is not however suitable in cases where analogies have to be kept in view between crystals chemically homotypic but belonging severally to the Monosymmetric and Anorthic systems. An orientation corresponding to that employed for the monosymmetric crystal is in such cases adopted in the Anorthic system.

350. The forms of the system being limited to pairs of faces, and related to each other only by the bond imposed on them by the crystalloid zone-law, a nomenclature distinguishing the forms which lie in particular zones, as for instance the zones containing two of the axial planes, possesses even less significance than is the case in the Mono-symmetric system. The sort of octahedrid figure produced by the union of all the four forms in the symbols of which the several indices have the same numerical values is one which has no crystallographic meaning; for the forms

$$\{h k l\}, \ \{\bar{h} k l\}, \ \{h \bar{k} l\},$$

are independent each of the other, and do not in fact concur with any regularity or frequency on anorthic crystals.

Hemiprismatic forms would include a *positive hemiprism* $\{hk0\}$ and a *negative hemiprism* $\{\bar{h}k0\}$, and of such united forms the *parametral hemiprisms* $\{110\}$ and $\{\bar{1}10\}$ are more frequent in their concurrence. So two concurrent hemidomes, consisting of a *positive pro-hemidome* $\{h0l\}$ and a *negative pro-hemidome* $\{h0\bar{l}\}$, would build a dome-form with its edges parallel to the axis Y; while a *positive para-hemidome* $\{0kl\}$, with a corresponding *negative* form $\{0k\bar{l}\}$, would constitute a dome-form with edges parallel to the X-axis.

Apart from the grounds on which particular faces are selected for pinakoid or for parametral forms in this system, such pairs of forms might be expected to be more frequent in their concurrence as quasi-dome forms than in the case of the quasi-octahedrid forms, inasmuch as in the one case only two hemidome or hemiprism forms with numerically identical indices must concur, while in the latter case four tetarto-octahedrid forms numerically identical in their indices must of necessity be extant together on the crystal. The three pinakoid forms parallel to the axial planes may be termed

the *para-pinakoid* $\{100\}$, parallel to the axes Y and Z;
the *pro-pinakoid* $\{010\}$, parallel to the axes X and Z;
and the *basal pinakoid* $\{001\}$, parallel to the axes X and Y:

the prefixes *para* and *pro* are suggested by the positions of the pinakoids relative to an observer looking at the crystal in the direction of the Y-axis; the faces of the para-pinakoid being then at the sides and those of the pro-pinakoid in front of the observer.

351. In selecting the faces that are to determine the axial elements in an anorthic crystal, and assigning to particular faces the character of axial pinakoids or prismatid or dome-forms, of course in the first place importance has to be attached to the faces which by the comparison of several crystals are recognisable as habitually prominent on them. Sometimes also crystals of this system—as is conspicuously the case with the felspars—present a close resemblance in the distribution of their forms with crystals of other systems, and more particularly with crystals of the Clino-rhombic system, with which they are at the same time allied in the type of their chemical composition. In such cases the orientation of the axes is determined in subordination to the principle necessitated by the symmetry of the system to which belong the crystals with which a comparison has to be instituted: in Fig. 322 is shown a stereographic projection of the more prominent forms exhibited

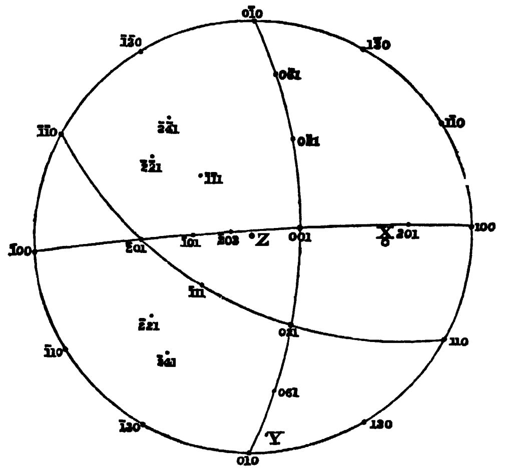

Fig. 322.

by the crystals of anorthite illustrated in Figs. 325 and 326, the axial system of which would be represented by the expression

$$\xi = 86^\circ\, 47', \; \eta = 115^\circ\, 55', \; \zeta = 88^\circ\, 48',$$
$$a : b : c :: 0{\cdot}635 : 1 : 0{\cdot}550.$$

Anorthic System. B.—*Mero-symmetrical Forms.*

352. *Hemi-symmetry.* The only conceivable case of mero-symmetry in the Anorthic system would be that of a haplohedral form, or forms represented each by a single face.

Anorthic System. C.—*Combinations of Forms.*

353. In the previous systems the symmetrical repetition of faces leads naturally to the grouping of forms in zones. In the Anorthic system there is no such necessary tendency. Nevertheless the aspect of an anorthic crystal very usually differs little in this respect from that of a mono-symmetric or ortho-symmetric crystal. The symbols of the forms in the Anorthic system are usually of a very simple kind, the indices rarely occurring with a higher numerical

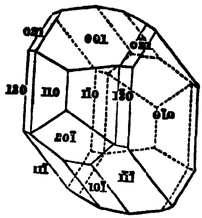

Fig. 323.

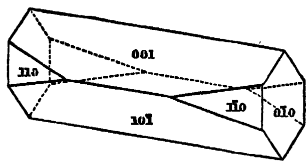

Fig. 324.

value than four. Indeed the number of minerals crystallising in the system is very small, though of organic products formed in the laboratory this number is considerable. The forms, however, that these present are usually few in number, and simple in their symbols.

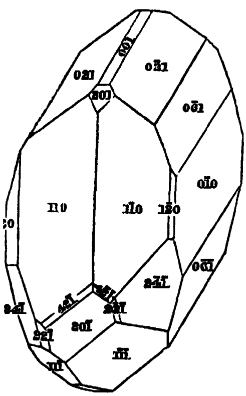

Fig. 325.

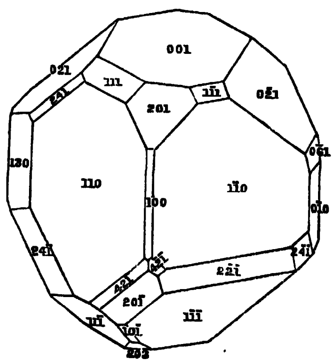

Fig. 326.

In Fig. 327 is illustrated a simple crystal of the felspar albite with the forms $\{001\}$, $\{010\}$, $\{110\}$, $\{1\bar{1}0\}$, $\{10\bar{1}\}$, and in Fig. 323 another crystal of the same mineral with forms additional to the above: Fig. 324 shows the form of an ideal simple crystal of albite elongated in the direction of the *Y*-axis, a variety to which the term pericline is applied. Figs. 325 and 326 represent crystals of the felspar anorthite from Vesuvius (after vom Rath) remarkable from the completeness of their developement.

Anorthic System. D.—*Twinned Forms.*

354. In the Anorthic system the twin-growths are of two kinds; in the one the twin-axis is the normal of a face, in the other it is parallel to a zone-axis; and as in this system, from the principles of symmetry, no face-normal can be a zone-axis, these two modes of twin-growth are distinct. Another kind of twin-growth has indeed been asserted to exist, in which the twin-axis is a line perpendicular to an edge and lies in a face belonging to the zone to the axis of which the edge is parallel; and Neumann, Kayser and Rose gave their great authority in favour of such a mode of twin-formation. A line of this kind is not however a crystallographic line in the Anorthic system, as neither the line itself nor the plane normal to it would have a rational symbol: and the line has therefore no significance other than what might be due to its supposed function as a twin-axis. The twins of albite and of the pericline variety of this mineral, for the explanation of which this mode of twinning was virtually called into existence and maintained, can be accounted for in a simpler manner, and will be found to be explained in a subsequent article by the mode of twin-growth round an edge. In other cases in which this mode of twinning has been called in, it will also be found that the two more simple kinds of twinning are competent to explain the complicated growths to which the former has been applied; and that where this more complex principle would seem more simply to explain them, accurate measurements, on which alone it could be established, are wanting or were not possible by reason of the impracticability of obtaining them. It is in fact only for complex structures consisting of several individuals, explicable by one or both of the two simpler methods of twin-growth, that this third mode of explanation has been invoked, and in most cases so invoked to account for the twin-relations of members of the group that are not in contact and are only mutually related through the intervention, so to speak, of one or more other individuals of the group.

355. The non-crystallographic line, round which the pericline-twin of two individuals was supposed by Kayser to be formed, is perpendicular to the X-axis and is inclined on the Y-axis, the zone-axis [010], at so very small an angle that it is practically impossible, in the case of a mineral presenting the striated faces common in pericline, to determine, by direct measurement of the angles between the planes of the twin-growth, which of these adjacent lines is the true twin-axis. The discussion of the argument in regard to these twins will be given later, when the twinning round a zone-axis is treated of in Art. **362**.

In the case of the twinned groups of albite, which were the first that suggested to Neumann recourse to this irrational axis, the crystals are twinned in pairs, each pair on a face-normal (010), and adjacent crystals of different pairs round a zone-axis [001]. The explanation of the relative position of the first and third individuals offered by Neumann, purely as a geometrical explanation, in which a line normal to a zone-axis and lying in a face is conceived to be a twin-axis, introduces in fact a greater difficulty than that of a double twin-law by giving to a line with an irrational symbol a crystallographic function.

356. Besides the felspar-twins alluded to in the last article, the only other cases adduced to support an irrational twin-axis are the following:—

1. A single specimen of labradorite[1] from Hafnefjord, consisting only of a termination of a twin-crystal formed by a thin lamina of only 1½ millimetre in length, and yielding only approximate measurements made with a source of light very close to the goniometer. Vom Rath described this crystal in 1871, but did not again refer to it when in 1879[2] he pointed out that the evidence for the variety of twinning under discussion was not sufficient to establish its recognition.

2. A complicated twin-growth of anorthite[3], which did not admit of accurate measurements and could be satisfactorily explained by the ordinary rational twin-axes, as was pointed out by vom Rath who first described it.

[1] Pogg. Annal. vol. 144, 1871, p. 277.
[2] Groth's Zeitschrift, vol. 6, 1879, p. 12.
[3] Pogg. Annal. vol. 147, 1872, p. 59.

3. A single crystal of kyanite described by Bauer[1]. In this mineral the twin-axis, if perpendicular to the Z-axis as supposed by Bauer, is inclined to a second edge at a small angle which has been determined by Bauer himself as 23′, by Brooke and Miller as 15′, and by vom Rath on one specimen as 5′, and on another as zero. It is therefore more probable that in the specimen described by Bauer this edge, and not the perpendicular to the Z-axis, is really the twin-axis; and in fact in two other very similar specimens, also described by Bauer, the Z-axes of the two individuals, instead of being quite parallel, were distinctly inclined to each other.

It is quite evident that no new mode of twin-growth could be established by any of these specimens, and that, had not the above mode been considered to be well-established by means of the specimens of pericline, a simpler explanation would have been suggested in each instance.

357. As the characters of the twin-growths observed in other anorthic crystals, as for instance those of brochantite and kyanite, are not essentially distinct from the characters presented by the anorthic felspars, the illustrations will be selected from the last-mentioned and more plentiful minerals.

I. Twin-plane a face; the face of union parallel to the twin-plane.

Since in this system there is no symmetral plane and the crystallographic axes are consequently selected more or less arbitrarily from the zone-axes, no essential difference in the twin-growths can be associated with a mere difference in the symbol of the twin-plane: as, however, the lines chosen for axes are usually the intersections with each other of planes either of cleavage or of prominent development, a numerical difference in the symbol will correspond to a distinction in the aspect of the twin-growth.

358. *Twin-face the pro-pinakoid* {010}.

This mode of twin-growth is very common in all the anorthic felspars. Fig. 327 represents a simple crystal of albite; two nearly perfect cleavages are parallel to the faces (010) and (001), and correspond to analogous faces of the mono-symmetric felspar

[1] Zeitsch. d. deutsch. geol. Gesell., vol. 30, 1878, p. 306.

orthoclase. If the crystal be bisected by a plane passing through its centre and parallel to the pro-pinakoid {010}, the section will have the form *defghk* and will be centro-symmetrical: hence the section will be congruent with itself after a rotation of 180° round the normal to its plane. If, therefore, the front half of the crystal of Fig. 327 be turned through two right angles round the normal to the face (010), the sections will again coincide and the resulting compound crystal will have the form shown in Fig. 328, and be symmetrical to the plane of union. No face being perpendicular to the twin-plane, with the exception of the faces of the pro-pinakoid {010}, no face of one individual is parallel to a face of the other. The normal-angles made by the planes 001, $\bar{1}01$ with the twin-plane $0\bar{1}0$ are 86° 24′ and 86° 21′ respectively:

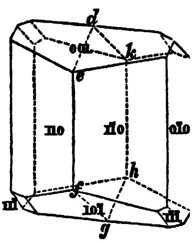

Fig. 327.

Fig. 328.

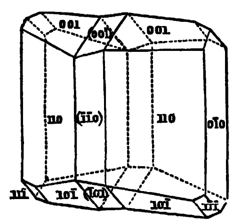
Fig. 329.

hence there will be two nearly equal re-entrant angles, 7° 12′ and 7° 18′, at the upper end of the twin, and two salient angles of the same magnitudes at the lower end: the angle 110 . $0\bar{1}0$ being 119° 40′, the angle 110 . ($\bar{1}\bar{1}0$) will be salient and equal to 59° 20′.

359. If the twinning be repeated the growth has sometimes the form shown in Fig. 329, and may then be otherwise regarded as a single crystal enclosing a twin-lamina: in most cases the twinning is repeated several times, as already illustrated in Fig. 59 and in some cases the twin-growth consists of a hundred or more laminæ, of which adjacent individuals are in twin- and alternate ones in parallel positions.

In the mono-symmetric felspar orthoclase the normal to the corresponding face (010) is an axis of diagonal symmetry, and is impossible as a twin-plane, for a rotation of the crystal through two right angles round such a line would bring it into a position not crystallographically distinct from the first: hence the presence of twinning about a face of the pro-pinakoid is sufficient evidence that a specimen of felspar belongs to the anorthic series.

360. Sometimes, as in the fine crystals of albite from the dolomite of Roc-tourné in Savoy, described by Gustav Rose[1], the individuals are interpenetrant. The crystals from this locality are quite tabular through the prominent developement of the pro-pinakoid, and have the habit shown in Fig. 330: these twin-crystals are remarkable in that the two angles formed at opposite ends by the basal pinakoids of the two individuals, instead of being one salient and the other re-entrant, are both re-entrant, while

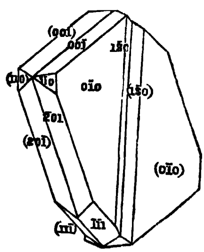

Fig. 330.

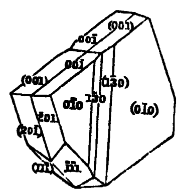

Fig. 331.

the angles formed at opposite sides by the faces of the form $\{20\bar{1}\}$ are both salient: further, a vertical furrow bounded by faces belonging to the form $\{\bar{1}30\}$ traverses each face of the pro-pinakoid $\{010\}$. If the twin-crystal be broken across so as to show the basal cleavages of the two individuals (Fig. 331), it is seen that on the broken part at one side of the furrow the cleavages form a re-entrant angle, and on the part at the other side of the furrow a salient angle, and that each of the cleavage-faces on one side of the furrow is parallel to the diagonally opposite face on the other side of the furrow. The individuals thus appear to cross over to opposite sides of the junction-plane at its intersection with the furrow.

361. Other faces presenting themselves as planes of twinning in

[1] Pogg. Annal. vol. 125, 1865, p. 460.

the anorthic felspars are comparatively rare. Lévy[1] and also Schrauf[2] have described specimens of albite twinned about the basal pinakoid (001) and similar in aspect to the orthoclase-twin of Fig. 308; growths of albite already twinned about the zone-axis [010] are likewise known in which there is a further twinning about the basal pinakoid (001), as shown in Fig. 340: Amazon-stone from Pike's Peak, a green variety of microcline, is sometimes twinned similarly to the so-called Baveno-twins of orthoclase, namely about a plane (021); an albite crystal has been described by Neumann[3], and one also by Brezina[4], twinned about a plane having the same indices.

In kyanite, crystals enclosing laminæ of which the position is due to twinning about a plane ($\bar{3}08$), not observed as a face of the crystals, have been described by vom Rath[5].

II. Twin-axis an edge: plane of union perpendicular to the twin-plane.

362. 1. *Twin-axis the Y-axis* [010].

The most important twin-growths assigned to this law are the common twins of pericline. Their most characteristic feature is the presence of an obtuse re-entrant edge of very small angle traversing a face of the pro-pinakoid {010} in a direction more or less nearly parallel to its intersection with the basal pinakoid {001}. The basal pinakoids of the two individuals being parallel, the twin-plane must be either parallel or perpendicular to these faces; that the latter is the true explanation is at once evident from the fact that corresponding faces of the two crystals, instead of being symmetrically disposed with respect to a plane parallel to the basal pinakoids, are diagonally symmetrical to a line lying in that plane: the exact position of this line, the twin-axis, has been long a matter of doubt.

363. The growth was first mentioned in 1824 by Mohs[6], whose

[1] Catalogue des minéraux de M. Turner, 1837, vol. 2, p. 194.
[2] Atlas der Krystall-formen, 1864.
[3] Abhandl. Ak. Berlin, vol. 19, 1830, p. 218.
[4] Tschermak's Min. Mittheil. 1873, p. 19.
[5] Groth's Zeitsch. vol. 3, 1879, p. 9.
[6] Grundriss der Mineralogie, 1824, vol. 2, p. 295.

statement of the law as twin-axis [010] was accompanied by a figure analogous to Fig. 332, in which the two individuals are represented as intersecting in a plane parallel to the faces of the basal pinakoid; thus the edge *X*, formed by the intersection of $0\bar{1}0 \,.\, 001$ and therefore parallel to the *X*-axis, is represented as parallel both to the corresponding edge X_1 and to the edge *R* formed by the intersection of $0\bar{1}0 \,.\, (0\bar{1}0)$, pro-pinakoid faces belonging to the different individuals. In 1835 Kayser[1] pointed out that as the line [010], the *Y*-axis, assumed as twin-axis by Mohs, is not at right angles to the *X*-axis but is inclined to it at an angle of $89^\circ\,13\frac{1}{3}'$ (according to Breithaupt), the edge X_1 instead of being parallel to *X* ought to be inclined to it at an angle of $1^\circ\,33\frac{1}{3}'$, and further that the edges *X* and *R* ought to have a mutual inclination of $13^\circ\,11\frac{1}{2}'$, as shown in Fig. 333. The question

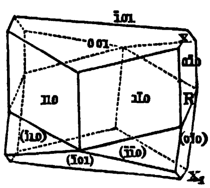

Fig. 332.

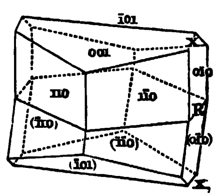

Fig. 333.

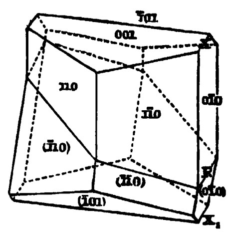

Fig. 334.

thus arose as to whether the statement of the law or the figure given by Mohs was most nearly in accordance with the features of the specimens themselves. Kayser, and later Rose[2], after examination of numerous specimens came to the conclusion that all deviations from parallelism presented by the edges *X*, *R*, X_1 were due to the striated and imperfect character of the faces, and that a line exactly perpendicular to the *Y*-axis, and not the *X*-axis which makes an angle of $89^\circ\,13\frac{1}{3}'$ with that line, must be regarded as the twin-axis: Fig. 334 illustrates the form of the ideal twin corresponding to this view. Kayser's explanation of the twins of pericline became very generally accepted, although three eminent

[1] Pogg. Annal. vol. 34, 1835, p. 108.
[2] Ibid. vol. 129, 1866, p. 1.

observers, Miller, Des Cloizeaux and Schrauf, adhered to the older statement given by Mohs.

By the publication of an elaborate memoir of Vom Rath in 1876[1] the accuracy of the older view may be regarded as now fully established. Vom Rath has pointed out that the approximate parallelism of the edges X, R, X_1 is only exceptional, and is then due to irregularity in the growth of the individuals beyond their plane of junction, and that the angle between the edges X and R changes considerably for a very slight change in the angles of the individuals, which in albite and its variety pericline vary somewhat with the locality of the specimens: he further states that Kayser's conviction of the parallelism of the above edges was arrived at chiefly from consideration of some excellent twin-growths not of pericline but of oligoclase, a felspar of which the angles are such that the Y-axis [010] is itself almost precisely normal to the edge X, so that the two modes of explaining the twin become in this particular case practically indistinguishable.

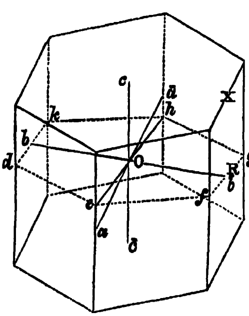

Fig. 335.

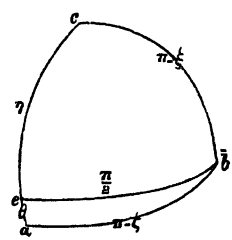

Fig. 336.

364. The relative positions of two individuals of an anorthic felspar after the rotation of one of them through 180° round the axis [010] may be determined as follows:—

In Fig. 335, a, $\bar{b}$, c are the points of intersection of the crystallographic axes OX, $O\bar{Y}$, OZ with the edge formed by the faces $1\bar{1}0 . 110$, with the face $0\bar{1}0$ and with the face 001 respectively: $aO\bar{a}$ is thus parallel to the edge X and cO to the edges of inter-

[1] Monatsber. d. Ak. Berlin, 1876, p. 147.

section of the prism-faces. Since the normals from a point in a twin-axis to two corresponding faces are co-planar with the twin-axis, the edge of intersection of the faces is perpendicular to that line: hence the edge of intersection of the face $0\bar{1}0$ with the corresponding face of the other individual must be the line in the face 010 which is perpendicular to the axis $bO\bar{b}$. Hence if the line $f\bar{b}g$ be in the face 010 and be at right angles to $bO\bar{b}$, and eOh, dbk be lines parallel to $f\bar{b}g$, the figure $defghk$ will be the intersection of corresponding faces of the two individuals. The position of the line fg, or of its parallel eh, may be thus calculated:—

Let the spherical triangle $a\bar{b}c$ (Fig. 336) be obtained from the intersections of the lines Oa, $O\bar{b}$, Oc with a sphere having its centre at O: e, the point of intersection of Qe with the sphere, will fall in the great circle passing through c and a, for the line Oe is in the plane aOc, parallel to $0\bar{1}0$: further, the arc $e\bar{b}$ is a quadrant, since Oe is at right angles to the axis Ob: whence, by the ordinary formula

$$\cos e\bar{b} = \cos ea \cos a\bar{b} + \sin ea \sin a\bar{b} \cos ea\bar{b},$$

we have

$$\cot ea = -\tan a\bar{b} \cos ea\bar{b}:$$

ea is the angle which Oe or fg makes with the line Oa, or the edge X, and may be denoted by θ; $ea\bar{b}$ is the angle between the planes aOc, $aO\bar{b}$, or the supplement of the angle ϕ between the normals to the faces $0\bar{1}0$, 001; $a\bar{b}$ is the supplement of the angle ζ between Oa and Ob:

hence $$\cot\theta = -\cos\phi\tan\zeta.$$

365. In the case of albite from Schmirn $\phi = 86° 30'$ and $\zeta = 92° 8'$, whence $\theta = +31° 23'$: in albite crystals measured by Breithaupt $\phi = 86° 41'$ and $\zeta = 90° 47'$, whence $\theta = +13° 18'$: in anorthite $\phi = 85° 50'$ and $\zeta = 88° 48'$, whence $\theta = -16° 5'$.

Thus while in the case of albite the edge of junction R of the pro-pinakoids of the two individuals would meet the edge X in front, in that of anorthite the two lines would meet if produced backwards, and in the one felspar the inclination may be as high as 31° or as low as 13° in one direction, and in the other felspar is 16° in the opposite direction. For the remaining felspars with intermediate parameters the line *R* will have an intermediate direction, and in oligoclase is almost exactly parallel to the edge *X*: hence vom Rath points out that the direction of the edge of junction on the pro-pinakoid face may be of occasional service for the determination of the kind of anorthic felspar to which the individuals of such a twin-growth belong.

366. The above section *defghk* determines the directions of the lines in which corresponding prism and pinakoid faces of the two individuals would meet, and is, so far at least, merely a geometrical fiction and analogous to the twin-axis as an axis of revolution: it is a constructional plane useful for the determination of the directions of the edges in which the corresponding faces would meet, if they met at all, and even in the latter case the whole surface of the figure might not be common to the two crystals. In fact Mohs regarded the plane parallel to the basal pinakoid as being in every case the plane of junction, while Kayser, who first pointed out that the edges of the basal pinakoid could not take up a congruent position on rotation through two right angles about any line in its plane, still considered that this plane was the plane of junction, holding that the frequent intersection in edges was due to the growth of parts which lay outside the boundaries of the surface common to both the individuals.

367. Similar growths to the twins of pericline are met with in anorthite, but in the latter felspar have a more precise character owing to the greater constancy of the angles of the individual crystals, and have thus been always and unhesitatingly assigned to this twin-law; in anorthite, too, the faces of the zone [010] are more numerous and are more prominent in their developement, so that the exact parallelism of the faces of this zone, corresponding to each other on the two individuals, is capable of easy demonstration. Vom Rath, to whom we are indebted for a careful description of these twin-growths[1], has described crystals enclosing numerous twin-laminæ of which the faces are parallel to the above section, and has thence inferred that, notwithstanding its irrational indices, it has generally a physical existence and is the plane of union, whenever corresponding faces of the two individuals actually

[1] Pogg. Annal. vol. 147, 1872, p. 42.

intersect, as in Fig. 337. Sometimes, however, the individuals have their basal pinakoids in contact, and present incongruent edges as illustrated in Fig. 338, which represents an actual twin-growth described by vom Rath.

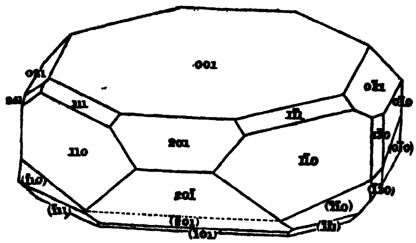

Fig. 337.

368. In the case of pericline itself the surface of junction is frequently irregular and is rarely quite plane: sometimes the individuals have their basal pinakoids in contact, and have incongruent edges similar to the anorthite-twin of Fig. 338: at other times they are more or less interpenetrant, and if the twin be broken parallel to the pro-pinakoid cleavages it presents a zigzag line of

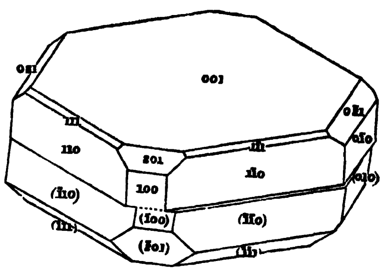

Fig. 338.

separation of the individuals, one side of the zigzag being determined by the above section and the other by the vertical axis: in two exceptionally good specimens the above section seemed to be actually the face of junction.

369. Simple twins of pericline have not been observed: the crystals always interpenetrate and appear to cross over to opposite sides of the junction-plane, as in the crystals of albite from

Roc-tourné described in Art. **360**, which are twinned about the pro-pinakoid (010). The edge traversing the pro-pinakoid face thus comes to be re-entrant at both sides of the twin, instead of being re-entrant at one side and salient at the other: such a twin is represented in Fig. 339.

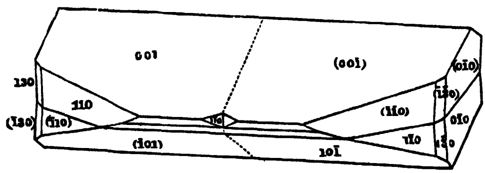

Fig. 339.

In Fig. 340 is illustrated a specimen of pericline, described by vom Rath, in which two twins of the kind just mentioned are united by their basal pinakoids and are symmetrical to their face of union, the two innermost individuals on the right or left sides of the growth being therefore twinned about the face (001): the equality of the inclinations of the edges R, R_1 to the edge X_1 in this specimen furnished a convincing proof that the obliquity is a fundamental, not an accidental, character of the pericline-twins.

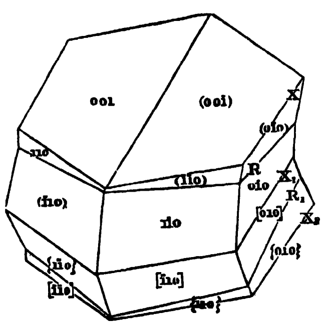

Fig. 340.

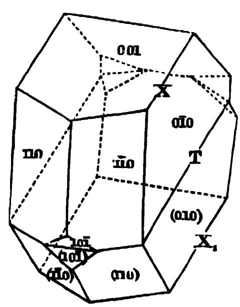

Fig. 341.

370. 2. *Twin-axis the X-axis* [100].

This mode of twinning is described by Des Cloizeaux as occurring, though very rarely, in pericline from Tyrol: in Fig. 341 is illustrated a simple twin figured by Schrauf. The plane of union is nearly parallel to the basal pinakoid and intersects the pro-pinakoids in an edge T parallel to the edges X and X_1: the twin-axis being parallel to the intersection of the basal pinakoid with the pro-pinakoid, a pro-pinakoid face of one individual will be co-planar with a pro-pinakoid face of the other, and the basal pinakoids of both will be parallel. The law is more often met with in complicated twin-growths of eight individuals, twinned in addition about the face (001), in the albite of Ala in Piedmont.

371. 3. *Twin-axis the Z-axis* [001].

This law of twinning, which is more rarely met with than the ordinary pericline law, corresponds to the Carlsbad law of the monosymmetric felspar orthoclase, and the growths are similar in aspect.

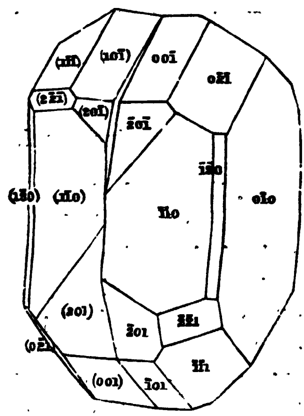

Fig. 342.

As before, the plane of junction is perpendicular to the twin-plane and intersects the crystal in a section which, though symmetrical to its centre, is not symmetrical to the twin-axis lying in it, whence the edges of the section are not congruent in the two positions. The twins are dissimilar in character from the pericline twins, for there is no growth of the individuals beyond the incongruent edges, nor do the individuals cross to opposite sides of the plane of junction: in Fig. 342 is illustrated such a twin-crystal of anorthite from Vesuvius, described by vom Rath.

CHAPTER VIII.

THE MEASUREMENT AND CALCULATION OF THE ANGLES OF CRYSTALS.

SECTION I.—**The Goniometer.**

372. Two methods have been employed for measuring the angle between the faces which form an edge on a crystal. The one is a mechanical contrivance by which the angle contained between two steel bars brought into contact with the faces of the crystal is read off on a graduated arc; the other consists in making the crystal revolve round the edge to be measured, and determining by the aid of a ray of light fixed in direction the angle through which it must be turned in order that each of the two faces in succession may be in a position to reflect the light in an identical direction. By the former method only approximately correct measurements can be obtained even from faces of exceptional smoothness and considerable size; while the accuracy of the latter method need only be limited by the precision with which the measuring instrument is graduated and the perfection as reflecting planes of the faces to be measured. The instrument employed in either case is termed a 'goniometer'.

373. *The contact- or hand-goniometer* (Fig. 343). As first employed by Romé de l'Isle, and subsequently modified by Haüy, it has the form of a flat semicircular arc of brass or silver graduated from 0° to 180° and subtended by a flat steel bar, which, for convenience in manipulation where the crystal is entangled with associated mineral, is able to move in the direction of its length, and is constrained so to move by two screw-pins which pass through two long slots cut in the bar parallel to its sides and clamp the bar to a corresponding bar which is fixed in the position of a radius to the graduated arc: one of these screw-pins is fixed at the centre of the arc, the other near its circumference. The edges of this bar are parallel to the diameter which passes through the zero of the graduation. A second moveable bar corresponding to the first has only one slot extending to about half its length, and being held by the central pin only, is capable of being rotated round it as a pivot with any desired amount of friction, while for convenience of adjustment it also can be moved in the direction of its own length. Of the two bars or arms of the instrument thus mounted, the parts of the narrower sides or edges that are to be brought into contact with the faces of the crystal are worked

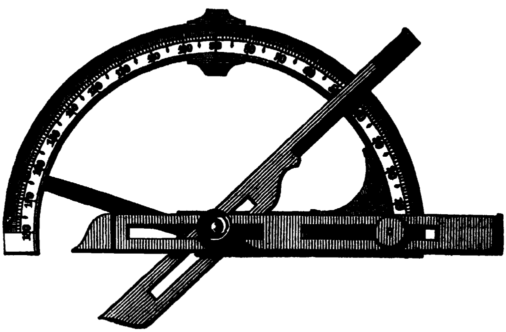

Fig. 343.

into perfectly squared and plane surfaces. The limb of the rotating arm which moves over the graduated face of the goniometer is cut away to one half its width and worked to a straight chisel-edge, the continuation of which would thus pass through the centre of the graduated semicircle; this edge therefore indicates the arc subtended by the two arms, and thus when the arms have been applied to two faces of a crystal indicates the angle between them: the instrument is more convenient when graduated in the opposite direction to that shown in the figure; the indicated angle will then be that between the *poles* of the faces.

The instrument is usually constructed with a hinge on the graduated arc, whereby one quadrant can be doubled back behind the other, so as not to impede the measurement of an embedded crystal; and to this purpose also the sliding motion of the arms subserves (Fig. 343).

In using the hand-goniometer the points to be attended to are the perpendicularity of the plane of the instrument and of the broader faces of the steel arms to the edge to be measured, and the complete simultaneous contact of *both* the straight narrower surfaces of the arms with the crystal-faces, as tested by examination between the eye and the light. And during the operation, and especially when removing the instrument from the crystal, it is well to maintain a slight pressure of the crystal on the arm that does not rotate, and none, or the least possible, on the crystal-face in contact with the rotating arm which indicates the arc on the graduated circle; the crystal being drawn away from the measured edge along and with a slight pressure against the fixed arm. With every precaution, and where the faces to be measured are well adapted by size, smoothness, and accessibility, the successive readings of the measured arc may accord within the limit of a quarter of a degree. Such cases are however very rare, and it is evident that even this is not a sufficient degree of accuracy for the purposes of exact crystallography.

The instrument described is often employed in a simpler form; the arms being separate from the arc but capable of being applied to it by inserting the pivot of rotation of the two arms in the centre of the metal arc, which then presents the character of an ordinary semicircular protractor.

374. *The reflection-goniometer* of Wollaston, in its original form, consists of a circular metallic disc four to eight inches in diameter, with a flanged rim graduated with degrees and subdivisions of a degree (Fig. 344). An axle at the centre of the disc revolves in a collar of metal fixed to the support of the instrument, and is firmly attached to a flat circular handle with a milled edge behind the disc, whereby the disc with its graduated arc is carried round. The graduations are read by the aid of fixed verniers. The axle of the disc is hollow, and a second and solid axle or core passing through it and ground accurately into it can be turned independently of the disc by a smaller handle similar to the one which turns the disc itself and parallel to but beyond it: the extremity of this axle, which projects beyond the face of the goniometer, bears a small and simple apparatus for the support and adjustment of the crystal to be measured.

In the adjustment of a crystal this independent rotation of the axle of the instrument is convenient, as it enables the crystal and the apparatus holding it to be turned without moving the graduated circle.

For use the goniometer is planted in front of a window, a bar of which may be taken as the object to be reflected in the crystal-faces; or, preferably, the object or *signal* may be a slit in a dark screen illuminated by the sky or by artificial light; or a beam of sunlight may be directed on the crystal by a heliostat and seen through a protecting glass held before the eye.

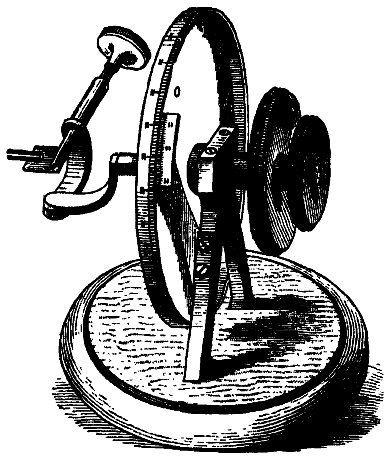

Fig. 344.

The crystal is now attached to the holder by a little plastic material formed by fusing together beeswax and pitch in suitable proportions; this may be kept at hand conveniently if rolled out into a pencil-form.

The holder is a small rectangle of thin sheet-brass which is gripped in a slit at the end of a rod, which rod serves to turn and to give a traversing motion to the holder by its play in a collar attached to a curved brass support, which again has a turning motion round a pivot in a second similar support which is fastened to the axle of the goniometer; see Fig. 344.

The variety of motions that can by these means be given to the holder enables the observer with a little practice to adjust the crystal in any position. For the purpose of measuring, however, especially for following the measurements of a zone, the position of the crystal usually presented by books on crystallography is impracticable. The crystal must in fact project clear of the apparatus and beyond the different portions of the support; the eye can then uninterruptedly observe the faces of the zone as they in succession come round and, if properly adjusted, reflect to it the signal, each in turn.

Assuming then that the crystal has been or is readily capable of being thus adjusted—since the precautions to be taken in the centering and adjustment will have to be discussed presently in detail—we may proceed now to the other conditions that have to be fulfilled if a correct measurement of the normal-angle of two of the faces of an adjusted zone is to be made. This can only be attained when the rotation of the crystal round the axis of the goniometer through this normal-angle shall have brought the two faces in succession into a position in which each reflects the axis of a pencil of rays, coming from the centre of the signal, in the same direction, a condition which necessitates the position into which the second face is brought being parallel to and, for exact measurement, coplanar with that previously occupied by the first face.

The direction of vision must evidently be unchanged during the measurement. Wollaston achieved this by placing in the line or vision of the reflected signal a second signal similar to it but more faintly illuminated. Then viewing this second signal by direct vision he would bring the image of the first signal, as seen by reflection from one crystal-face, into contact with it, and when the one signal, seen so through a portion of the pupil of the eye, exactly covered the other as seen directly through the remaining portion of the pupil, he arrested the turning motion. A second crystal-face being treated in the same way, the difference of the readings on the circle gave the normal-angle between the faces. Kupffer substituted for the second signal the image of the first signal as seen by reflection in a mirror of black glass.

375. But the most important modification that Wollaston's goniometer has received is that which, in the hands of Mitscherlich, has made it a new instrument by endowing it with a telescope fitted with cross-wires, by the aid of which the direction of the rays reflected from successive crystal-faces was fixed with great precision.

The greater rigidity of direction thus imparted to the axis of vision involved the necessity of a motion either of the telescope or of the crystal in a direction parallel to the axis of the instrument, so that the faces of the zone to be measured might be brought with precision into the field of view. This is usually effected by giving the telescope a sliding movement parallel to itself on a bar that supports it. But Professor Viktor von Lang introduced an improvement on this method by fixing the 'head' (or part of the goniometer which carries the crystal and the screws for its adjustment) on the end of a hollow cylinder, which fits on and can be clamped immoveably to an accurately turned solid cylinder of steel, forming a continuous piece with the solid core of the axle of the goniometer: the hollow cylinder can thus slide on the solid one and be fixed by the clamp. This method offers the advantage that the goniometer being adjusted and fixed in position the telescope may be also adjusted once for all to the centre of the reflected signal; any variations in situation of the edge to be measured, whether from the magnitude of the crystal or from the conditions under which it is mounted, being compensated by the motion not of the telescope but of the 'head' and of the crystal carried by it.

If a collimator be used, the signal introduced by Websky is very convenient: it is limited by two brass discs of which the distance from each other can be altered by means of a screw, and by two straight edges parallel to the line joining the centres of the discs.

In Wollaston's and Mitscherlich's goniometers the axis is horizontal and the plane of the graduated circle is vertical. Professor Miller, however, adopted a form of the instrument in which the axis is vertical and the disc horizontal: a form very simple in adjustment, and furthermore one especially adapted for measuring such crystals as are either too heavy of themselves, or are attached to masses of mineral and rock too considerable, to be carried by a vertical goniometer without causing strains that must conduce to flexure in the support and to very appreciable inaccuracy in the results obtained.

376. We shall now proceed to discuss the process of putting the goniometer into proper adjustment for measuring a crystal, and for this purpose we shall have recourse to methods independent of the use of a mounted crystal. Certain qualities have of course to be presupposed as being presented by the instrument itself. These include accurate graduation involving a power of reading by the aid of the verniers *v* (Fig. 345) to 15″; the perpendicularity

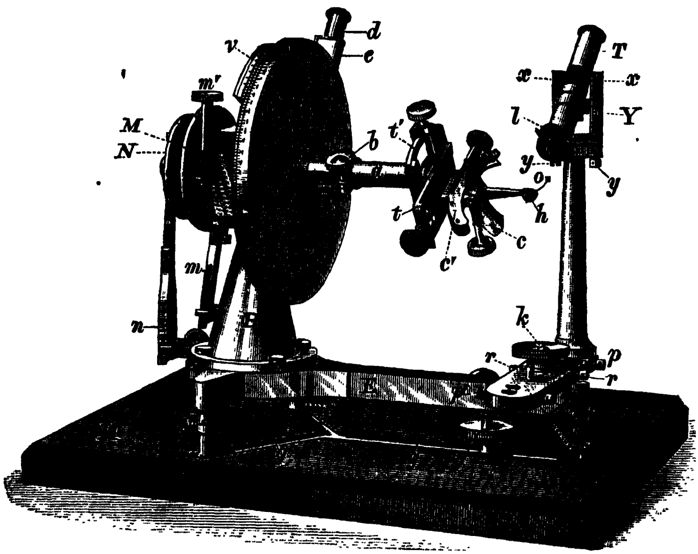

Fig. 345.

of the axle to the face of the goniometer disc; a minimum of 'backlash' in all the screws that impart motions; and a firm fastening of the handles to the axles which they turn, so that any motion of either handle is instantly responded to by the graduated circle or by the mounted crystal, as the case may be.

In order to proceed in a defined manner we shall assume the goniometer, in the first place, to be one of the vertical construction with the telescope permanent in its position and the head moveable (Fig. 345). The foot on which the telescope is mounted has adjustment-screws *pp*, to regulate in azimuth the position of the plane of observation relatively to that of the goniometer-disc *D*. There are also screws *yy* for regulating the horizontality of the cradles or *Y*'s in which rest the trunnions *xx* on which the telescope *T* turns.

Now the following are the ends to be attained in planting the goniometer and adjusting the telescope:—

a. The goniometer-axis is to be horizontal.

b. The plane of observation which passes through the centre of the signal and the optic axis of the telescope is to be vertical, and is to be parallel to that of the goniometer-disc.

c. And it is convenient that it should be perpendicular to the plane of the signal.

d. The optic axis of the telescope is to intersect the axis of the goniometer.

In order to effect these adjustments, and to do so in a certain order, the following adjuncts to the signals employed are of the greatest use:—

A long wire of substantial thickness, swinging freely and as close to the plane of the signal as is consistent with its carrying a suspended plumb-weight: it must exactly intersect the centre of the signal. A second and similarly weighted wire (for which however may be substituted a distinctly seen vertical line on the wall or window) placed at a distance to the right of the former wire *equal to twice the distance of the goniometer-face from the intersection of the cross-wires of the telescope*. The latter will be termed the *adjustment-wire*, the former the *signal-wire*. A flat dish or tray containing clean mercury. A metal pin *w* (Fig. 347), one end of which fits into the 'head' of the instrument while the other terminates in an accurately turned point. A lens *l* (Fig. 345) of such focal length that when applied in front of the object-glass of the telescope the crystal can be observed as by a microscope of low power. This lens turning on a pivot can be applied or removed at will: the telescope when used with the lens will be termed the micro-telescope. Where a collimator is not employed it is important that the signal should be at a considerable distance—about 20 feet to

30 feet from the telescope is sufficient—from the goniometer, and, where this instrument is one of the vertical construction, it is desirable that the signal should be at such a height that the angle of incidence and reflection from a crystal-face may be about 45°: and the slit or bar employed as a signal should be quite horizontal.

377. In the first place, the goniometer is planted on a firm table, in such a position that the plane of observation is approximately perpendicular to the window or plane of the signal. A method will be given for obtaining this perpendicularity with accuracy, but a first approximation attained by any simple geometrical method that may suggest itself according to circumstances is desirable as simplifying the processes.

a. Horizontality of axis. The goniometer is to be turned round upon the table until the image of the signal-wire can be seen in the lacquered face of the disc *D*, and the turning movement continued until this image is seen to be in contact at the edge of the disc with the adjustment-wire as seen directly. If the adjustment-wire and the image of the signal-wire are not in contact throughout their length but are seen to be inclined to each other, the levelling screws *S* are set in motion to bring them into parallelism when a little separated; after a slight turning of the whole instrument the one wire will then be seen to eclipse the other. A small spirit-level with a stirrup mounting is also convenient for securing the horizontality of the steel core of the instrument.

Two objects will now have been effected: the axis of the goniometer will be horizontal and the disc will lie in a vertical plane which bisects the space between the signal- and the adjustment-wires, so that if the telescope is in correct adjustment, on directing it to the signal, the signal-wire should be seen passing through the cross-point of the telescope-wires in every position of the telescope as it turns on its trunnions.

b. Adjustment of plane of observation. But if the centre of the signal lie on one side of this point when the telescope is turned to it, the adjustment-screws *pp* are brought into use to turn the telescope in azimuth until the cross-wires and the signal-centre are seen as if in contact. The effect is of course to bring the optic axis of the telescope, when it is in this one position, into parallelism with the plane of the disc.

The mercurial trough is now placed in such a position on the floor of the room that the signal as reflected in the mercurial surface can be seen in the telescope; and if the centre of the reflected signal is found to fall on the cross-wire, there is nothing more needed in this part of the adjustment: the telescope obviously must have been moved in a plane perpendicular to the surface of the mercury and therefore containing the signal-wire; and if the vertical wire of the telescope be correctly placed to cover any part of this signal wire it will continue to do so when the telescope is directed to any other part of that wire and its reflected continuation.

But if these events do not take place and the signal-centre is on the cross-wire when seen directly and to the side of it when seen reflected in the mercurial surface, the adjustment screws *yy* that regulate the relative elevation of the trunnions of the telescope are set in action to bring the axis on which the telescope turns into parallelism with that of the goniometer. In fact the correction has to be made to half the extent of the error as shown in the mercurial mirror. If this and the previous operation have been correctly performed, and have in fact been repeated to ensure greater exactitude, the telescope will now be in accurate adjustment as regards the parallelism of the plane of observation with that of the disc of the goniometer and therefore with the plane containing the graduated circle.

c. Plane of observation and plane of signal made perpendicular. So far we have been content with only an approximately perpendicular position for the plane of observation in respect to the plane containing the signal.

The cross-wires in a telescope are rarely strictly perpendicular to each other; it is not always that they intersect exactly in the centre of the field of view. Adjustments to the true centre can however be effected by turning the eye-piece through 180°, and taking for the 'cross-wire' the point on the horizontal wire to which the two positions of the actual wire-crossing are symmetrical.

In order to make the plane of observation perpendicular to the plane of the signal, let the horizontal wire be first turned into a position of exact parallelism with the sides of the signal; which is assumed to have been previously mounted in a horizontal position by the aid of a good spirit-level. On observing the image of the slit from the mercurial surface, if the plane of observation is not strictly perpendicular to that of the signal, the image will now be seen inclined at a certain angle to the horizontal wire of the telescope. Let therefore the eye-piece be turned till the horizontal wire bisects this angle.

If now the goniometer be moved in such a manner that on the one hand the adjustment-wire is kept in apparent contact with the image of the signal-wire as in the adjustment (*a*), and so that, on the other hand, the image of the signal-slit seen on the mercury is brought into parallelism with the horizontal wire, the adjustment of the instrument in respect to the signal will be found to be complete.

d. Direction of telescope-axis. We have only, in fine, to be careful that the optic axis of the telescope accurately intersects the axis of the goniometer: and this is readily effected by first mounting the pointed metal pin *w* (Fig. 347), to which allusion has been made, so centrically that when the goniometer-disc is turned the point makes no excursion from the axis; and then by means of the adjustment-screw (not visible in the figure) which regulates the inclination of the telescope the horizontal wire is brought accurately to cover the point of the metal pin as seen in the micro-telescope. Or if, first, one of the traversing screws be placed in a vertical position, and the point of the metal pin be brought into apparent contact with the horizontal wire of the micro-telescope by turning the traversing screw; and if, secondly, on bringing the second traversing screw into the vertical position the point has not moved away; or, if it has moved away and after being brought back to the wire by the second traversing screw it remains permanently in contact with the wire as the goniometer-disc is turned, the adjustment is complete. If, however, in the last event the position of the point is not permanent, the point should be turned by means of the axle into the position of its maximum excursion from the centre, that namely in which the goniometer has revolved through 180° from the last position. The declination-screw of the telescope should now be turned until the point is moved through one half the distance separating it from the horizontal wire, and the point itself should be brought down by the second traversing screw into contact with the wire. The adjustment will be found to be now complete.

The best assurance that all the corrections have been accurately made will be obtained when a zone, preferably an ortho-symmetrical zone, has been adjusted on the goniometer, and the images from the successive faces have been found to pass through the field of view, each moving symmetrically in respect to the vertical wire and in parallelism to the horizontal. Any deviation from this symmetry or parallelism should be at once traced to its source in one or other of the causes that have been discussed.

378. Where a collimator is used, it is advantageous to have a small triangular prism fixed to it, just above the objective, in such a position that the image of the signal given by a face may be viewed after total reflection in the prism. A pencil of rays from any point of the signal is first reflected from the face in a direction nearly perpendicular to it, then totally reflected in the prism, and next enters the telescope, which in this case is so arranged that its axis can be directed to intersect the axis of the goniometer either directly or after total reflection in the prism. This arrangement is occasionally convenient when the faces are much striated or the crystal is attached to its matrix.

379. Before passing from the adjustment of the goniometer to the mounting of a crystal on the adjusted instrument it should be remarked that it is very desirable either to have both the goniometer and the signal fixed permanently in position, or to have in front of the permanent signal a fixed stand to which the goniometer can be transferred without involving the necessity of readjustment.

The three-armed groove proposed by Sir W. Thomson forms an admirable stand for this purpose. The top of the goniometer-table or a slab *H* that can be fixed to it is formed of hard wood or slate, and has three grooves *ggg* (Fig. 345) cut in its surface which meet at angles of 120°: their section is a right angle, and one of the three cylindrical feet of the goniometer is placed in each groove.

In planting the goniometer in position before the signal and in levelling it, this stand has to be shifted until the adjustment is complete and is then fixed in the position thus determined. The goniometer may now be removed and replaced on its stand in identically the same position as often as is requisite.

380. It has already been indicated that, in order to ensure accuracy in the process of measuring the angles of a crystal, the precautions to be taken relate not only to those preparatory adjustments of the goniometer and the telescope which have been

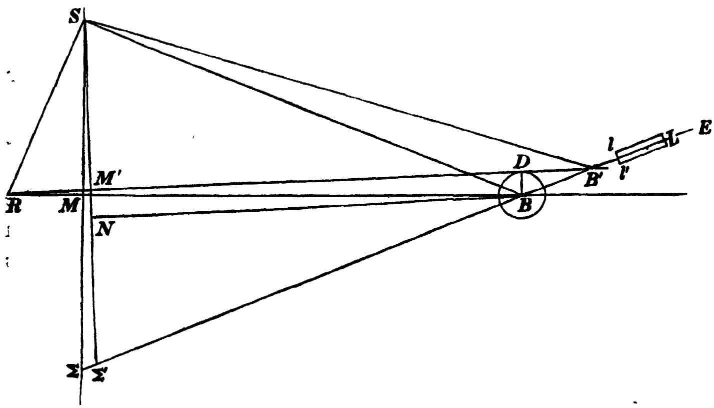

Fig. 346.

discussed in detail, but also to the proper mounting of the crystal itself; but before defining these latter precautions it will be well to discuss the optical principles involved in the method of measurement by a reflection-goniometer, certain of which have been in fact already assumed by anticipation during the discussion of the adjustments of the instrument.

Let *S*, Fig. 346, be the centre of the signal, *B*, a point in the axis of the goniometer, and *EL* the optic axis of the observing telescope of the instrument of Mitscherlich, or the line of sight of the naked eye as in the method of Wollaston, and so directed that if continued it passes through *B*. If then *S*, the centre of the slit, is seen reflected by a crystal-face so as to coincide with the intersections of the cross-wire in the telescope, the following statements follow from the elementary principles of optics:—

(1) Every image of the centre of the signal S that is to be seen placed at the intersection of the cross-wires must be due to rays of which the axis is reflected in the direction ΣBE.

(2) If Σ be the image of S, S and Σ will be symmetrically situate in regard to the reflecting plane. So that if M be the intersection of the line $S\Sigma$ with the plane of the crystal-face, $SM = \Sigma M$ and $S\Sigma$ is perpendicular to the reflecting plane; and this plane forms an oblique section with the cone of rays of which Σ is the apex and ll' the object-glass of the telescope or the pupil of the eye is the base: the brightness of the image will therefore increase as more of this oblique section is a reflecting surface, and will therefore increase with the magnitude of the crystal-face so long as this is not greater than that of the section.

(3) If the crystal-face lie in a plane BM passing through the axis of the goniometer, its position must be at or near B and within the limits of the cone of rays $\Sigma ll'$; the direction of MB being determined by the angle SBE, since the angle SBM is the complement of $\frac{1}{2} SBE$.

In the case of the crystal-face lying in a plane not passing through the axis of the goniometer (though necessarily parallel to it), if the plane be parallel to the plane BM the axis of the reflected rays will not intersect the horizontal wire of the telescope, for the rays will appear to proceed from a point not lying in the line ΣBE.

In order that the image of the slit may be reflected on to the horizontal wire of the telescope by the crystal-face, this face must be turned into such a position that the image of S will lie on the line $E\Sigma$ though it will be at a different point from Σ; say at Σ'. Let $B'M'$ be the trace of this plane perpendicular to and bisecting $S\Sigma'$. The crystal-face will now lie in some part of this plane, as at B', and will have been turned from its previous position, round an edge perpendicular to the plane of the figure (and therefore parallel to the axis of the goniometer), through an angle which may be denoted by θ. Let BM and $B'M'$ intersect in R.

Then S and Σ' will be symmetrical in respect to $B'M'$, and

$B'R$ will bisect $S\Sigma'$ perpendicularly in M'. From B draw perpendiculars BN on $S\Sigma'$ and BD on $B'R$. Let M be supposed joined to M' and N.

Let $BD = NM' = d$, $SB = r$, and $SBR = \omega$;

then $$\theta = BRD = \Sigma S \Sigma' = MBN.$$

Since the angles at M and N are right angles, $SMNB$ are points on a circle, and $$MNM' = MBS = \omega.$$

Also since M and M' bisect $S\Sigma$ and $S\Sigma'$, MM' is parallel to $\Sigma\Sigma'$, and

$$MM'N = B\Sigma'N = B\Sigma S + \Sigma S \Sigma' = \frac{\pi}{2} - \omega + \theta,$$

and $$M'MN = \pi - MNM' - MM'N = \frac{\pi}{2} - \theta.$$

Also $$\frac{MM'}{NM'} = \frac{\sin M'NM}{\sin M'MN} = \frac{\sin \omega}{\sin\left(\frac{\pi}{2} - \theta\right)}$$

Therefore $$MM' = \sin\omega \frac{d}{\cos\theta}.$$

Also $$\frac{MM'}{MS} = \frac{\sin MSM'}{\sin MM'S} = \frac{\sin\theta}{\cos(\omega - \theta)}.$$

Therefore $$MM' = \sin\omega \frac{r\sin\theta}{\cos(\omega - \theta)}.$$

Equating these values, we have

$$\frac{d}{\cos\theta} = \frac{r\sin\theta}{\cos(\omega - \theta)},$$

whatever be the value of θ.

This angle θ is however necessarily very small, while, in relation to d, r is always very large; whence, assuming

$$\sin\theta = \theta \quad \text{and} \quad \cos\theta = 1,$$

we have $$d = r\frac{\theta}{\cos\omega} \quad \text{or} \quad \theta = \frac{d}{r}\cos\omega.$$

It may be observed that, since $\sin\theta = \frac{BD}{BR}$, we have within the assumed degree of approximation, $\theta = \frac{d}{BR}$;

and, substituting this value of θ in the above expression, we have

$$\frac{SB}{RB} = \cos \omega,$$

whence $BSR = 90°$, to the same degree of approximation.

Hence if d, the distance from the axis of the goniometer of a plane passing through the crystal-face that gives an image of S on the horizontal wire of the telescope, be given, we can find the direction of the plane by drawing a line through S at right angles to BS and meeting BM in R, and drawing from R a line RD tangent to the circle described with radius d round B as a centre: the position of the reflecting crystal-face in this plane RD will be determined by the intersection of this plane with the cone $\Sigma' l I'$.

381. It will be seen then that the position of a reflecting face as determined by the reflection-goniometer is different according as the plane of the face contains the axis of the instrument or is only parallel to it: and the correction for the error—an error of θ if we estimate the angle and of d if we consider the position of the plane—must be treated as positive if it lie on one side of BM, negative if on the opposite side of that plane.

If then we are measuring the angle between two faces on a crystal, it is clear (1) that their edge must be parallel to the axis of the instrument, (2) that it should be coincident with this axis. Where however it is not coincident the correction for the errors of angle will be given by

$$\theta = \frac{d}{r} \cos \omega \text{ for the one plane,}$$

$$\theta' = \frac{d'}{r} \cos \omega \text{ for the other.}$$

And as we can always in practice so centre the crystal as that the axis may fall within the angle formed by the two faces, the correction can be put into the form $\frac{d-d'}{r} \cos \omega$; and this can always be made less than $\frac{t}{r} \cos \omega$, where t is the greatest thickness of the crystal; *a fortiori* less than $\frac{t}{r}$, or the maximum angle which the crystal when placed at B subtends at S.

And, further, the amount of error will depend on the magnitude of ω, becoming smaller in proportion as the directions of the ray from the signal and of the axis of the telescope approach to coincidence with the normal to the reflecting face; and also on the magnitude of r, vanishing when r is infinite and the rays from the signal S become parallel.

The latter case, where the rays from S are parallel, is a condition which can be induced by the use of a collimator; the illuminated slit being placed in the geometrical focus of the collimator-lens, and the focus of the telescope adjusted for parallel rays.

382. In the discussion of the adjustments needed for the goniometer, the processes employed for detecting and correcting error did not necessarily involve the employment of a mounted crystal: and the method for attaching and properly mounting a crystal remains to be considered. It will have been seen from the result of the theoretical discussion of the last articles that the conditions to be fulfilled are that the edge to be measured should coincide as nearly as may be with the axis of the goniometer, and that the part of that edge selected for measurement should be intersected perpendicularly by the plane of observation.

The instrumental means at our disposal for effecting the different motions of the crystal are the following, and they belong entirely to the head or moveable apparatus for adjusting and centering the crystal.

(1) Two circular motions cc' in planes perpendicular to each other may be imparted to the crystal by reason of its being placed, when mounted, at or near the common centre of two toothed circular arcs worked by tangent screws VV'; see Fig. 347. The crystal can thus receive a partial rotation, approximately round its own centre, in either or both of two perpendicular planes which are *parallel* to the axis.

(2) Two traversing motions tt' perpendicular to each other and to the axis of the instrument. These are worked by two screws, UU', parallel to the tangent screws that impart the rotatory movements, and they operate in the directions of these tangent screws.

(3) A motion already alluded to whereby the head can be moved along the cylindrical core A of the goniometer away from or

towards the disc; and a means *b* of clamping the head at any point firmly to the steel core (Figs. 345, 347).

(4) A supply of small brass crystal-holders in the form of cubes or flattened prisms, each with a projecting peg fitting into a corresponding socket drilled into the 'head' in such a manner that the axis of the peg is in the axis of the instrument when the traversing and tangent screws are in their normal position:

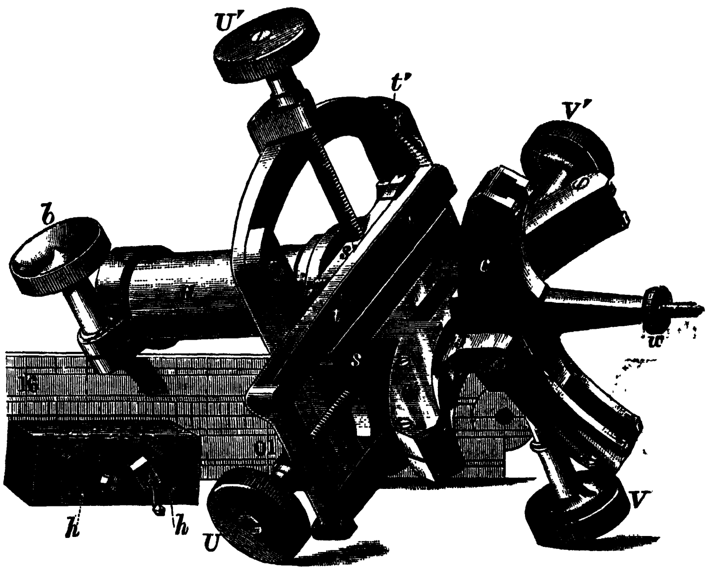

Fig. 347.

with the crystal it carries the holder is subject to the motions imparted by these screws.

The forms *h h* in Fig. 347 are the most useful; the one when a wax cement is the adhesive material employed, the other when the crystal is attached to a platinum wire. The latter is one of the most convenient modes of mounting a small crystal, as then the wire can be screwed firmly to the support and can be bent into any position required for bringing different zones of the crystal into adjustment. When a wire is thus employed the best cement wherewith to effect the fastening is formed of gelatine and acetic acid, just sufficiently viscous to become hard in the course of two or three hours. Or, as recommended by Klein, gum-arabic thickened with milk-sugar, kept dried and moistened when needed, will answer the purpose.

383. Let it be supposed, as an example, that we have to measure the angles of a zone in which two good reflecting faces P and Q meet in an edge. In the first place, the goniometer itself (Fig. 345) having been adjusted, the apparatus forming the head is brought into the normal position, that namely in which the circular arcs cc' worked by the tangent screws are both symmetrically divided by the axis, so that when the crystal-holder is in its place its edges are parallel to the axis; further, the head is turned round until the traversing screws $t't$ are respectively vertical and horizontal, the former being on the upper side and the latter on the side away from the observer.

The crystal o is so adjusted on the brass holder h, that the face P is as near as may be parallel to the flat side of the holder, and that the edge to be measured is parallel to its axis; and care is afterwards taken, when fixing the crystal-holder in its socket, that the flat side is as near as may be horizontal, and therefore perpendicular to a definite tangent screw. The head is next clamped to the steel core A in a position in which the crystal can be seen in the micro-telescope T. By means of the traversing screws the edge to be measured is brought as near as possible to the horizontal cross-wire and in focus; one method of quickly effecting this is as follows:—the inner axle is turned until one of the traversing screws is perpendicular to the axis of the telescope, and by means of this screw the edge is brought to the horizontal wire; the inner axle is then turned until the other traversing screw is perpendicular to the axis of the telescope and the edge is again brought to the horizontal wire in the same way as before: the edge of the crystal is now close to the axis of the goniometer, but is only approximately parallel to it.

384. The inner axle is next turned round until the face P is as nearly as may be in position to reflect light from the signal into the telescope: the face P having been arranged to be approximately perpendicular to a definite tangent screw, the requisite position of the head is a constant of the instrument and becomes familiar to the observer. By means of the tangent screw perpendicular to *P*, the edge to be measured is now brought into parallelism with the horizontal cross-wire, and next into apparent concidence with it by the opposite traversing screw: on turning aside the lens *l* from the object-glass an image of the signal should now be visible, and its centre may be brought on to the vertical cross-wire by a further rotation of the same tangent screw: the edge will require to be once more brought to the horizontal wire by means of the opposite traversing screw. The inner axle is now turned round until the face *Q* is seen in the micro-telescope to be illuminated with light from the signal: by a rotation of the second tangent screw the position of *Q* is altered until the face is at its brightest, the lens *l* is again turned away from the object-glass, and the centre of the image of the signal then visible is brought on to the vertical cross-wire by means of the same tangent screw, and the edge as seen in the micro-telescope is again brought to the horizontal wire by means of the opposite traversing screw. Since, in the preliminary arrangement of the crystal, the face *P* was made approximately parallel to the plane of the circular motion produced by the second tangent screw, the adjustment of the face *Q* by means of this screw will little affect the direction of the face *P*.

385. These operations must be repeated until, when the goniometer disc makes a complete rotation (1) the crystal is seen to turn round the edge as if hinged on the horizontal wire, and (2) the images of the signal reflected from the faces *P* and *Q*, and indeed from all faces of the zone, pass in succession through the field of view symmetrically to the vertical wire, and are ortho-symmetrically divided by the two wires when at the centre of the field.

The nearer the angle between the faces *P* and *Q* approaches to a right angle, the more quickly is the accurate adjustment of the zone arrived at: for, by the preliminary arrangement of the crystal, the face *Q* will in such cases become more nearly parallel to a tangent screw, and thus be less affected by the adjustment of the other face.

386. The chief advantages of an instrument of the above construction over one with a horizontal disc are the following: (1) the graduation being on the edge of the disc is more easily read; (2) owing to the inclined position of the telescope the work is less fatiguing to the observer; (3) the crystal, the traversing and adjustment screws on the head, and the milled heads of the axles are all more easily accessible than when they are above or beneath the instrument; (4) the sliding of the hollow cylinder which carries the 'head' renders the adjustment more easy and speedy; (5) the correctness of adjustment of the goniometer itself is easily verified or restored. On the other hand, a goniometer with a horizontal disc can be used for the measurement of larger crystals, and the adjustment of the crystal is less liable to disturbance, through the yielding of the wax support during the measurement: also the motion of the telescope renders it possible to alter, when desirable, the angle of incidence of the light upon the reflecting faces; further, the instrument can be used for the determination of indices of refraction.

387. In the vertical goniometer (Fig. 348) the 'head' is fixed to a hollow cylinder which can be clamped to a solid core continuous with the inner axle, but in the horizontal goniometer such an arrangement is impracticable: in the latter instrument the head is fixed to a solid steel cylinder which passes concentrically through the inner axle, and is terminated by a screw: it can be raised or lowered by the rotation of a milled head through which this screw passes.

When the crystal has been brought to the height of the horizontal cross-wire, the cylinder is clamped to the inner axle by tightening a collar situated just above the graduated disc. During the measurement of a zone, the inner and outer axles are fixed together by the pressure of a screw. A slow motion of the graduated circle can be produced by an ordinary tangent screw. Finally, the telescope and also the verniers of the graduated circle are fixed to another concentric hollow axle, external to the other two; this moves in a hollow cone fixed to the three legs of the instrument: hence, if the graduated circle be clamped to the tripod, the angle of rotation of the telescope can be measured: the telescope can be fixed in any of its positions and is provided with a tangent screw for slow motions. The collimator is fixed to the tripod.

Corresponding parts of the vertical and horizontal goniometers (Figs. 345 and 348) are indicated by identical letters.

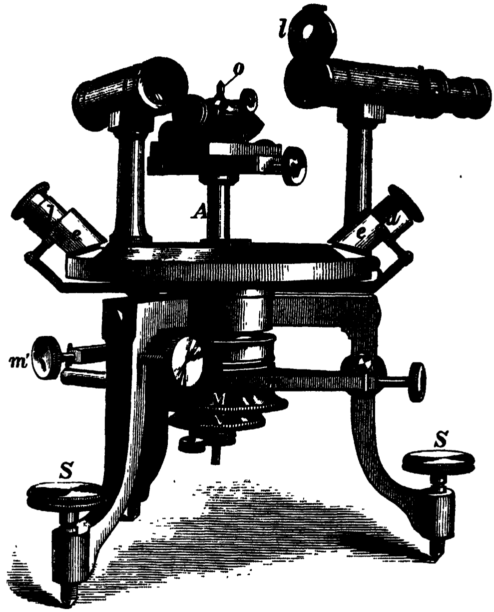

Fig. 348.

388. In the horizontal goniometer, as made by Fuess, the telescope, instead of being capable of adjustment, is screwed fast to its carrier: the instrument is so carefully constructed that adjustment can be made precise by a simple motion of the cross-wires, and is therefore less easily disturbed. The process of adjustment of the instrument, recommended by Websky, is the following:—

(1) One of the eyepieces having been adjusted so that its cross-wires are distinctly visible, is inserted in the tube of the telescope until a distant object is in focus.

(2) A needle is adjusted in the axis of the instrument, and the eyepiece is turned round until one of its cross-wires is as near as may be parallel to the needle, as seen in the micro-telescope: the cross-wires are then moved horizontally until this wire is in apparent coincidence with the axis of the needle.

(3) A plate of 'parallel glass' is adjusted in approximate parallelism to the axis of the goniometer, and the axle is rotated until the plate is as near as may be perpendicular to the axis of the telescope.

(4) A small plate of glass is now fastened in front of the eyepiece at an inclination of about 45° to the axis of the telescope: a lamp having been so placed that light is reflected from this plate into the telescope, the cross-wires of the eyepiece are seen both directly by dispersed light and by reflection from the parallel glass: if the latter be adjusted by means of the circular motions so that the images of the cross-wires, due to reflection from opposite sides of the parallel glass, take in turn exactly the same position in the field, the plate will be exactly parallel to the axis of the goniometer. By raising or lowering the cross-wires their centre may be brought into coincidence with its image as seen after reflection: the axis of the telescope is then exactly normal to the plate and therefore to the axis of the instrument.

(5) When the outer-axle is rotated, the image of the intersection of the cross-wires will move across the field in a direction exactly perpendicular to the axis of the goniometer: if this direction be inclined to the 'horizontal' wire, the eyepiece must be turned until parallelism is produced: coincidence may be obtained by a vertical translation of the cross-wires. The 'horizontal' wire and the axis of the telescope are now exactly normal to the axis of the goniometer. The glass-plate in front of the eyepiece is now removed.

(6) Four tubes with signals are generally supplied with the instrument; the signals being respectively, a cross-wire, a small circular hole, a straight-edge slit, and a Websky's slit. The second of these is first inserted in the collimator: the telescope is then turned until the image is seen either directly or after reflection from the parallel glass: the signal is moved about in the tube of the collimator until the image is in focus: the screws which fasten the pillar of the collimator to the tripod are adjusted so that the centre of the image is seen on the horizontal cross-wire. The straight-edge slit and the Websky's slit are next focussed. Finally, the cross-wire signal is inserted and focussed, and then by means

of their traversing screws the cross-wires of the collimator are brought into apparent coincidence with those of the telescope. The remaining eyepieces of different magnifying powers are now adjusted by means of the cross-wire signal: in fact, the chief use of the latter is for the verification of the adjustment, since few crystals present faces so perfect as to give images of such a signal. All the eyepieces and signals are provided with sliding collars with a projecting piece which fits into a corresponding notch in the collar of the tube of the telescope or collimator: if this collar be tightened when the tube is in its final position, the eyepiece or signal may be taken away and afterwards replaced in exact adjustment without necessitating the removal of an adjusted crystal.

In performing the actual measurements of the successive angles between the faces of a zone, each edge has to be separately adjusted by means of the traversing screws; unless the crystal is very minute in its dimensions, in which case the crystal, after one of the edges of the zone has been adjusted, may be brought by the traversing screws into a position in which the zone to be measured is seen always symmetrical to the horizontal wire. In this case the resulting error is too minute to be estimated. If the edge itself cannot be seen, from injury or other cause, the centering may be effected by making the profile of each face coincide with the cross-wire parallel to the axis of the goniometer.

389. *Error of adjustment.* It has been seen that, if A and B be two faces of a zone, their edge will be seen in the field of the micro-telescope to be precisely parallel to the horizontal wire, and can be brought into apparent contact with it when the zone is correctly adjusted; and the images of the signal reflected from the two faces will each be seen in the telescope to pass over the field of view symmetrically to the vertical wire, and to be ortho-symmetrically divided by both wires when the centre of the image has reached the horizontal wire.

Neither of these statements holds good when the axis of the zone $[AB]$ is not strictly parallel to the axis of the goniometer. Let the angle of inclination of the zone-axis to the axis of the goniometer be a; and if the face A be adjusted so that the image reflected from A has its centre at the crossing of the wires in the telescope, let δ be the distance from this crossing of the centre of the image reflected from B when moved through the field till it is in contact with the horizontal wire. The value of δ in any particular case can be determined by finding out the angle which a responding distance on the vertical wire represents when an image from a correctly adjusted crystal-face traverses that distance; i.e. finding the difference of the two readings on the goniometer when the image is central and when its centre is on the vertical wire at the distance δ from the crossing; δ is twice this difference. Let T, Fig. 349, be the projection of the telescope-axis perpendicularly to the plane of the figure, LT the axis of the goniometer, ST the direction of the axis of the collimator, or of the ray from the centre of the signal which intersects the goniometer-axis in the plane of observation.

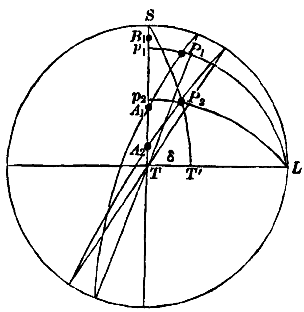

Fig. 349.

For simplicity, let the axis of the telescope be assumed to be perpendicular to ST; hence L, T, and S may be three points on a sphere separated by quadrants.

If now A and B are in correct adjustment, and the letters A_1, B_1, &c. in the figure represent the projections on this sphere of the poles of the planes in question, it is evident that for the image reflected from the first face A to be central we must have

$$A_1 T = A_1 S = 45^\circ.$$

And similarly for B_1, since the pole of B must be at the point A_1 when the image from it is central, if the zone be in correct adjustment. Where however the zone-axis is inclined to the goniometer-axis at an angle a, let P_1 be the pole of B when the centre of the image from A is on the horizontal wire.

Let the angle of the edge $[AB]$ be γ; then, since the angle between the zone-circles $A_1 S$ and $A_1 P_1$ will be the same angle a as that of the inclination of their zone-axes, and $A_1B_1 = A_1P_1 = \gamma$, if the position of the pole A_1 is given anywhere on the great circle ST the corresponding position of P_1 can be determined. When the image from A is on the horizontal wire the pole of A is at A_1 and the pole of B at P_1, and when the image from B is at T' on the horizontal wire the pole of B is at P_2. Then we have

$$P_2S = P_2T' = 45^\circ \quad \text{and} \quad TT' = \delta.$$

And if the great circles LP_1 and LP_2 intersect ST in p_1 and p_2,

$$LP_1 = LP_2,$$

and $\qquad \sin P_2p_2 = \sin P_1p_1 = \sin A_1P_1 \sin P_1A_1p_1$

$$= \sin\gamma \sin a;$$

also $\qquad \sin P_2p_2 = \sin SP_2 \sin P_2Sp_2$

$$= \sin 45^\circ \sin\delta = \frac{\sin\delta}{\sqrt{2}};$$

whence $\qquad \sin a = \dfrac{\sin\delta}{\sqrt{2}\sin\gamma} \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$

The angle, as measured on the instrument, between A and B will be

further we have

$$\tan p_1A_1 = \tan P_1A_1 \cos P_1A_1p_1 = \tan\gamma \cos a,$$

$$\tan p_2S = \tan P_2S \cos P_2Sp_2 = \cos\delta.$$

Let E be the error in the measurement of the angle, i.e. the true angle—the measured angle; then

Now $\qquad \tan(\gamma - A_1p_1) = \dfrac{\tan\gamma - \tan A_1p_1}{1 + \tan\gamma \tan A_1p_1} = \dfrac{\tan\gamma(1 - \cos a)}{1 + \tan^2\gamma.\cos a}$

$$= 2\,\frac{\tan\gamma \sin^2\frac{a}{2}}{\cdot\tan^2\gamma\cos a},$$

$1 - \cos\delta \qquad \delta$

and

a and δ are small quantities and their cubes may be neglected;

$$\tan(\gamma - A_1p_1) = \frac{\frac{a^2}{2}\tan\gamma}{1 + \tan^2\gamma} = \frac{a^2}{4}\sin 2\gamma,$$

$$\tan(45^\circ - Sp_2) = \frac{\delta^2}{4}.$$

The product of these two tangents will thus be of the fourth order, and may be neglected.

So $$\tan E = \tan(\gamma - A_1\beta_1) + \tan(45^\circ - Sp_2)$$
$$= \frac{a^2}{4}\sin 2\gamma + \frac{\delta^2}{4},$$

or to the same degree of approximation,

$$E = \frac{a^2 \sin 2\gamma + \delta^2}{4}.$$

From (1) $$\sin^2\delta = 2\sin^2\gamma \sin^2 a,$$
and $$\delta^2 = 2a^2\sin^2\gamma.$$

Whence $$E = \frac{2a^2\sin\gamma\cos\gamma + 2a^2\sin^2\gamma}{4}$$
$$= \frac{a^2}{\sqrt{2}}\sin\gamma\sin(\gamma + 45^\circ), \quad \ldots\ldots\ldots\ldots \quad (2)$$

or $$E = \frac{\delta^2}{2\sqrt{2}}\,\frac{\sin(\gamma + 45^\circ)}{\sin\gamma}. \quad \ldots\ldots\ldots\ldots \quad (3)$$

E, a, and δ are in circular measure, and if E_1, a_1, δ_1 be the corresponding angles in degrees, we shall have

$$E_1 = \frac{\pi}{180}\,\frac{a_1^{\,2}}{\sqrt{2}}\sin\gamma\sin(\gamma + 45^\circ),$$

$$E_1 = \frac{\pi}{180}\,\frac{\delta_1^{\,2}}{2\sqrt{2}}\,\frac{\sin(\gamma + 45^\circ)}{\sin\gamma}.$$

This is an error clearly small in quantity and of the second order. If a be given we may determine the value of γ for which E is a maximum or minimum.

From (2) we have

$$E = \frac{a^2}{2\sqrt{2}}[\cos 45^\circ - \cos(2\gamma + 45^\circ)].$$

For a given value of a, E will be a maximum when

$$\cos(2\gamma + 45^\circ) = -1, \text{ or when } \gamma = 67\tfrac{1}{2}^\circ,$$

and a minimum when

$$\cos(2\gamma + 45^\circ) = 1, \quad \text{or when} \quad \gamma = 157\tfrac{1}{2}^\circ;$$

that is to say, with a given error of adjustment a, the error in the angle measured increases with the angle until that angle is $67\frac{1}{2}°$, and diminishes until the angle is $157\frac{1}{2}°$, &c., &c.

390. Before proceeding to measure any particular zones on a crystal it is best to examine the crystal carefully and to assign letters to prominent and distinctly recognisable faces, after having made a few small free-hand sketches of it in its different aspects. Thus, in subsequently noting the results, one has in the letters assigned to these faces a means of provisionally designating the various zones. It is well in entering measurements in a note-book to head the page with this symbol of the zone, and to make a sketch of each face measured, as seen in the micro-telescope, noting any peculiarities it may exhibit. Opposite the entry of the arc measured, an abbreviation (im., m. l., &c.) should indicate that the observation was made from an image, maximum light, striæ, bands, a broad belt, &c.; a good image being indicated by one or by two straight lines scored under it, an indifferent or bad one by a waved or more or less jagged line; and similarly for the quality of the readings. Every fresh adjustment of an edge should be noted in the margin.

Finally, a certain relative weight is assigned to the readings for each edge in the zone while the observations are still fresh and the crystal is under the eye and in adjustment, and an average is taken of the different measurements thus weighted of any symmetrically repeated edges.

It cannot however be too decidedly dwelt on that one fine measurement with distinct images from the faces of a well-adjusted edge has an altogether different intrinsic value from that of any average obtained from less reliable measurements made on faces that, from whatever cause, give imperfect or confused images.

In noting the nature of the observation, other terms than an 'image' have been alluded to; and in fact one of the great difficulties that baffles the crystallographer is the comparative rarity with which he is able to record an observation from two consecutive faces in a zone with the note that they gave perfect images. Sometimes in lieu of a single image two or three are seen, of which perhaps some lie slightly out of the zone to left or right, the images in fact overlapping, but with their centres not on the vertical wire. And not rarely these take the form of a long ribboned band made up of an indefinite number of such images, images in fact reflected from a legion of striations, or of the numerous pseudo-parallel tesseræ of a tesselated face. In other cases, again, they form a broad band of light, the result either of the massing together of striæ, in which case they may be resolved by narrowing the slit, or else of the confused character of the image reflected from a face which presents unevenness of reflecting surface.

391. It is in dealing with such difficulties that the qualities of the good observer come into play.

A whole zone should be considered in determining which images may indicate the faces that are most to be relied on for giving direction to the zone-axis on the goniometer. Sometimes one half of the zone can be determined consistently from one group of images, while no good images can be found from faces parallel to them in the other half of the zone; and in this other half again a different set of images may give results concordant with the former, although representing faces which are slightly inclined to those which gave the former images; indicating in fact a small inclination of the two hemi-zone axes on each other. This is sometimes traceable to a twinning of two forms, but oftener to a composite structure in which the numerous crystal-individuals are in approximately but not completely parallel union.

392. When there are numerous separate images it may be hopeless to identify those belonging to two faces which correspond to each other. Sometimes however by patience and scrutiny such correspondence can be found, evidenced by the repetition of an identical angle measured over other symmetrically repeated edges in the zone; and such corresponding striæ are by no means as a rule those of equal brightness or breadth. They are however generally to be found in the same zone except in the case alluded to in the last article.

But often we get no image at all from a face, and are driven to the unsatisfactory expedient of estimating the position in which the face under measurement gives in the micro-telescope the brightest illumination. For this purpose the slit should be narrowed to the smallest width that will give any appreciable illumination of the face: and in measuring from a face that gives an image to one that does not, this method of maximum illumination should be employed for both faces. In every case the measurement should be repeated several times with the object of obtaining an average result; and, even in determining an angle where the faces both give perfect images, it is always well to take the mean of three measurements on successive portions of the graduated circle.

SECTION II.—**Crystallographic Calculation.**

General Problems.

393. From the instrumental means of measuring the angles between the faces of a crystal, we pass to the consideration of the crystallographic problems by which the results of these measurements are to be interpreted.

These problems are of two kinds; involving, first, the determination of the elements of the crystal and therewith its symmetry; and secondly, the assignment to each form and face of the crystal of its proper symbol and position.

In proportion as the data obtained by the aid of the goniometer are exact and numerous are these problems simplified; and where, from the imperfection or the absence of faces, the observations taken are few or are wanting in precision, the results arrived at are necessarily incomplete.

The crystals which belong to the more complete types of symmetry are so far the more simply dealt with that the elements to be determined are few, there being none in the case of the Cubic system, while from the symmetrical recurrence of the same angles the opportunities of obtaining good measurements of any angle are multiplied. Crystals presenting a lower order of symmetry require a greater number of independent angles to be measured, at the same time that the calculations become more complex, involving the more frequent determination of oblique triangles and purely trigonometrical processes.

We have then in the investigation of a crystal to employ methods which deal with the general relations of the planes of a crystalloid system to each other and to the axial planes, combined with special methods which belong especially to each particular type of symmetry. Those of the former class have been already to a large extent considered: they fall under the following heads:—

(1) The general equations (A) connecting the symbols of the different faces belonging to a crystalloid system (Art. **17**).

(2) The expressions connecting the symbol of a zone with those of faces belonging to it and of a face with those of zones containing it.

(3) The relations connecting three faces in a zone.

(4) The problems relating to four tautozonal faces and their anharmonic ratios.

(5) The expressions involved in the transformation of an axial system.

(6) Those dealing with the faces of twinned crystals.

And certain other general problems still remain to be considered before we pass to the discussion of problems peculiar to each different system.

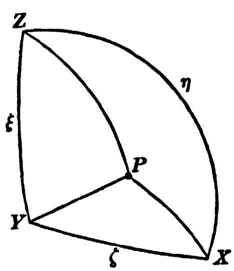

Fig. 350.

394. In the first of these general problems it is sought to represent the angles between the normal of a face and the several axes as a general function of the parameters of the system, the indices of the face, and of the angles between the axes; or in the equivalent form, as a function of the parameters and indices, and of the angles between the normals of the axial-planes.

I. *To establish the relation connecting the arcs* PX, PY, PZ, YZ, ZX, XY (Fig. 350).

$$YPZ = 2\pi - YPX - XPZ,$$

$$\cos YPZ = \cos(YPX + XPZ),$$

and

$$= \cos YPX \cos XPZ - \sin YPX \sin XPZ,$$

Whence

$$1 = \cos^2 YPZ + \cos^2 ZPX + \cos^2 XPY - 2 \cos YPZ \cos ZPX \cos XPY, \qquad \text{(i)}$$

a condition which holds true whatever the position of P.

Also we have

$$\cos YPZ = \frac{\cos\xi - \cos PY \cos PZ}{\sin PY \sin PZ},$$

and the other expressions symmetrical therewith.

Substituting these values in (i), transforming the sines into cosines, and reducing, we get for the required relation,

$$\begin{aligned}
&1 - \cos^2\xi - \cos^2\eta - \cos^2\zeta + 2 \cos\xi \cos\eta \cos\zeta \\
&= \cos^2 PX + \cos^2 PY + \cos^2 PZ - \cos^2\xi \cos^2 PX - \cos^2\eta \cos^2 PY - \cos^2\zeta \cos^2 PZ \\
&- 2 (\cos\xi \cos PY \cos PZ + \cos\eta \cos PZ \cos PX + \cos\zeta \cos PX \cos PY) \\
&+ 2 (\cos\xi \cos\eta \cos PX \cos PY + \cos\eta \cos\zeta \cos PY \cos PZ + \cos\zeta \cos\xi \cos PZ \cos PX), \quad \text{(ii)}
\end{aligned}$$

an expression which is symmetrical in respect to PX, PY, PZ, and also to ξ, η, ζ.

395. *To express* $\cos PX$, $\cos PY$, *and* $\cos PZ$ *in terms of* h, k, l, a, b, c, X, Y, Z.

From the general equations

$$\frac{a}{h} \cos PX = \frac{b}{k} \cos PY = \frac{c}{l} \cos PZ = p,$$

we have

$$\cos PX = p\frac{h}{a}; \quad \cos PY = p\frac{k}{b}; \quad \cos PZ = p\frac{l}{c}.$$

Substituting these values in (ii) we get, to determine p,

$$\begin{aligned}
&1 - \cos^2\xi - \cos^2\eta - \cos^2\zeta + 2 \cos\xi \cos\eta \cos\zeta \\
&= p^2 \left\{ \frac{h^2}{a^2} \sin^2\xi + \frac{k^2}{b^2} \sin^2\eta + \frac{l^2}{c^2} \sin^2\zeta - 2\frac{kl}{bc}(\cos\xi - \cos\eta\cos\zeta) \right. \\
&\qquad \left. - 2\frac{lh}{ca}(\cos\eta - \cos\zeta\cos\xi) - 2\frac{hk}{ab}(\cos\zeta - \cos\xi\cos\eta) \right\}; \quad \text{(iii)}
\end{aligned}$$

whence, if h, k, l, ξ, η, ζ, a, b, c be given, p is known, and therefore also $\cos PX$, $\cos PY$, $\cos PZ$.

If A, B, C, Fig. 351, be the poles of the axial planes YZ, ZX, XY, and therefore ABC, XYZ be polar triangles, we have, if BC, CA, AB be α, β, γ, respectively,

$$\alpha = \pi - X, \quad \beta = \pi - Y, \quad \gamma = \pi - Z,$$
$$A = \pi - \xi, \quad B = \pi - \eta, \quad C = \pi - \zeta.$$

Since $\cos\xi = \dfrac{\cos X + \cos Y \cos Z}{\sin Y \sin Z}$

$$\sin^2\xi = \frac{1-\cos^2 X-\cos^2 Y-\cos^2 Z-2\cos X\cos Y\cos Z}{\sin^2 Y\sin^2 Z}.$$

Let $4N^2 = 1-\cos^2 X-\cos^2 Y-\cos^2 Z-2\cos X\cos Y\cos Z,$

$$= 1-\cos^2 a-\cos^2\beta-\cos^2\gamma+2\cos a\cos\beta\cos\gamma;$$

then

$$\sin\xi = \frac{2N}{\sin Y\sin Z} = \frac{2N}{\sin\beta\sin\gamma}.$$

Similarly, $\sin\eta = \dfrac{2N}{\sin\gamma\sin a}$, $\sin\zeta = \dfrac{2N}{\sin a\sin\beta}$

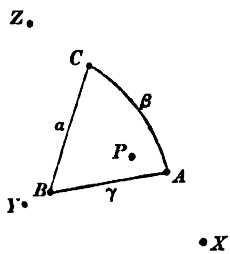

Fig. 351.

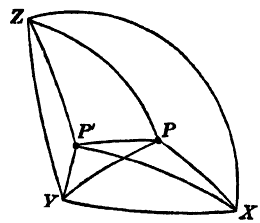

Fig. 352.

Also $\cos\xi-\cos\eta\cos\zeta = \sin\eta\sin\zeta\cos X = -\dfrac{4N^2\cos a}{\sin^2 a\sin\beta\sin\gamma}$, and other expressions symmetrical therewith.

Substituting in (iii) we get

$$p^2 = \frac{1-\cos^2 A-\cos^2 B-\cos^2 C-2\cos A\cos B\cos C}{\dfrac{h^2}{a^2}\dfrac{4N^2}{\sin^2\beta\sin^2\gamma}+\ldots+\dfrac{2kl}{bc}\dfrac{4N^2\cos a}{\sin^2 a\sin\beta\sin\gamma}+\ldots}$$

$$= \frac{(1-\cos^2 A-\cos^2 B-\cos^2 C-2\cos A\cos B\cos C)\sin^2 a\sin^2\beta\sin^2\gamma}{4N^2\left\{\Sigma\dfrac{h^2}{a^2}\sin^2 a+\Sigma\dfrac{2kl}{bc}\cos a\sin\beta\sin\gamma\right\}};$$

where the series indicated by $\Sigma\dfrac{h^2}{a^2}\sin^2 a$ is

$$\frac{h^2}{a^2}\sin^2 a+\frac{k^2}{b^2}\sin^2\beta+\frac{l^2}{c^2}\sin^2\gamma;$$

and that by $\Sigma \frac{2kl}{bc} \cos a \sin\beta \sin\gamma$ is

$$\frac{2kl}{bc}\cos a \sin\beta\sin\gamma + \frac{2lh}{ca}\cos\beta\sin\gamma\sin a + \frac{2hk}{ab}\cos\gamma\sin a\sin\beta.$$

But

$$1 - \cos^2 A - \cos^2 B - \cos^2 C - 2\cos A\cos B\cos C = \sin^2 a \sin^2 B \sin^2 C;$$

and since

$$\sin^2 B = \frac{4N^2}{\sin^2\gamma\sin^2 a}, \text{ and } \sin^2 C = \frac{4N^2}{\sin^2 a\sin^2\beta}.$$

we have
$$p^2 = \frac{4N^2}{\Sigma\frac{h^2}{a^2}\sin^2 a + \Sigma\frac{2kl}{bc}\cos a\sin\beta\sin\gamma}.$$

Substituting in the original expression, we have

$$\left.\begin{aligned}
\cos PX &= \frac{h}{a}\sqrt{\frac{4N^2}{\Sigma\frac{h^2}{a^2}\sin^2 a + \Sigma\frac{2kl}{bc}\cos a\sin\beta\sin\gamma}}\\
\cos PY &= \frac{k}{b}\sqrt{\frac{4N^2}{\Sigma\frac{h^2}{a^2}\sin^2 a + \Sigma\frac{2kl}{bc}\cos a\sin\beta\sin\gamma}}\\
\cos PZ &= \frac{l}{c}\sqrt{\frac{4N^2}{\Sigma\frac{h^2}{a^2}\sin^2 a + \Sigma\frac{2kl}{bc}\cos a\sin\beta\sin\gamma}}
\end{aligned}\right\} \quad \ldots \text{(v)}$$

396. In the second problem we have *to find the angle between any two normals of a crystal in terms of the parameters, the normal-angles of the axial planes, and the indices of the two faces.*

If $P(hkl)$ and $P'(h'k'l')$ be two poles, the axial points being as before X, Y, Z (Fig. 352); then

$$XPY = XPP' - P'PY,$$

and
$$\cos XPY = \cos XPP'\cos P'PY + \sin XPP'\sin P'PY \ldots \text{(vi)}$$

But also
$$\cos XPY = \frac{\cos\zeta - \cos PX\cos PY}{\sin PX\sin PY}. \quad \ldots \ldots \ldots \text{(vii)}$$

In the triangles XPP', $YP'P$,

$$\cos XPP' = \frac{\cos P'X - \cos PX\cos PP'}{\sin PX\sin PP'},$$

Since $\cos\xi = \dfrac{\cos X + \cos Y \cos Z}{\sin Y \sin Z}$

$$\sin^2\xi = \frac{1 - \cos^2 X - \cos^2 Y - \cos^2 Z - 2\cos X \cos Y \cos Z}{\sin^2 Y \sin^2 Z}.$$

Let $4N^2 = 1 - \cos^2 X - \cos^2 Y - \cos^2 Z - 2\cos X \cos Y \cos Z$,

$$= 1 - \cos^2 a - \cos^2\beta - \cos^2\gamma + 2\cos a \cos\beta \cos\gamma;$$

then

$$\sin\xi = \frac{2N}{\sin Y \sin Z} = \frac{2N}{\sin\beta \sin\gamma}.$$

Similarly, $\sin\eta = \dfrac{2N}{\sin\gamma \sin a}$, $\sin\zeta = \dfrac{2N}{\sin a \sin\beta}$.

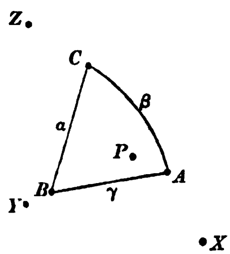

Fig. 351.

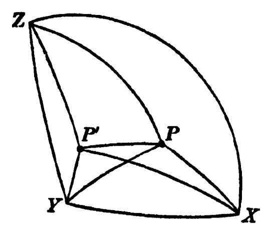

Fig. 352.

Also $\cos\xi - \cos\eta\cos\zeta = \sin\eta\sin\zeta\cos X = -\dfrac{4N^2\cos a}{\sin^2 a \sin\beta \sin\gamma}$,

and other expressions symmetrical therewith.

Substituting in (iii) we get

$$p^2 = \frac{1 - \cos^2 A - \cos^2 B - \cos^2 C - 2\cos A \cos B \cos C}{\dfrac{h^2}{a^2}\dfrac{4N^2}{\sin^2\beta\sin^2\gamma} + \ldots + \dfrac{2kl}{bc}\dfrac{4N^2\cos a}{\sin^2 a \sin\beta\sin\gamma} + \ldots}$$

$$= \frac{(1 - \cos^2 A - \cos^2 B - \cos^2 C - 2\cos A\cos B\cos C)\sin^2 a \sin^2\beta \sin^2\gamma}{4N^2\left\{\Sigma\dfrac{h^2}{a^2}\sin^2 a + \Sigma\dfrac{2kl}{bc}\cos a \sin\beta\sin\gamma\right\}};$$

where the series indicated by $\Sigma\dfrac{h^2}{a^2}\sin^2 a$ is

$$\frac{h^2}{a^2}\sin^2 a + \frac{k^2}{b^2}\sin^2\beta + \frac{l^2}{c^2}\sin^2\gamma;$$

and that by $\Sigma \frac{2kl}{bc} \cos a \sin \beta \sin \gamma$ is

$$\frac{2kl}{bc} \cos a \sin \beta \sin \gamma + \frac{2lh}{ca} \cos \beta \sin \gamma \sin a + \frac{2hk}{ab} \cos \gamma \sin a \sin \beta.$$

But

$$1 - \cos^2 A - \cos^2 B - \cos^2 C - 2 \cos A \cos B \cos C = \sin^2 a \sin^2 B \sin^2 C;$$

and since

$$\sin^2 B = \frac{4N^2}{\sin^2 \gamma \sin^2 a}, \text{ and } \sin^2 C = \frac{4N^2}{\sin^2 a \sin^2 \beta}.$$

we have
$$\frac{4N^2}{\Sigma \frac{h^2}{a^2} \sin^2 a + \Sigma \frac{2kl}{bc} \cos a \sin \beta \sin \gamma}.$$

Substituting in the original expression, we have

$$\left.\begin{aligned}
\cos PX &= \frac{h}{a} \sqrt{\frac{4N^2}{\Sigma \frac{h^2}{a^2} \sin^2 a + \Sigma \frac{2kl}{bc} \cos a \sin \beta \sin \gamma}} \\
\cos PY &= \frac{k}{b} \sqrt{\frac{4N^2}{\Sigma \frac{h^2}{a^2} \sin^2 a + \Sigma \frac{2kl}{bc} \cos a \sin \beta \sin \gamma}} \\
\cos PZ &= \frac{l}{c} \sqrt{\frac{4N^2}{\Sigma \frac{h^2}{a^2} \sin^2 a + \Sigma \frac{2kl}{bc} \cos a \sin \beta \sin \gamma}}
\end{aligned}\right\} \quad \ldots \text{(v)}$$

396. In the second problem we have *to find the angle between any two normals of a crystal in terms of the parameters, the normal-angles of the axial planes, and the indices of the two faces.*

If $P (h k l)$ and $P' (h' k' l')$ be two poles, the axial points being as before X, Y, Z (Fig. 352); then

$$XPY = XPP' - P'PY,$$

and $\cos XPY = \cos XPP' \cos P'PY + \sin XPP' \sin P'PY \ldots \text{(vi)}$

But also $$\cos XPY = \frac{\cos \zeta - \cos PX \cos PY}{\sin PX \sin PY}. \quad \ldots \ldots \ldots \text{(vii)}$$

In the triangles XPP', $YP'P$,

$$\cos XPP' = \frac{\cos P'X - \cos PX \cos PP'}{\sin PX \sin PP'},$$

$$\sin XPP' = \frac{\sqrt{\sin^2 PX \sin^2 PP' - (\cos P'X - \cos PX \cos PP')^2}}{\sin PX \sin PP'},$$

$$\cos YPP' = \frac{\quad}{\sin PY \sin PP'},$$

$$\sin YPP' = \frac{\sqrt{\sin^2 PY \sin^2 PP' - (\cos P'Y - \cos PY \cos PP')^2}}{\quad}.$$

By substituting these values and that in equation (vii) in equation (vi), we obtain a quadratic equation in cos PP', the coefficients of which are the cosines and squares of the cosines of PX, PY, $P'X$, $P'Y$ and ζ. This equation may be solved in the ordinary way: the solution, however, is more readily arrived at by performing operations identical with the above for the triangles XPZ, $XP'Z$, whereby a second quadratic equation in cos PP' is obtained, having for coefficients the cosines and squares of cosines of PX, PZ, $P'X$, $P'Z$ and η; from the two quadratics the term $\cos^2 PP'$ may be eliminated, and there will remain a simple equation giving cos PP' in terms of the cosines and squares of cosines of PX, PY, PZ, $P'X$, $P'Y$, $P'Z$, η and ζ.

Also, since the arcs ξ, η, ζ represent the angles of inclination of the axes, and the arcs a, β, γ are supplementary to those representing the inclinations of the axial planes, we have the equations

$$\left.\begin{aligned}
\cos\xi &= \frac{\cos\beta\cos\gamma - \cos a}{\sin\beta\sin\gamma}; \quad & \cos a &= \frac{\cos\eta\cos\zeta - \cos\xi}{\sin\eta\sin\zeta}; \\
\cos\eta &= \frac{\cos\gamma\cos a - \cos\beta}{\sin\gamma\sin a}; \quad & \cos\beta &= \frac{\quad}{\sin\zeta\sin\xi} \\
\cos\zeta &= \frac{\cos a\cos\beta - \cos\gamma}{\sin a\sin\beta}; \quad & \cos\gamma &= \frac{\cos\xi\cos\eta - \cos}{\sin\xi\sin\eta}
\end{aligned}\right\} \text{(viii)}$$

If now in the above simple equation in cos PP', the values of $\cos PX$, $\cos PY$, $\cos PZ$, $\cos P'X$, $\cos P'Y$, $\cos P'Z$ given in equations (v) and the values of $\cos\eta$ and $\cos\zeta$ given in equations (viii), are substituted, we ultimately obtain on reducing the expressions the equation

$$\cos PP' = \frac{\Sigma hh' \frac{bc \sin a}{a \sin\beta \sin\gamma} + \Sigma (kl' + lk') \frac{a}{\tan a}}{\sqrt{\Sigma h^2 \frac{bc \sin a}{a \sin\beta \sin\gamma} + 2\Sigma kl \frac{a}{\tan a}} \sqrt{\Sigma h'^2 \frac{bc \sin a}{a \sin\beta \sin\gamma} + 2\Sigma k'l' \frac{a}{\tan a}}} \quad \text{(ix)}$$

in which

$$\Sigma hh' \frac{bc \sin a}{a \sin\beta \sin\gamma} = hh' \frac{bc \sin a}{a \sin\beta \sin\gamma} + kk' \frac{ca \sin\beta}{b \sin\gamma \sin a} + ll' \frac{ab \sin\gamma}{c \sin a \sin\beta},$$

$$\Sigma h^2 \frac{bc \sin a}{a \sin\beta \sin\gamma} = h^2 \frac{bc \sin a}{a \sin\beta \sin\gamma} + k^2 \frac{ca \sin\beta}{b \sin\gamma \sin a} + l^2 \frac{ab \sin\gamma}{c \sin a \sin\beta},$$

$$\Sigma h'^2 \frac{bc \sin a}{a \sin\beta \sin\gamma} = h'^2 \frac{bc \sin a}{a \sin\beta \sin\gamma} + k'^2 \frac{ca \sin\beta}{b \sin\gamma \sin a} + l'^2 \frac{ab \sin\gamma}{c \sin a \sin\beta},$$

$$\Sigma (kl' + lk') \frac{a}{\tan a} = (kl' + lk') \frac{a}{\tan a} + (lh' + hl') \frac{b}{\tan\beta} + (hk' + kh') \frac{c}{\tan\gamma},$$

$$\Sigma kl \frac{a}{\tan a} = kl \frac{a}{\tan a} + lh \frac{b}{\tan\beta} + hk \frac{c}{\tan\gamma},$$

$$\Sigma k'l' \frac{a}{\tan a} = k'l' \frac{a}{\tan a} + l'h' \frac{b}{\tan\beta} + h'k' \frac{c}{\tan\gamma}.$$

The equation may also be put into the form

$$\cos PP' = \frac{\Sigma hh' \frac{bc}{a} \sin^2\xi - \Sigma (kl' + lk')\, a (\cos\xi - \cos\eta \cos\zeta)}{\sqrt{S}\sqrt{S'}}, \quad \text{(ix } a\text{)}$$

in which

$$S = \Sigma h^2 \frac{bc}{a} \sin^2\xi - 2\,\Sigma kla (\cos\xi - \cos\eta \cos\zeta)$$

$$S' = \Sigma h'^2 \frac{bc}{a} \sin^2\xi - 2\,\Sigma k'l'a (\cos\xi - \cos\eta \cos\zeta).$$

397. The equations (ix) or (ix a) represent in the most general form, though not a form directly adapted for logarithmic computation, the arc-distance of any two poles whatever of a crystal; and in the most general case, that namely of the Anorthic system, in order to determine the five elements of a crystal there are required five such equations connecting the elements with the indices of the faces between the poles of which five independent arc-distances have been measured, the symbols of the faces being of course supposed to be known.

The use of spherical trigonometry, however, combined with the application of the rules of zones and the methods that will be presented in the discussion of the various systems, render the problem of finding the elements of an anorthic crystal less laborious than it would be if the only method adopted were that of dealing with a series of such equations. Where however the data obtained by the goniometer are not sufficiently exact, and the determination of the elements from these data with the closest attainable precision has to be sought in the employment of the method of least squares, the application of the equation (ix) to the determination of a large series of angles will be found the most direct, since, in the form in which it is here given, the aid of logarithms in determining the recurring constants can be advantageously brought into the calculations of these angles.

For other systems than the Anorthic the general equation (ix) assumes forms which become simpler in proportion as the arcs α, β, γ become two or three of them quadrants, and as the parametral ratios become reduced to a single ratio or to equality.

Thus, in the Mono-symmetric system, where the arcs α and γ are quadrants and the arc β is the arc-distance AC of the poles 100, 001, the equation is

$$\cos PP' = \frac{hh'\frac{bc}{a} + kk'\frac{ca}{b}\sin^2\beta + ll'\frac{ab}{c} + (hl' + lh')\,b\cos\beta}{\sqrt{h^2\frac{bc}{a} + k^2\frac{ca}{b}\sin^2\beta + l^2\frac{ab}{c} + 2hlb\cos\beta}\ \sqrt{h'^2\frac{bc}{a} + \&c.}};$$

where the expression under the second square root is identical with that under the first, but with $h'k'l'$ substituted for hkl.

In the Ortho-rhombic system the equation becomes

$$\cos PP' = \frac{hh'\frac{bc}{a} + kk'\frac{ca}{b} + ll'\frac{ab}{c}}{\sqrt{h^2\frac{bc}{a} + k^2\frac{ca}{b} + l^2\frac{ab}{c}}\ \sqrt{h'^2\frac{bc}{a} + k'^2\frac{ca}{b} + l'^2\frac{ab}{c}}};$$

and in the Tetragonal and Cubic systems the expression for $\cos PP'$

becomes further simplified as a, b, and c are two of them or are all three of them equal.

398. It is often necessary in crystallographic problems to assign symbols to particular faces of a crystal the relative positions of which have been already determined by measurement and projection, and to proceed to deal with the remaining faces or with the determination of the crystallographic elements by methods which involve the symbols and positions established for these particular faces. For this purpose certain definitions and propositions are of general application, and receive special forms in the different systems.

399. A pole will be spoken of as being given *in position* when its arc-distance from a certain given pole has been ascertained, together with the angle contained by two zone-circles, of which one traverses the two poles and the other passes through the given pole, but is only known in symbol; and where this angle is not a right angle it may always be taken as the acute angle in which the zones meet, care being had as to the positive or negative character of the angles as ascertained for different faces referred to a common face and zone; positive angles being always measured in one direction from the zone of reference.

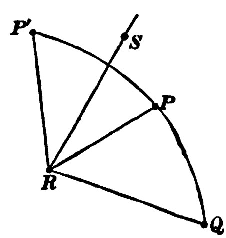

Fig. 353.

In discussing each system methods will be sought for finding the symbol of a face the pole of which is given in position in regard to some determined pole and zone-circle of the crystal. The following propositions will therefore deal with the general methods for determining the position of a pole or poles: the elements of the crystal are in each case supposed to have been previously determined.

I. Given the elements of a crystal and the symbols of two of its poles R and S; given, further, the symbols of two poles P and P': it is required to find the magnitude of the arc PP'.

Instead of calculating PP' directly by means of equation (ix), it will often be more convenient to calculate, by means of that equation or otherwise, the magnitudes of *RP*, *PS*, *SR*, *RP′*, *P′S*, Fig. 353, and thence to deduce the angles *PRS*, *P′RS*, and the arc *PP′* by means of spherical trigonometry.

In any system but the Anorthic a pole can be found to represent *R*, which is also the pole of a zone-circle identical with it in the indices of its symbol. The zone-circle [*PP′*] will in such case intersect this zone-circle—say in a pole *Q*—so that, taking *R* and [*RQ*] as the pole and zone-circle of reference for determining the positions of *P* and *P′*, the arcs *PQ* and *P′Q* are each a side of a triangle, of which another side *RQ* is a quadrant, and the arc *PP′* is the sum or the difference of these two arcs.

II. *T* and *T′* being two poles of which the symbols are known, *R*, [*RS*] being respectively a pole and zone given in symbols and position; let *P* be a pole of which the position relative to

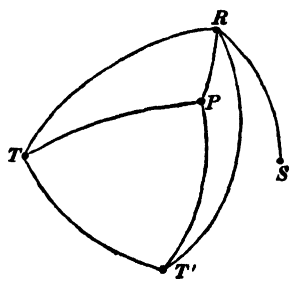

Fig. 354.

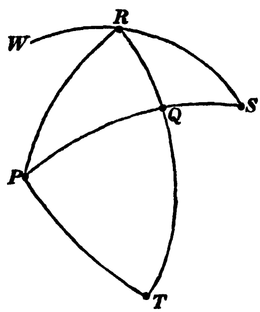

Fig. 355.

the pole *R* and the zone [*RS*] is to be determined, the arcs *PT* and *PT′* being the data as regards *P*.

Here (Fig. 354) *RT*, *SRT*, *RT′*, *SRT′*, and *TT′* are to be determined from the symbols of *T* and *T′*, and the known elements of the crystal by means of equation (ix) or otherwise, and by spherical trigonometry *RTT′*, *RT′T*, *PTT′*, *PT′T*, and so *RTP* can be computed, and therefore also the arc *RP* and the angle *TRP* in the triangle *RPT*. Hence *SRP* and the arc *RP* which give the position of *P* are known.

The symbol of P can now be determined by the methods special to the system.

III. Let T be a pole of which the symbol is known, and let the arc-distance of T from a pole P be given: the symbol of [u v w] a zone traversing P is given, R and RW being given as the pole and zone of reference. The position of P is required with a view to the determination of its symbol.

Let [u v w] intersect the zone-circles $[RT]$ in Q and $[RW]$ in S (Fig. 355). The symbols of Q, R, S and T being known, the arcs RS, RQ, QT and TS can be found from equation (ix), and then the angles SRT and RQS or PQT can be calculated by spherical trigonometry. In the triangle PQT the arcs PT, TQ and the angle PQT being known, the angle QTP or RTP can be calculated: and finally, from the arcs RT, TP and the angle RTP of the triangle RTP, the arc RP and the angle PRT can be found. The position of P being thus established in reference to R and $[RW]$, its symbol may be determined by the methods special to the system.

Section III.—Crystallographic Calculation.

The Oblique Systems.

I. *The Anorthic System.*

400. In the Anorthic system the five axial elements have to be determined irrespectively of symmetry, for the only symmetry in a crystal belonging to this system is that to a centre.

A, B, C, Fig. 356, being the positive poles of the axial planes YZ, ZX, XY, their symbols are 100, 010, 001, respectively; and the arcs α, β, γ, or BC, CA, AB, form a triangle polar to that formed by the arcs ξ, η, ζ, or YZ, ZX, XY, traversing the axial points X, Y and Z, arcs which measure the axial angles of the crystal. Thus the angle

$$A = 180° - \xi, \quad B = 180° - \eta, \quad C = 180° - \zeta; \; \ldots \ldots \quad \text{(i)}$$

and for the sides of the triangles XYZ and ABC we have, from spherical trigonometry,

$$\left.\begin{aligned}\cot\frac{}{2} &= \sqrt{\frac{}{\sin S\sin(S-\alpha)}},\\ \cot\frac{\eta}{2} &= \sqrt{\frac{\sin(S-\gamma)\sin(S-\alpha)}{\sin S\sin(S-\beta)}},\\ \cot\frac{\zeta}{2} &= \sqrt{\frac{\sin(S-\alpha)\sin(S-\beta)}{\sin S\sin(S-\gamma)}},\end{aligned}\right\}\quad \ldots\ldots\quad (\text{ii}\,a)$$

where $S = \frac{1}{2}(\alpha+\beta+\gamma)$;

and

$$\left.\begin{aligned}\cot\frac{\alpha}{2} &= \sqrt{\frac{\sin(\Sigma-\eta)\sin(\Sigma-\zeta)}{\sin\Sigma\sin(\Sigma-\xi)}},\\ \cot\frac{\beta}{2} &= \sqrt{\frac{\sin(\Sigma-\zeta)\sin(\Sigma-\xi)}{\sin\Sigma\sin(\Sigma-\eta)}},\\ \cot\frac{\gamma}{2} &= \sqrt{\frac{\sin(\Sigma-\xi)\sin(\Sigma-\eta)}{\sin\Sigma\sin(\Sigma-\zeta)}},\end{aligned}\right\}\quad \ldots\ldots$$

where $\Sigma = \frac{1}{2}(\xi+\eta+\zeta)$.

401. Let P, Fig. 356, be the pole of hkl, and let the zone-circles $[AP]$, $[BP]$, $[CP]$ intersect $[BC]$, $[CA]$, $[AB]$ in the

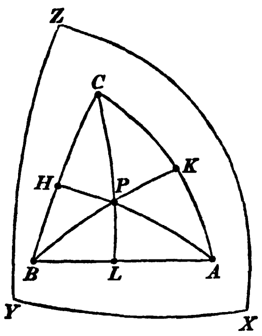

Fig. 356.

poles H, K, L. Then H is $o\,k\,l$, K is $h\,o\,l$, and L is $h\,k\,o$, and from the fundamental equation,

$$= \frac{a}{h}\cos LX,$$

$$\frac{a}{\cdot}\cos KX = \frac{c}{\cdot}$$

Further, from the triangles *HAY*, *HAZ*, since

$$AY = AZ = \frac{\pi}{2} = CY = CAY = BZ = BAZ,$$

we have

$$\cos HY = \sin HA \cos HAY = \sin HA \sin HAC,$$
$$\cos HZ = \sin HA \cos HAZ = \sin HA \sin HAB;$$

and by substitution from these and similar expressions in the preceding equations,

$$\frac{b}{c}\frac{l}{k} = \frac{\sin HAB}{\sin HAC}, \quad \frac{c}{a}\frac{h}{l} = \frac{\sin KBC}{\sin KBA}, \quad \frac{a}{b}\frac{k}{h} = \frac{\sin LCA}{\sin LCB} \qquad \text{(iii)}$$

For logarithmic computation these expressions may be put into the form

$$\tan(HAB - \tfrac{1}{2}CAB) = \tan\tfrac{1}{2}CAB \tan(45^\circ - \theta),$$
$$\tan(KBC - \tfrac{1}{2}ABC) = \tan\tfrac{1}{2}ABC \tan(45^\circ - \phi),$$
$$\tan(LCA - \tfrac{1}{2}BCA) = \tan\tfrac{1}{2}BCA \tan(45^\circ - \psi \qquad \text{(iv}a\text{)}$$

where $\tan\theta = \frac{c}{b}\frac{k}{l}, \quad \tan\phi = \frac{a}{c}\frac{l}{h}, \quad \tan\psi = \frac{b}{a}\frac{h}{k};$

or also into the equivalent form

$$= \tan\tfrac{1}{2}\xi \tan(135^\circ - \theta),$$

$$= \tan\tfrac{1}{2}\zeta \tan(135^\circ - \psi).$$

The arcs *HB*, *KC*, *LA* can now be readily determined. For *HB*, for instance, we know the angles *HBA* and *HAB* and the arc *AB*; and similarly for *KC* and *LA*.

402. From the six triangles into which the triangle *ABC* is divided by the arcs *APH*, *BPK*, *CPL*, we obtain the equations

$$\sin AB \sin BAP = \sin BH \sin BHP,$$
$$\sin CA \sin CAP = \sin CH \sin CHP,$$
$$\sin BC \sin CBP = \sin CK \sin ($$
$$\sin AB \sin ABP = \text{sir}$$

$$\sin BC \sin BCP = \sin BL$$

where

Hence, from equations (iii),

$$\left.\begin{aligned} \frac{\sin AB \sin CH}{\sin CA \sin BH} &= \frac{\sin CAP}{\sin BAP} = \frac{c}{b}\,\frac{k}{l}, \\ \frac{\sin BC \sin AK}{\sin AB \sin CK} &= \frac{\sin ABP}{\sin CBP} = \frac{a}{c}\,\frac{l}{h}, \\ \frac{\sin CA \sin BL}{\sin BC \sin AL} &= \frac{\sin BCP}{\sin ACP} = \frac{b}{a}\,\frac{h}{k}, \end{aligned}\right\} \quad \ldots\ldots \quad \text{(v)}$$

and $\sin BH \sin CK \sin AL = \sin CH \sin AK \sin BL$; .. .. (vi)

an equation which is true for every point P of the sphere, and is therefore true for every pole P of the crystal.

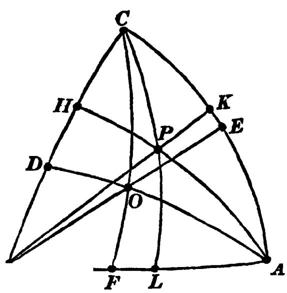

Fig. 357.

If any five of the arcs it involves are given, the sixth is known.

403. Let O be the parametral face 111, and the poles corresponding to H, K, L be D, E, F; so that (Fig. 357),

D is $01\bar{1}$, E is 101, and F is 110.

The axial angles may be obtained in terms of the arcs AB, BC, CA by means of equations (ii$a$), and the parametral ratios in terms of the arcs AB, BC, CA, and of the segments of these arcs as formed by the poles D, E, F, by means of equations (v). Thus

$$\left.\begin{aligned} \frac{a}{b} &= \frac{\sin BC}{\sin CA}\,\frac{\sin AF}{\sin BF} = \frac{\sin ACO}{\sin BCO}, \\ \frac{b}{c} &= \frac{\sin CA}{\sin AB}\,\frac{\sin BD}{\sin CD} = \frac{\sin BAO}{\sin CAO}, \\ \frac{c}{a} &= \frac{\sin AB}{\sin BC}\,\frac{\sin CE}{\sin AE} = \frac{\sin CBO}{\sin ABO}. \end{aligned}\right\} \quad \ldots \quad \text{(vii)}$$

The segmental arcs AF, FB, BD, DC, CE, EA may thus be taken as the arc-elements of the crystal, any five of them serving to give the five axial elements, independently of the sixth segment.

404. If the two segments of one of the three arcs α, β, γ are unknown, and the other four segments are given, the two unknown segments can be readily found; for, from (vi), if the segments of AB and CA are given,

$$\frac{\sin BH}{\sin CH} = \frac{\sin AK}{\sin CK}\,\frac{\sin BL}{\sin AL},$$

which may be written as

$$\frac{\sin (BC - CH)}{\sin CH} = \tan\theta;$$

whence

$$\tan (CH - \tfrac{1}{2} BC) = \tan \tfrac{1}{2} BC \tan (45^\circ - \theta);$$

an equation which gives CH, and therefore BH, since BC is known.

405. The position of a pole H or (okl), K or (hol), L or (hko) in a zone containing two of the poles A, B, C is determinable by the equations (v), supposing the elements of the crystal and the symbol of the pole to be given. And if the position of a pole on one of these zone-circles is known, the same equations give the symbol of the pole.

When the elements of a crystal are given, and it is required to determine the position of a pole P of known symbol not lying on one of the arcs α, β, γ, instead of making use of the general equation (ix) of Art. **396** it will usually be more convenient to project the great circles APH, BPK, CPL, and find the arc-distances from P of the poles A, B, C, and the angles which the zone-circles PA, PB, PC make with the arcs α, β, γ. These values are in fact deducible by spherical trigonometry from any two of the angles PAB, PBC, PCA, or any two of the arcs PA, PB, PC, or from one of the angles together with the corresponding arc; so that if any pair of these data be given, the position of the pole is determined: but the equations (iv b), and the values assigned in them to $\tan\theta$, $\tan\phi$, and $\tan\psi$, enable us to determine the angles PAB, PBC, PCA at once from the symbol of the pole. Hence when this symbol can be assigned in consequence of the known distances of the pole from each of three other poles of known symbols in one zone-circle, or from other data, the position of the pole in question relative to the poles A, B, C can be found by any two of the equations (iv *b*). The inverse problem of finding the symbol of a pole given in position, the elements of the crystal being known, differs from the last in that here the position of the pole may be given in reference to some two poles or to a pole and zone of the crystal not belonging to the elementary triangle *ABC*, though known themselves in position relatively to the sides and angles of that triangle. This in fact is the problem for finding the position of a pole in respect to a given pole and zone-circle which has already been considered in Article **399**.

Taking in place of the pole *R* and zone of reference [*RS*], in Article **399**, a pole and a side of the elementary triangle, for instance *C* and [*AC*]; and, in place of *T* and *T'*, or *T* and [**u v w**], taking the poles or pole and zone to which the position of *P* has been referred, we have to proceed, as in that article, to find the position of *P* in respect to all the poles and zone-circles of the elementary triangle. And the symbol of *P* can then be calculated by means of equations (v).

406. *To find the arc-distance of two poles with known symbols, the elements of the crystal being given.*

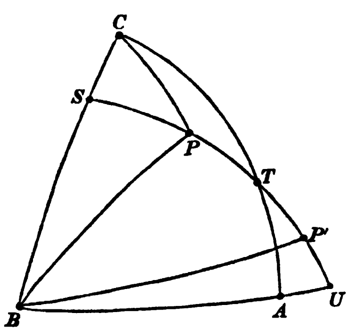

Fig. 358.

Instead of calculating the arc-distance by help of the general equation (ix) of Art. **396**, we may proceed thus:

The poles *P* and *P'* can be severally determined in position in regard to a pole and adjoining side of the elementary triangle, by equations (iv *a*).

Thus, for instance, *PAC* and *P'AC* as well as *PA* and *P'A* being known, the arc *PP'* is found by spherical trigonometry.

Or, also, Fig. 358, [*PP'*] intersects the zone-circles α, β, γ in poles *S*, *T*, *U* to which the symbols can be assigned. The arcs *SC*, *SB*, *TA*, *TC*, *UB*, *UA* can be computed by equations (v);

and the arcs *ST* and *TU* from any two of the triangles *STC*, *TAU*, *SBU*. We have then two groups of four tautozonal faces *UTPS* and *UTP′S* from which the arcs *UP*, *UP′* and therefore *PP′* can be determined: see also Art. **408**.

407. In the previous articles the position of a pole has been determined relatively to a pole and side of the elemental triangle, the elements of the crystal being given. The problem however of determining the position of a pole when its symbol is given, or of finding its symbol when its position is given relatively to four other known poles, may be put into a completely general form.

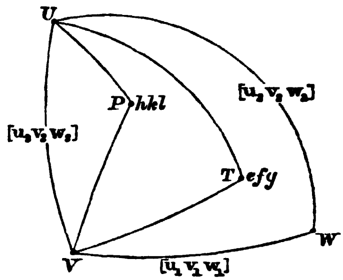

Fig. 359.

Let *U*, *V*, *W* and *T* (Fig. 359) be four heterozonal poles, the symbols of the zones [*VW*], [*WU*], [*UV*] being $[u_1 v_1 w_1]$, $[u_2 v_2 w_2]$, $[u_3 v_3 w_3]$ severally, and (efg) being the symbol of *T*. Then, by spherical trigonometry, if five of the six arcs joining the four poles are given, the sixth can be found, as well as the angles at *U*, *V*, *W* and the angles which [*TU*], [*TV*], [*TW*] make with the arcs joining *U* *V* and *W*.

Let *P* be a fifth pole with the symbol (hkl). Then, if

$$\tan\phi = [illegible] \; \frac{\sin UVT}{\sin(UVT - UVW)},$$

we have, by articles **48** and **51**,

$$\tan(VUP - \tfrac{1}{2}VUW) = \tan\tfrac{1}{2}VUW \tan(135^\circ - \theta),$$

$$\tan(UVP - \tfrac{1}{2}UVW) = \tan\tfrac{1}{2}UVW \tan(135^\circ - \phi);$$

whence the angles which [*PU*] and [*PV*] make with [*UV*], and so the arcs *PU* and *PV*, and if needed *PW*, can be found, and the position of *P* in respect to any pole and adjacent arc of the triangle *UVW* is determined.

408. If *P′* or $h'k'l'$ be a sixth pole of the system, it can also be found in position by an identical method; and the arc-distance

PP′ between two poles of known symbol can then be directly found by the methods of spherical trigonometry.

409. The symbol of a pole *P* can be found if its arc-distances from four poles of given positions and symbols are known. For if [*UT*] and [*VT*], and [*UP*] and [*VP*] (Fig. 360), intersect the zone-circles [*VW*] and [*UW*] in poles *R*, *R′*, *Q*, and *Q′* respectively, the positions of these poles on *UW* and *VW* can be determined by spherical trigonometry, and therewith the symbols of *Q* and *Q′*, since the anharmonic ratios of [*U Q′ R′ W*] and [*V Q R W*] are then known: the symbol of *P*, in which the zones [*UQ*] and [*VQ′*] of known symbols are tautohedral, is then straightway deduced.

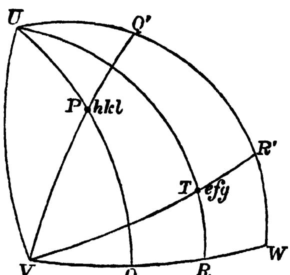

Fig. 360.

410. *To determine the elements.* Since in the Anorthic system the elements of a crystal are five independent quantities, the requisite data for determining them are five measured angles between poles of which the symbols are known but of which none are deducible from the rest.

For this purpose therefore three poles forming the angles of a spherical triangle, together with a fourth pole heterozonal to them, and therefore not on a great circle containing a side of the triangle, are available; since five out of the six angles they form are quite independent: or, the three poles forming the angles and two other poles each lying on a great circle containing a different side of the triangle may be employed, the arcs joining all the five poles being known.

If however the faces thus to be employed are initially to have known symbols, it will need a preliminary investigation of the crystal with a view to selecting faces to which it may be possible to assign such symbols, and of which some at least may be employed in determining the axial system, either by taking the axial planes parallel to them or by treating them as parametral faces.

For such a selection, the measurement and projection of the zones containing the faces which are the most prominent or which give the best images, will be undertaken first: and, after independent symbols have been assigned to certain of the faces, it is generally possible, by the rules of zones and of the tautohedral law, or by the application of the problem of four planes, to determine the symbols for other faces of which the poles are projected. Usually this assignment of symbols leads directly to the determination of certain arc-elements for the crystal. But sometimes it is not possible to assign at once the poles most suitable for the elementary triangle, or those which offer the best data for determining the elements; in which case these elementary planes have, in the first place, to be, as it were, tentatively selected and, even then perhaps only approximately, determined. Nevertheless, when they have been so determined, the symbols assigned by inspection and the application of the zone-laws to the different poles of the crystal, or calculated for them by aid of the provisional elements, will generally have validity; and ultimately, if a different axial system be adopted, the application of the formulæ for the transformation of all or of part of the elements of an axial system will readily convert the symbols of these faces to the new symbols representing them in the transformed system.

If however the elementary triangle and the arc-elements cannot all, or some of them, be thus directly determined, they may be deduced from a series of measurements between faces heterozonal to the elementary triangle, to which the symbols have been tentatively assigned consistently with the zone-laws. It is however a case of most improbable occurrence that no one of the zones α, β, γ, and at the same time no one of the poles A, B, C, can be identified with zones or faces determinable by the goniometer.

The simplest method to be followed in such case is that indicated in Articles **407–8**, the expressions in which are independent of the elements of the crystal.

Thus, where five of the six arcs joining the four poles U, V, W, T having known symbols are given by measurement, but none of these poles are identified with A, B or C, the arc AC between the two poles A, C having the known symbols 100, 001 can be calculated; and so on, for the remaining arc-elements. Usually however V, or U and V, can be taken to represent one or two of the poles A, B, C, and not unfrequently also one of the quasi-octahedral parametral forms can be represented by one of the remaining poles in the figure: and the process for determining the elements is proportionately simplified.

The Oblique Systems.

II. *The Mono-symmetric System.*

411. In this system, the normal of the plane of symmetry is taken for the axis Y. The axial points Z and X lie, as do the poles C, 001, and A, 100, of the axial planes XY and YZ, in the plane of symmetry ZX, while the pole B, or 010, of that plane lies on the axis Y; so that in trigonometrical problems recourse can frequently be had to quadrantal and right-angled triangles.

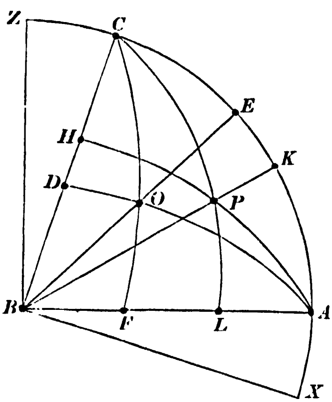

Fig. 361.

The normal-angle AC, i.e. (100.001), or β, is the supplement of the axial angle η, and $AZ = CX = 90°$.

To determine the position on the sphere of a pole P or hkl, we have from the quadrantal triangles PBZ and PBX (Fig. 361),

$$\cos PZ = \sin BP \cos PBZ = \sin BP \sin PBA,$$
$$\cos PX = \sin BP \cos PBX = \sin BP \sin (AC - PBA),$$

But $$\frac{a}{h}\cos PX = \frac{b}{k}\cos PY = \frac{c}{l}\cos PZ;$$

whence $$\frac{a}{h}\sin (PBA + \eta) = \frac{b}{k}\cot PB = \frac{c}{l}\sin PBA, \quad \ldots \ldots \ldots \text{(i)}$$

and $$\frac{\sin (PBA + \eta)}{\sin PBA} = \frac{c}{a}\frac{h}{l}.$$

Let O be the pole of 111 and let zone-circles through BO

and BP meet the zone-circle $[AC]$ in E and K respectively. Then E is 101, K is $h\,o\,l$, $OBA = AE$, and $PBA = AK$.

For O, $\quad \sin(\eta + AE) = \sin(180° - EC) = \sin EC$,
and for P, $\quad \sin(\eta + AK) = \sin(180° - KC) = \sin KC$,

and equation (i) becomes

$$\left.\begin{array}{ll} \text{for } O, & a\sin EC = b\cot OB = c\sin AE, \\ \text{for } P, & \frac{}{h}\sin KC = \frac{}{k}\cot PB = \frac{}{l} \qquad , \end{array}\right\} \quad \ldots\ldots\ldots\ldots\text{(ii)}$$

whence $\dfrac{h}{l} = \dfrac{\sin AE}{\sin EC}\,\dfrac{\sin KC}{\sin AK}$ and $\dfrac{k}{l} = \dfrac{\sin AE}{\sin AK}\,\dfrac{\tan BO}{\tan BP}$.

The arcs EC, AE, OB are parametral, and may be employed as the arc-elements of the crystal in place of the axial elements $\frac{a}{b}$, $\frac{c}{b}$, and η.

412. If

$$\tan\theta = \frac{c}{a}\frac{h}{l} = \frac{h}{l}\frac{\sin EC}{\sin AE} = \frac{\sin(PBA + \eta)}{\sin PBA} = \frac{\sin KC}{\sin AK},$$

then $\quad \tan(PBA + \tfrac{1}{2}\eta) = \tan\tfrac{1}{2}\eta\,\tan(135° - \theta)$,

and $\quad \cot PB = \dfrac{a}{b}\dfrac{k}{h}\sin PBC = \dfrac{c}{b}\dfrac{k}{l}\sin PBA, \quad \Big\} \ldots$ (iii)

$$\cot OB \quad .$$

From the triangles PBA, PBC,

$$\left.\begin{array}{l} \cos PA = \sin PB\cos PBA = \sin PB\cos AK, \\ \cos PC = \sin PB\cos PBC = \sin PB\cos KC, \\ \qquad = -\sin PB\cos(AK + \eta). \end{array}\right\} \ldots\ldots\text{(iv)}$$

413. The position of a pole K of a hemidome $\{h\,o\,l\}$ is given on the zone-circle [010] by equations (iii). PB is a quadrant since P coincides with K,

$$\tan(AK + \tfrac{1}{2}\eta) = \tan\tfrac{1}{2}\eta\,\tan(135° - \theta),$$

where $\qquad \tan\theta = \dfrac{h}{l}\dfrac{c}{a}$;

or, where AK is known, the symbol is given by the expression from (ii),

$$\frac{h}{l} = \frac{a}{c}\,\frac{\sin KC}{\sin AK} \cdot \quad \sin AE \sin KC$$

414. For a face L of the prism $\{hk0\}$, P in the foregoing formulæ becomes L and PBA vanishes, while

$$LBC = AC, \quad AL + LB = 90^\circ;$$

and
$$\cot LB = \frac{k}{h}\,\frac{a}{b}\sin AC = \frac{k}{h}\,\frac{\cot OB}{\sin FC}\sin AC,$$

$$\cos LC = \sin LB \cos AC.$$

Also, if F is the pole of the parametral face 110,

$$\frac{a}{b} = \frac{h}{k}\,\frac{\tan AL}{\sin AC} = \frac{\tan AF}{\sin AC},$$

and
$$\frac{h}{k} = \frac{a}{b}\,\frac{\sin AC}{\tan AL} = \frac{\tan AF}{\tan AL}.$$

415. For a face H of an ortho-dome $\{0kl\}$, $BH + HC = 90^\circ$,

$$\frac{\ }{l}\,\frac{\ }{b}\sin AC = \frac{\ }{l}$$

$$\cos HA = \sin HB \cos AC,$$

and
$$\frac{c}{b} = \frac{\tan DC}{\sin AC}, \text{ if } D \text{ is } 011;$$

also we have

$$c \ \sin.$$

and
$$\frac{l}{k} = \frac{\tan CD}{\tan}.$$

416. The poles of a form $\{hkl\}$ are symmetrical in pairs, where the signs of their k indices are the same, to the axis [010], and where the signs of their h and l indices are the same, to the systematic plane (010). Hence if the symbol of P be $h\,k\,l$, the angle between the faces

and the angle between the faces

$$hkl, h\bar{k}l =$$

The equations (iii) may be written

$$\frac{\tan BP}{} = \frac{h}{k}\,\frac{\sin EC}{\sin KC} : \quad l \sin AE \qquad \text{(v)}$$

where AK and KC are obtained as in Article **413**; the arcs PA and PC being given by equation (iv).

417. If P' or $h'k'l'$ be a pole in a zone passing through (010) and P or (hkl), we have by equating values for cot BP and cot BP', from the third equation in (iii),

from which, P being given in position, the position of P' can be found if its symbol is known, or its symbol can be found if its distance from B is given.

418. If K and K' be any two poles in the zone [010], we have, from Article **413**,

$$\frac{l}{h}\,\frac{\sin(AC-AK)}{\sin AK} = \frac{l'}{h'}\,\frac{\text{si}}{\sin AK'},$$

whence

$$\frac{l}{h}\cot AK - \frac{l'}{h'}\cot AK' = \frac{lh'-hl'}{hh'}\cot AC;$$

an expression which is often convenient in computing such angles when a table of natural cotangents can be employed.

For the parametral hemidome faces E, 101 and E', $\bar{1}01$ it becomes, if A' be $\bar{1}00$,

$$\cot AE - 2\cot AC = \cot E'A',$$

and for a plane K, $h0l$ of a single hemidome form

$$\cot AE - \frac{l}{h}\cot AK = \frac{h-l}{h}\cot AC.$$

419. *The elements of the crystal being given and the symbols of two poles, to find the arc joining these poles.*

$P\,(hkl)$ and $P'\,(h'k'l')$ being the two poles, their positions may be determined in reference to B and $[BA]$ by the equations (iii). Thus, PB, $P'B$, PBA, and $P'BA$ are known, and the triangle PBP' can be solved.

420. *To find the arc-elements of a mono-symmetric crystal.* Since the elements in this system involve three unknown magnitudes, three equations of the form of equation (ix), Art. **396**, would be required in the most general case; that, namely, where the poles, of which the arc-distances are given, are heterozonal and have no zero in their indices.

In fact however the problem always presents itself in a more simple form than this, since the preliminary investigation of the crystal in deciding the character of the symmetry will have always given measurements between poles which, by their symmetrical repetition, or by the position of one or more of them in the plane of symmetry, or, again, by the identification of one of them with the pole of that plane, lead to the direct application of the equations (ii) or (iii) (Arts. **411–12**). And where the application is not so simple it is frequently possible to obtain the arcs between poles in the zone [100, 001] by aid of the 'problem of four planes' or of the expressions in Article **396**; and where the arc-distance of a pole from the pole of symmetry 010 is not obtained by direct measurement (as for instance in the absence of this pole), it can frequently be deduced from a measured angle between the normals of two symmetrically repeated faces.

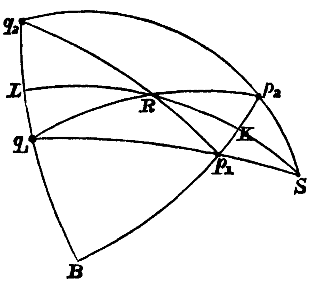

Fig. 362.

Let the distances of four poles p_1, p_2, q_1, q_2 be given; and let $p_1 q_2 = p_2 q_1$ and $p_1 q_1 = p_2 q_2$, Fig. 362,

Let $[p_1 q_1]\,[p_2 q_2]$ intersect in S,

$[p_1 q_2]\,[p_2 q_1]$ „ in R,

$[p_1 p_2]\,[q_1 q_2]$ „ in B,

and $[RS]$ intersect $[p_1 p_2]\,[q_1 q_2]$ in K and L.

Then $[KS]$ lies in a plane of symmetry to the crystal, and B is its pole and is (010).

Also $p_1 K = \frac{1}{2} p_1 p_2$ and

Project the poles on the zone-plane [*KS*]. *BK* and *BL* are quadrants (Fig. 363); hence Bp_1 and Bq_1 are known, and from the triangle p_1q_1B, the angles p_1Bq_1 (and therefore arc *KL*) and $Bp_1q_1 (= Kp_1S)$ may be computed. Hence also in the right-angled triangle Kp_1S, *KS* can be determined.

Let p_1 be 111 and q_1 be $\bar{1}11$. Then *K* is 101, *L* is $\bar{1}01$, *R* is 001, and *S* is 100; and from the four poles *S*, *K*, *R*, *L* known in symbol and known in distance as regards *SK* and *KL* the angle *KR* can be computed; whence the arc-elements *SK*, *KR*, and

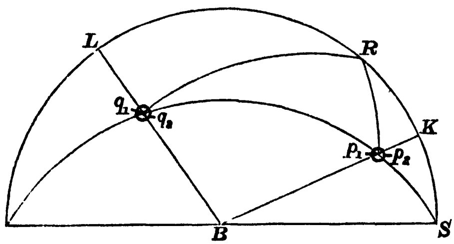

Fig. 363.

Bp_1 are found. If the relative positions of p_1 and q_1 had led to a different appropriation of symbols to those poles, the poles *S*, *K*, *R*, and *L* would also have different symbols, but the method of procedure would be the same.

For instance, if p_1 be assumed to be 111 and q_1 021, *L* becomes 001; *R*, 203; *K*, 101; *S*, 201; and the distance of a pole *A*, or 100, from *K* has to be found.

Section IV.—Crystallographic Calculation.

The Rectangular-axed Systems.

421. The Ortho-symmetric or Ortho-rhombic system represents the most general case of a crystallographic plane-system referred to rectangular axes; the case, namely, in which the parameters are all different, and the three axes are axes of ortho-symmetry.

The three systems which have been referred in this treatise to a rectangular axial system have the directions of their axes determined by considerations of physical as well as morphological symmetry, though in no system are the three directions so emphatically fixed by physical conditions as in the Ortho-rhombic system; since all the physical characters which distinguish a crystal in this system are ortho-symmetrical to the three perpendicular axial directions to which the forms of the crystal are referred, and to these alone.

The characters which are common to the different systems of planes referred to a rectangular axial system may be considered as *introductory to those which more particularly bear on the crystallography of the Ortho-symmetric system itself.*

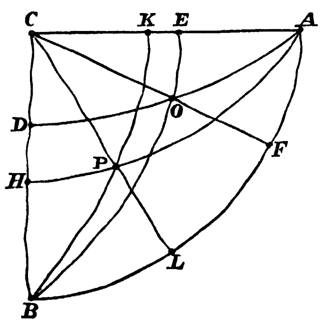

Fig. 364.

422. In a system in which the *faces of a crystal are referred to* three perpendicular axes the directions of the axes X, Y, Z coincide with those of the normals to the axial planes, and the axial points X, Y, Z fall on the poles A, B, C of the axial planes; which poles are therefore those of the faces belonging to the forms {100}, {010}, and {001}.

Hence the fundamental equation (A) becomes

a [illegible] b [illegible] c [illegible] . .. . (*a*)

From the triangles PCA, PCB, (Fig. 364)

$$\left.\begin{aligned}\cos PA &= \sin PC \cos PCA, \\ \cos PB &= \sin PC \sin PCA,\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (b)$$

and $\cos PC = \sin PA \cos PAC,$

ch

bl

whence, $\cot PC = \frac{[illegible]}{ch} \cos PCA = \frac{[illegible]}{ck} \sin PCA,$

$$\tan PCA = \frac{ak}{bh};$$

symmetry, though in no system are the three directions so emphatically fixed by physical conditions as in the Ortho-rhombic system; since all the physical characters which distinguish a crystal in this system are ortho-symmetrical to the three perpendicular axial directions to which the forms of the crystal are referred, and to these alone.

The characters which are common to the different systems of planes referred to a rectangular axial system may be considered as *introductory to those which more particularly bear on the crystallography of the Ortho-symmetric* system itself.

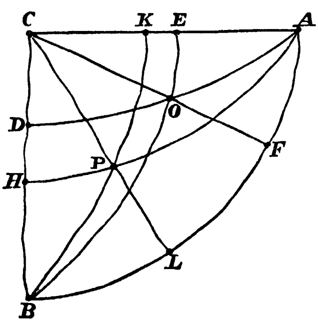

Fig. 364.

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Hence the fundamental equation (A) becomes

$$a \qquad b \qquad c \qquad \ldots\ldots\ldots\ldots \quad (a)$$

From the triangles PCA, PCB, (Fig. 364)

$$\left.\begin{aligned} \cos PA &= \sin PC \cos PCA, \\ \cos PB &= \sin PC \sin PCA, \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (b)$$

and $\qquad \cos PC = \sin PA \cos PAC,$

$$ch$$
$$bl$$

whence, $\qquad \cot PC = \frac{\ }{ch} \cos PCA = \frac{\ }{ck} \sin PCA,$

$$\tan PCA = \frac{ak}{bh};$$

and similarly, we obtain

$$\tan PAB = \frac{bl}{ck},$$

and

$$\tan PBC = \frac{ch}{al}.$$

Squaring equations (b) and adding,

$$\left.\begin{array}{l}\cos^2 PA + \cos^2 PB = \sin^2 PC = 1 - \cos^2 PC, \\ \text{and} \quad \cos^2 PA + \cos^2 PB + \cos^2 PC = 1.\end{array}\right\} \dots \dots \dots (d)$$

From (c), and substituting values from (b) and (a), we have

$$\left.\begin{array}{l}\cot PC = \dfrac{al}{ch} \dfrac{\cos PA}{\sqrt{\cos^2 PA + \cos^2 PB}}, \\ \qquad = \dfrac{ab}{c} \dfrac{l}{\sqrt{b^2 h^2 + a^2 k^2}};\end{array}\right\} \dots \dots \dots \dots \dots (e)$$

and similarly,

$$\cot PA = \frac{bc}{a} \frac{h}{\sqrt{c^2 k^2 + b^2 l^2}},$$

and

$$\cot PB = \frac{ca}{b} \frac{k}{\sqrt{a^2 l^2 + c^2 h^2}}$$

423. If H, K, L be the poles $0kl$, $h0l$, $hk0$ they will be the poles in which the zones AP, BP, CP intersect the arcs BC, CA, AB respectively; and O being the pole 111, and D, E, F being the poles 011, 101, 110 in which the zone-circles OA, OB, OC intersect the arcs BC, CA, AB, we have from (a)

$$\frac{\cos^2 PA}{\frac{h^2}{a^2}} = \frac{\cos^2 PB}{\frac{k^2}{b^2}} = \frac{\cos^2 PC}{\frac{l^2}{c^2}} = \frac{1}{\frac{h^2}{a^2} + \frac{k^2}{b^2} + \frac{l^2}{c^2}},$$

and

$$\frac{\cos PA}{\frac{h}{a}} = \frac{\cos PB}{\frac{k}{b}} = \frac{\cos PC}{\frac{l}{c}} \quad \frac{abc}{\sqrt{h^2 b^2 c^2 + k^2 c^2 a^2 + l^2 a^2 b^2}} = \frac{abc}{\sqrt{S}}.$$

where $S = h^2 b^2 c^2 + k^2 c^2 a^2 + l^2 a^2 b^2$;

whence

$$\cos AP = \sin PH = \frac{hbc}{\sqrt{S}},$$

$$\cos BP = \sin PK = \frac{kca}{\sqrt{S}}, \qquad (f)$$

$$\cos CP = \sin PL = \frac{lab}{\sqrt{S}}.$$

For the parametral plane O,

$$\left.\begin{aligned}\cos AO &= \sin OD = \frac{bc}{\sqrt{S}},\\ \cos BO &= \sin OE = \frac{ca}{\sqrt{S}},\\ \cos CO &= \sin OF = \frac{ab}{\sqrt{S}};\end{aligned}\right\} \qquad (f')$$

These equations may be put into the form

$$a:b:c = h\cos PB\cos PC : k\cos PC\cos PA : l\cos PA\cos PB, \quad (g)$$
$$= \cos OB\cos OC : \cos OC\cos OA : \cos OA\cos OB. \quad (g')$$

424. If P and P' be the poles of two faces hkl and $h'k'l'$, we have, from the spherical triangle PCP' formed by the arcs CP, CP', PP', and by substitution from (b) and from (f),

$$\begin{aligned}\cos PP' &= \cos PC\cos P'C + \sin PC\sin P'C\cos(P'CA - PCA)\\ &= \cos PC\cos P'C + \sin PC\cos PCA\sin P'C\cos P'CA\\ &\qquad + \sin PC\sin PCA\sin P'C\sin P'CA\\ &= \cos PA\cos P'A + \cos PB\cos P'B + \cos PC\cos P'C\\ &= \frac{hh'b^2c^2 + kk'c^2a^2 + ll'a^2b^2}{\sqrt{h^2b^2c^2 + k^2c^2a^2 + l^2a^2b^2}\sqrt{h'^2b^2c^2 + k'^2c^2a^2 + l'^2a^2b^2}}\\ &= \frac{\Sigma\frac{bc}{a}hh'}{\sqrt{\Sigma\frac{bc}{a}h^2}\sqrt{\Sigma\frac{bc}{a}h'^2}} \quad \ldots\ldots\ldots\ldots\ldots (h)\end{aligned}$$

an equation identical with that to which the general equation (ix), Article **396**, becomes reduced by the condition of the axes being perpendicular and the arcs α, β, γ, which also measure the angles ξ, η, ζ, being quadrants. The equation (h), though not directly adapted for logarithmic computation, involves only terms which are readily so computed.

The equations in the last article are evidently directly deducible from the above form (h) of the general equation by substituting in it the different values of the indices of P and P'. Evidently also the following expressions will hold good:—

If P' coincide with 111, i.e. with O, P being any pole hkl, the expression (h) becomes

$$\cos PO = \frac{\Sigma \frac{bc}{a} h}{\sqrt{\Sigma \frac{bc}{a} h^2}\sqrt{\Sigma \frac{bc}{a}}}; \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (h')$$

and if P' be one of the poles 011, 101, or 110, i.e. D, E, or F, we have

$$\cos PD = a\,\frac{lb^2 + kc^2}{\sqrt{S}\sqrt{b^2+c^2}},$$

$$\cos PE = b\,\frac{hc^2 + la^2}{\sqrt{S}\sqrt{c^2+a^2}},$$

$$\cos PF = c\,\frac{ka^2 + hb^2}{\sqrt{S}\sqrt{a^2+b^2}};$$

and if also P be one of the poles D, E, or F,

$$\cos EF = \frac{bc}{\sqrt{c^2+a^2}\sqrt{a^2+b^2}}$$

$$\cos FD = \frac{ca}{\sqrt{a^2+b^2}\sqrt{b^2+c^2}}$$

$$\cos DE = \frac{ab}{\sqrt{b^2+c^2}\sqrt{c^2+a^2}}.$$

425. Since, by equation (d), in any system referred to rectangular axes,

$$\cos^2 PA + \cos^2 PB + \cos^2 PC = 1,$$

$$\left.\begin{aligned} \cos^2 PA &= -\cos(PB+PC)\cos(PB-PC), \\ \cos^2 PB &= -\cos(PC+PA)\cos(PC-PA), \\ \cos^2 PC &= -\cos(PA+PB)\cos(PA-PB); \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (A)$$

and since

$$a : b : c = \cos OB \cos OC : \cos OC \cos OA : \cos OA \cos OB,$$

by substituting in any two of the last ratios,

for $\cos OA$, $\sqrt{-\cos(OB+OC)\cos(OB-OC)}$,

for $\cos OB$, $\sqrt{-\cos(OC+OA)\cos(OC-OA)}$,

for $\cos OC$, $\sqrt{-\cos(OA+OB)\cos(OA-OB)}$, we can obtain the parametral ratios for a crystal whenever two of the arcs OA, OB, OC have been determined.

Similarly, by substituting the square roots of the quantities in (A) in the expression

$$a:b:c = h \cos PB \cos PC : k \cos PC \cos PA : l \cos PA \cos PB,$$

we can find the parametral ratios from two measured arcs between P and A, B, or C, if the symbol of P be given.

And the parametral ratios can equally be found if a pole P or hkl be given in position in reference to one of the poles A, B, or C, and to an arc BC, CA or AB—for instance if PC and AL (L being hko) are given—since, by equations (b), the arc-distance of P from all the poles A, B, and C can then be determined.

So if two of the arcs PH, PK, PL be given, since they are the complements of the arcs PA, PB, PC, the parameters may be found.

The Rectangular-axed Systems.

I. *The Ortho-rhombic System.*

426. In considering the application of these general results to the particular problems arising in the investigation of an ortho-rhombic crystal, it will be seen that the equations (c) give the position of any pole in respect to the pole C and the arc CA, when the elements of the crystal and the symbol of P are known; and that by employing corresponding expressions the position of the pole may be determined in regard to A and $[AB]$, or to B and $[BC]$.

427. *Prism- and dome-forms.* For a pole H belonging to the proto-dome {okl} the angle PAB becomes HAB or HB: hence from equation (c) of Art. **422**

$$\tan BH = \frac{b}{c}\frac{l}{k}. \quad \ldots\ldots\ldots\ldots \quad \text{(i)}$$

Similarly, if K and L belong respectively to the deutero-dome {hol} and the prism {hko} we have

$$\tan CK = \frac{c}{a}\frac{h}{l}, \tan AL = \frac{a}{b}\frac{k}{h}. \quad \ldots\ldots\ldots \quad \text{(ii)}$$

428. O being as before a pole of the parametral scalene octahedron {111}, and D, E; F adjacent poles of the parametral domes and prism {011}, {101}, {110}, we have

$$\cos OF = \frac{\cos OA}{\cos FA} = \frac{\cos OB}{\sin FA}; \quad \ldots\ldots\ldots\ldots \quad \text{(iii } a)$$

hence

$$\frac{\cos OB}{\cos OA} = \tan FA = \frac{a}{b} \text{ from equations (ii).}$$

Similarly

$$\left.\begin{aligned} \frac{\cos OC}{\cos OB} &= \tan DB = \frac{b}{c}, \\ \text{and} \quad \frac{\cos OA}{\cos OC} &= \tan EC = \frac{c}{a}; \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad \text{(iii } b)$$

also

$$\tan FA \tan DB \tan EC = 1.$$

Any two of the arcs FA, DB, EC may thus be taken to determine the parameters and serve for the arc-elements of the crystal.

Further, from (i) and (ii),

$$\tan AL = \frac{k}{h} \tan FA, \quad \tan BH = \frac{l}{k} \tan DB, \quad \tan CK = \frac{h}{l} \tan EC. \quad \text{(iv)}$$

429. The parameters of the crystal being given, let P and P' be the poles of two faces $(h\,k\,l)$ and $(h'\,k'\,l')$.

If now the zone-circle $[PP']$ traverse one of the poles A, B, or C—and let it for example pass through C—then, from equation (c),

$$\frac{\cot CP}{\cot CP'} = \frac{lh'}{hl'} = \frac{lk'}{kl'},$$

and similar expressions hold where A or B lie in the zone $[PP']$. Whence

$$\left.\begin{aligned} &(1) \text{ for a zone } [APP'], \quad \frac{h' \tan AP'}{h \tan AP} = \frac{k'}{k} = \frac{l'}{l}, \\ &(2) \text{ for a zone } [BPP'], \quad \frac{k' \tan BP'}{k \tan BP} = \frac{l'}{l} = \frac{h'}{h}, \\ &(3) \text{ for a zone } [CPP'], \quad \frac{l' \tan CP'}{l \tan CP} = \frac{h'}{h} = \frac{k'}{k}. \end{aligned}\right\} \quad \text{(v)}$$

430. If, again, the symbol has been determined of a second pole P', representing a face $(h'k'l')$, where PP' is heterozonal to

A, B, and C, the arc PP' may be found by spherical trigonometry, after the poles P and P' have been severally determined in position relatively to C and CA as in the last article, by the solution of the right-angled triangles PCA and $P'CA$.

431. A measurement between faces belonging to the prism-zone, combined with one or more in a dome-zone, suffices for determining the parameters of an ortho-rhombic crystal. Or it may be that the angles between the faces of an octahedrid form offer the only or the best images on the goniometer, and an arc can generally be measured between the poles of two adjacent faces of such a form; as, for instance, between hkl and $\bar{h}kl$, where

$$\tfrac{1}{2}(hkl,\ \bar{h}kl) = PH = 90^\circ - AP.$$

When a second arc, such as

$$(hkl,\ hk\bar{l}) = 2\,PL, \text{ or } (hkl,\ h\bar{k}l) = 2\,PK,$$

can also be obtained, this second arc conjoined with the first gives the position of P; so that the parameters of the crystal can be determined.

The Rectangular-axed Systems.

II. *The Tetragonal System.*

432. In a tetragonal crystal the morphological axis is an axis of symmetry for all the physical characters of the crystal; whereas the axes of ortho-symmetry in which the trito-systematic zone-plane intersects the proto- and the deutero-systematic planes are only to be distinguished from the other possible zone-axes lying in that zone-plane by the fact that they are the normals of actual, not of abortive, planes of symmetry for the crystallographic forms. The symmetry of the crystal thus leads to the morphological axis being taken for one, namely, for the Z-axis, one of the other pairs of perpendicular axes of symmetry being taken for those of X and Y.

433. *The di-prism.* The fundamental equation (A) takes for a tetragonal crystal the form

$$\frac{a}{h}\cos PA = \frac{a}{k}\cos PB = \frac{c}{l}\cos PC; \quad \ldots\ldots\ldots\ldots (1)$$

which for the pole L of a form $\{h\,k\,o\}$ belonging to the zone [001] becomes

$$\frac{a}{h}\cos AL = \frac{a}{k}\sin AL, \quad \text{or} \quad \tan AL = \frac{k}{h}. \quad \ldots\ldots\ldots\ldots (2)$$

434. *The proto-prism* {100}, *and deutero-prism* {110}, are particular cases of the form $\{h\,k\,o\}$ in which the poles of the form lie on the proto- and deutero-systematic zone-circles respectively. In the one case $\tan AL = 0$; in the other $\tan AL = 1$, and $AL = \frac{\pi}{4}$.

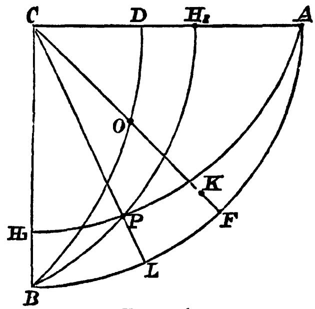


Fig. 365.

435. *The di-octahedron* $\{h\,k\,l\}$. If P (Fig. 365) be the pole of a face of a general scalenohedral form $\{h\,k\,l\}$, H_1 and H_2 the poles in which [AP], [BC] and [BP], [AC] respectively intersect, and L be the pole $h\,k\,o$ in which the zone-circles [001, $h\,k\,l$] and [001] intersect, then, following similar reasoning to that in the last section, or, directly from equation (c), (Art. **422**); since in a tetragonal crystal $a = b$, we have

$$\left.\begin{aligned} \cot AL &= \cot ACP = \frac{h}{k}, \\ \tan PL &= \cot CP = \frac{al}{ch}\cos ACP. \end{aligned}\right\} \quad \ldots\ldots\ldots (3)$$

Similarly, $$\tan PH_1 = \cot AP = \frac{h}{k}\cos BAP,$$

$$\tan PH_2 = \cot BP = \frac{ck}{al}\cos CBP.$$

Equations by means of which, if the parametral ratio be given and the symbol of a pole, the position of the pole can be determined, or its position being given its symbol can be computed.

436. *The proto-octahedron and deutero-octahedron.* H being a pole of the proto-octahedron $\{h\,o\,l\}$, and A the nearest pole to it of the form $\{100\}$ on the same proto-systematic zone-circle, since $\cos ACP = 1$, we have

$$\cot CH = \frac{a}{c}\frac{l}{h} = \tan AH; \quad \ldots \ldots \ldots \ldots \quad (4)$$

and if D be the pole of the form (101) with the same signs to its indices,

$$\left.\begin{aligned} \cot DC &= \frac{a}{c} = \tan AD, \\ \text{and} \qquad \tan AH &= \frac{l}{h}\tan AD. \end{aligned}\right\} \quad \ldots \ldots \ldots \ldots \quad (5)$$

Evidently then DC may be taken as the arc-element of a tetragonal crystal, since its tangent gives the value of the parametral ratio.

If K be a pole of the deutero-octahedron $h\,h\,l$, and F the nearest pole of the form $\{110\}$,

$$\cos KCA = \cos 45^\circ = \frac{1}{\sqrt{2}},$$

and

$$\cot CK = \frac{1}{\sqrt{2}}\frac{al}{ch} = \tan KF; \quad \ldots \ldots \ldots \ldots \quad (6)$$

and for O the pole of the form $\{111\}$ with the same signs to its indices as K,

$$\cot CO = \tan OF = \frac{1}{\sqrt{2}}\frac{a}{c};$$

whence

$$\tan AD = \frac{a}{c} = \sqrt{2}\tan OF.$$

From the expressions (5), the elements of the crystal being given, the position of a pole on a proto-systematic zone-circle can be determined if its symbol be known; or its symbol be found if its position on the zone-circle be known: and the equation (6) gives the same result in the case of a pole situate on a deutero-systematic zone-circle.

437. If P and P' be any two poles the symbols of which are $h\,k\,l$ and $h'k'l'$, equations (f), Article **423**, take the forms

$$\cos AP = \sin PH_1 = h\frac{c}{\sqrt{F}},$$

$$\cos BP = \sin PH_2 = k\frac{c}{\sqrt{F}},$$

$$\cos CP = \sin PL \;= l\frac{a}{\sqrt{F}};$$

where $$F = \frac{S}{a^2} = c^2(h^2+k^2)+a^2l^2.$$

And, if $$F' = \frac{S'}{a^2} = c^2(h'^2+k'^2)+a^2l'^2,$$

we have similar expressions for P'; namely,

$$\cos AP' = \sin P'H_1' = h'\frac{c}{\sqrt{F'}},$$
&c. &c. &c.

Whence also, as in equation (h), Article **424**, we obtain

$$\cos PP' = \frac{c^2(hh'+kk')+a^2ll'}{\sqrt{c^2(h^2+k^2)+a^2l^2}\sqrt{c^2(h'^2+k'^2)+a^2l'^2}}$$
$$= \frac{c^2(hh'+kk')+a^2ll'}{\sqrt{FF'}}; \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \quad (7)$$

a result directly deducible from the equation (ix), Article **396**, by making $a = b$ and α, β, γ quadrants.

438. By assigning special values to the indices in the symbols of P and P' the arc-distance of any two poles may be computed if the parametral ratio of the crystal be given.

And for any two poles mno and $m'n'$o in the equatorial zone [001], we have

$$\cos(mn\text{o},\ m'n'\text{o}) = \frac{mm'+nn'}{\sqrt{(m^2+n^2)(m'^2+n'^2)}}; \quad \ldots \ldots \ldots \ldots \ldots \ldots \quad (8)$$

an expression which is independent of the parametral element of the crystal; so that the normal-angles between faces perpendicular to the trito-systematic plane must be the same for faces with the same symbols in all crystals of the Tetragonal system, and depend solely on the indices in these symbols; and further, if mno and $n'm'$o are poles of the same form, in which the two symbols have their similar indices in the inverse order, but differ in the sign of one of them,

$$\cos(mn\text{o},\ n'm'\text{o}) = 0,$$

and the arc-distance is for every form $\frac{\pi}{2}$. The zone is in fact a zone of abortive symmetry.

439. If P be a pole hkl, Q a pole mno, and hko be L, CQP is a quadrantal triangle;

$$\begin{aligned}\cos PQ &= \sin CP \cos PCQ \\ &= \sin CP \cos LQ \\ &= \sin CP \frac{hm+kn}{\sqrt{h^2+k^2}\sqrt{m^2+n^2}}.\end{aligned}$$

440. If H and H' be two poles of forms $\{hol\}$ and $\{h'ol'\}$, K and K' two poles of forms $\{hhl\}$ and $\{h'h'l\}$,

$$\cos HH' = \frac{c^2hh'+a^2ll'}{\sqrt{c^2h^2+a^2l^2}\sqrt{c^2h'^2+a^2l'^2}}; \quad \cos CH = \frac{al}{\sqrt{c^2h^2+a^2l^2}},$$

$$\cos KK' = \frac{2c^2hh'+a^2ll'}{\sqrt{2c^2h^2+a^2l^2}\sqrt{2c^2h'^2+a^2l'^2}}; \quad \cos CK = \frac{al}{\sqrt{2c^2h^2+a^2l^2}}.$$

441. *To find the element of a tetragonal crystal.*

Let P and P' be poles of which the known symbols are hkl and $h'k'l'$ respectively, and let their arc-distance from each other be given (Fig. 366).

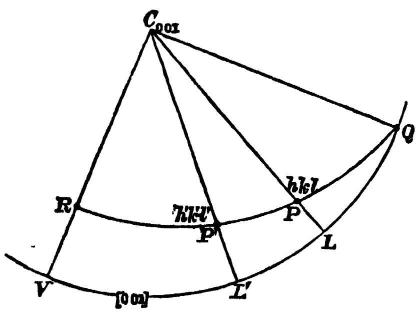


Fig. 366.

Let the zone-circle [001] intersect [CP] in L, i.e. in hko; [CP'] in L', or $h'k'o$; and PP' in Q, or mno: V a pole at a quadrant's distance from Q on [001] is (nmo), and R or (uvw) the pole in which [PP'] and [CV] intersect is a quadrant's distance from Q. The arc QP can therefore be found by the expression in Article **52**. Also,

$$\tan AL = \frac{k}{h}, \text{ and } \tan AQ = \frac{n}{m}, \text{ from (3); whence } LQ \text{ is known.}$$

Hence PL can be found from the right-angled triangle PLQ; thus CP the complement of PL is known.

Hence the position of P relatively to C and an arc CA (or CB) is known, and by equations (3),

$$\frac{c}{a} = \frac{l}{h} \tan PC \cos ACP.$$

The Rectangular-axed Systems.

III. *The Cubic System.*

442. The general expressions, obtained in Articles **422** and **423**, for the relations of the faces of a crystal referred to rectangular axes, assume their simplest forms when applied to the Cubic system, in which, the parameters being equal, all the elements of the crystal are fixed.

Thus the fundamental equations for the direction-cosines of a pole P with the symbol hkl become

$$\frac{\cos AP}{h} = \frac{\cos BP}{k} = \frac{\cos CP}{l},$$

A, B, C being as before 100, 010, 001; these poles now belong to a single form, the cube {100}. The equations (e) in Article **422** take the forms

$$\cot CP = \frac{l}{\sqrt{h^2+k^2}}$$

$$\tan ACP = \frac{k}{h}, \quad \tan BAP = \frac{l}{k}, \quad \tan CBP = \frac{h}{l};$$

and corresponding expressions give the position of P in respect to the other poles of the cube. So that the position of P being thus given, its indices can be determined; or, the indices being given, its position is known.

443. As in equations (f) of the general expressions for crystals referred to rectangular axes, we have

$$\cos AP = \sin PH = \frac{h}{\sqrt{S}},$$

$$\cos BP = \sin PK = \frac{k}{\sqrt{S}},$$

$$\cos CP = \sin PL = \frac{l}{\sqrt{S}};$$

where $S = h^2 + k^2 + l^2$ and H, K, L are poles of the form $\{hk0\}$, in which the zone-circles $[AP]$ and $[100]$, $[BP]$ and $[010]$, $[CP]$ and $[001]$ respectively intersect.

If O be a face of the octahedron $\{111\}$, and Ω be a face of that form symmetrical with O in regard to a proto-systematic plane and lying therefore in an adjacent octant, while O' is a face symmetrical with O in regard to a deutero-systematic plane, and lying therefore in an octant attingent to that containing O; since the pole of O is equidistant from three poles of the cube, we have

$$\cos AO = \sin OD = \frac{1}{\sqrt{3}},$$

D being a face of one of the three equidistant adjacent faces of the rhomb-dodecahedron.

Hence the normal-angles between adjacent faces of the octahedron and cube, and octahedron and dodecahedron, are

$$AO = 54^\circ\, 44{\cdot}14', \quad OD = 35^\circ\, 15{\cdot}86'.$$

And, as has been otherwise proved, article **125**, the normal-angle between the faces forming the edges of the octahedron is

$$O\Omega = 2\, OD = 70^\circ\, 31{\cdot}73',$$

while $$OO' = 2\, AO = 109^\circ\, 28{\cdot}27' = \Omega\Omega';$$

the latter being the normal-angle between the faces of the octahedron which meet oppositely at a quoin, and being equally the normal-angle of an edge of the tetrahedron $\sigma\{111\}$.

444. *To calculate the arc between two poles in the Cubic system.* If P be a pole of the form $\{hkl\}$, and P' a pole of the form $\{h'k'l'\}$, where $S = h^2 + k^2 + l^2$, and $S' = h'^2 + k'^2 + l'^2$; by substitution in the equation (h), Article **424**, we have

$$\cos PP' = \cos AP \cos AP' + \cos BP \cos BP' + \cos CP \cos CP'$$

$$= \frac{hh' + kk' + ll'}{\sqrt{SS'}},$$

an equation by which the normal-angle between any two faces in the Cubic system of which the symbols are given can be computed; the indices hkl, $h'k'l'$ being taken absolutely as regards sign and independently of relative magnitude.

In the following articles the values for the normal-angles

between faces of the different forms of the Cubic system will be obtained by application of this equation.

445. The proto-systematic zone-circles S contain the poles of the cube, the rhomb-dodecahedron, the tetrakis-hexahedron, and its hemi-systematic form the pentagon-dodecahedron. If H_1 and H_2 be two faces the poles of which lie on one of these zone-circles, we have for their normal angle

$$\cos H_1 H_2 = \frac{hh' + kk'}{\sqrt{h^2+k^2}\sqrt{h'^2+k'^2}}.$$

If H_1 and H_2 be faces of the same form and their poles lie on the same zone-circle S, they may be symmetrical to a second plane S and their symbols will only differ in the signs of one index. Then for adjacent faces,

$$\cos H_1 H_2 = \frac{h^2-k^2}{h^2+k^2};$$

and for faces that are not adjacent the angle is supplementary, or

$$\cos H_1 H_2 = \frac{k^2-h^2}{h^2+k^2}.$$

If the poles of H_1 and H_2 are situate as before on a zone-circle S but are symmetrical to a deutero-systematic plane their symbols differ only in the transposition of their indices, and

$$\cos H_1 H_2 = \frac{2hk}{h^2+k^2};$$

or their symbols also differ in the signs of both their indices, and

$$\cos H_1 H_2 = -\frac{2hk}{h^2+k^2}.$$

If the poles of $H_1\,H_2$ lying on a zone-circle S are not symmetrical in respect to any systematic plane, the indices of their symbols are transposed and differ in one sign, and $\cos H_1 H_2 = 0$; the two faces are perpendicular and belong in fact to a zone of abortive symmetry, a condition shewn to be true of every zone of the Cubic system.

For adjacent faces of the tetrakis-hexahedron, cube, and rhomb-dodecahedron, we have

$$\cos AH = \frac{h}{\sqrt{h^2+k^2}}, \qquad \cos AH = \cos(45^\circ - DH);$$

$$\cos AD = \frac{1}{\sqrt{2}}, \qquad \cos AB = 0.$$

Since $$\cos^2 AH = \frac{h^2}{h^2+k^2},$$

we have $$\frac{k^2}{h^2} = \frac{1-\cos^2 AH}{\cos^2 AH}, \text{ or } \frac{k}{h} = \tan AH;$$

whence the normal-angle AH being given the symbol of H is known.

446. The deutero-systematic zone-circles Σ contain the poles of the cube, icositetrahedron, octahedron, triakis-octahedron, and rhomb-dodecahedron.

If R and R' be faces the poles of which lie on a deutero-systematic zone-circle, and have for their symbols mmn and $m'm'n'$, we have for their normal-angle

$$\cos RR' = \frac{2mm'+nn'}{\sqrt{2m^2+n^2}\sqrt{2m'^2+n'^2}}.$$

If R and R' belong to the same form, the poles $\overline{R}$ and $\overline{R'}$ lie on the same zone-circle Σ. If R and R' are symmetrical to a proto-systematic plane,

$$\cos RR' = \frac{2m^2-n^2}{2m^2+n^2}.$$

Also we have $$\cos CR = \frac{n}{\sqrt{2m^2+n^2}} = \sin RD;$$

whence $$\cos OR = \frac{2m+n}{\sqrt{3}\sqrt{2m^2+n^2}}.$$

Also $$\frac{m}{n} = \frac{\tan CR}{\sqrt{2}}.$$

447. *The cube, octahedron, and rhomb-dodecahedron.* Of these three fundamental figures of the Cubic system it has been already shewn, with regard to the cube and octahedron, that their edges are truncated by the faces of the rhomb-dodecahedron, and that the quoins of the one form are truncated by the faces of the other; and further, that each face of the cube is perpendicular to a zone in which lie four faces of the cube and four faces of the dodecahedron: each face is also inclined at a normal-angle of $54° 44{\cdot}14'$ on four faces of the octahedron, and of $125° 15{\cdot}86'$ on the remaining four faces of that form.

Each face of the *octahedron* is perpendicular to a zone in which six faces of the rhomb-dodecahedron lie, and is inclined at $35^\circ 15{\cdot}86'$ on three faces of that figure symmetrically grouped round it, and $144^\circ 44{\cdot}14'$ on the faces opposite to these.

The normal-angle of an edge of the *rhomb-dodecahedron* is 60°, since $\cos(110, 101) = \frac{1}{2}$; and that of the faces which meet oppositely at a tetragonal quoin is 90°. The edges of this form are truncated by those of the icosi-tetrahedron $\{211\}$. They are bevilled by the faces of a hexakis-octahedron of the series (Art. **60**) in which the symbols are of the form $\{\mu+\lambda, \mu, \lambda\}$.

The following table gives the normal-angles between adjacent faces in the zone $[110, 101]$ of certain of these forms:

$\{321\}$, $321, 312 = 21^\circ 47' 12''$; $321, 23\bar{1} = 38^\circ 12' 48''$;

$\{431\}$, $431, 413 = 32^\circ 12' 15''$; $431, 34\bar{1} = 27^\circ 47' 45''$;

$\{532\}$, $532, 523 = 13^\circ 10' 25''$; $532, 35\bar{2} = 46^\circ 49' 35''$;

$\{541\}$, $541, 514 = 38^\circ 12' 48''$; $541, 45\bar{1} = 21^\circ 47' 12''$;

$\{743\}$, $743, 734 = 9^\circ 25' 48''$; $743, 47\bar{3} = 50^\circ 34' 12''$;

$\{752\}$, $752, 725 = 27^\circ 47' 45''$; $752, 57\bar{2} = 32^\circ 12' 15''$.

448. Of the three fundamental forms of the Cubic system the octahedron is the only one which can undergo a hemi-symmetrical suppression in the number of its faces, the tetrahedron $\sigma\{111\}$ being its haplohedral semiform; the normal-angle of each of the six edges D of the tetrahedron is $109^\circ 28{\cdot}27'$.

449. *The tetrakis-hexahedron.* The edges of the cube are bevilled by the faces of the *cube-pyramidion* or *tetrakis-hexahedron* $\{hk0\}$. This figure has two kinds of edges, each face of the form being an isosceles triangle, the edge H, which forms the base of the triangle, being parallel to one of the crystallographic axes, but lying, like each of the remaining edges D, in a deutero-systematic plane.

The following angles for different varieties of the form follow from the equations

$$\cos H = \frac{2hk}{h^2+k^2}; \qquad \cos D = \frac{h^2}{h^2+k^2}:—$$

For the form $\{210\}$, $H = 36^\circ 52' 12'' = D$;

$\{530\}$, $H = 28^\circ\ 4' 21''$, $D = 42^\circ 40'\ 5''$;
$\{320\}$, $H = 22^\circ 37' 12''$, $D = 46^\circ 11' 13''$;
$\{520\}$, $H = 46^\circ 23' 50''$, $D = 30^\circ 27'\ 1''$;
$\{310\}$, $H = 53^\circ\ 7' 48''$, $D = 25^\circ 50' 31''$;
$\{410\}$, $H = 61^\circ 55' 39''$, $D = 19^\circ 45'\ 0''$;
$\{720\}$, $H = 58^\circ\ 6' 33''$, $D = 22^\circ 24' 10''$.

The pentagon-dodecahedron. The only hemi-symmetrical form of the tetrakis-hexahedron exhibiting a defalcation in the number of the faces of that figure is the *pentagon-dodecahedron* $\pi\{hk0\}$ or *pyritohedron* (Article **183**). The symmetrical pentagons which form its faces are euthy-symmetrical to a proto-systematic plane and have one edge O dissimilar to the other four: it lies in a proto-systematic plane perpendicular to the former plane. The remaining edges lie in no systematic plane.

450. *The triakis-octahedron.* The faces of the *triakis-octahedron* or *octahedrid pyramidion* $\{hhk\}$ bevil the edges of the octahedron: the edges O, which are parallel to those of the octahedron, lie each in a proto-systematic plane S; the edges D, which form the isosceles sides of the faces of the form, lie in the Σ-planes: Article **174**.

The equations

$$\cos O = \frac{2h^2 - k^2}{2h^2 + k^2}, \quad \cos D = \frac{h^2 + 2hk}{2h^2 + k^2}$$

give the following normal-angles for the two kinds of edges in some of the more usual varieties of the form:

$\{332\}$, $O = 50^\circ 28' 44''$, $D = 17^\circ 20' 29''$;
$\{221\}$, $O = 38^\circ 56' 33''$, $D = 27^\circ 15' 58''$;
$\{331\}$, $O = 26^\circ 31' 31''$, $D = 37^\circ 51' 49''$;
$\{441\}$, $O = 20^\circ\ 3'\ 0''$, $D = 43^\circ 20' 30''$.

The *hemi-triakisoctahedron* or *twelve-deltohedron* $\sigma\{hhk\}$, Article **179**, is the only form hemi-symmetrically derived from the triakis-octahedron by the extinction of half its faces. It has two sets of edges, one identical with the edges D of the triakis-octahedron, and the other denoted by Δ, the normal-angle of which is determined by the equation

$$\cos \Delta = \frac{h^2 - 2hk}{2h^2 + k^2}.$$

The following are the normal-angles between the faces forming the edges Δ for different values of h and k:

$$\sigma\{332\},\ \Delta = 97^\circ 50' 15'';$$
$$\sigma\{221\},\ \Delta = 90^\circ;$$
$$\sigma\{331\},\ \Delta = 80^\circ 54' 55'';$$
$$\sigma\{441\},\ \Delta = 75^\circ 58' 13''.$$

In consequence of the edges Δ of the form σ{221} being right-angles, it will be seen that the faces of that form are those of four cube-quoins symmetrically grouped in respect to the axes of symmetry of the system. Their poles are at a normal-distance of 15° 48′ from those of the adjacent faces of the octahedron, at 70° 31′ from the more remote of the three nearest poles of the cube; furthermore, each pole of the dodecahedron is 45° from four poles, and each pole of the form {411} is 45° from two poles of the form {221}.

451. *The icositetrahedron.* The faces of the *icositetrahedron* {*hkk*} replace symmetrically the quoins of the octahedron and of the cube (Art. **173**).

Of the two kinds of edges *O* and *H* which form the sides of the deltoid faces of a form {*hkk*}, the normal-angle of the faces meeting in an edge *O* is given by the equation

$$\cos O = \frac{h^2}{h^2 + 2k^2},$$

the two faces being symmetrical to a plane *S*, that of a pair of faces symmetrical to a plane Σ and meeting in an edge *H* is obtained from the expression

$$\cos H = \frac{2hk + k^2}{h^2 + 2k^2};$$

the values of these angles for some of the more common varieties of the icositetrahedron are,

$$\{322\},\ O = 58^\circ 2' 3'',\quad H = 19^\circ 45';$$
$$\{211\},\ O = 48^\circ 11' 23'',\ H = 33^\circ 33' 26'';$$
$$\{311\},\ O = 35^\circ 5' 48'',\quad H = 50^\circ 28' 44;$$

$\{411\}$, $O = 27° 15' 58''$, $H = 60°$;
$\{511\}$, $O = 22° 11' 30''$, $H = 65° 57' 29''$;
$\{611\}$, $O = 18° 40' 18''$, $H = 69° 59' 41''$.

The only hemi-symmetrical form of the icositetrahedron presenting a defalcation in the number of faces belonging to that figure is the *hemi-icositetrahedron* or *tetrahedrid-pyramidion* $\sigma \{h k k\}$ (Art. **180**). Its faces are isosceles triangles, and the edges H forming the similar sides of each face are identical with the edges H of the holo-symmetrical figure.

The normal-angle of the remaining edge, D, of each face may be computed by the formula

$$\cos D = \frac{h^2 - 2k^2}{h^2 + 2k^2}.$$

Thus for the forms, $\sigma \{322\}$, $D = 86° 37' 40''$,
$\sigma \{211\}$, $D = 70° 31' 44''$,
$\sigma \{311\}$, $D = 50° 28' 44''$,
$\sigma \{411\}$, $D = 38° 56' 33''$,
$\sigma \{511\}$, $D = 31° 35' 11''$,
$\sigma \{611\}$, $D = 26° 31' 31''$.

452. *The forty-eight scalenohedron* or *hexakis-octahedron*, $\{h k l\}$ (Art. **175**). Each face of this form is a triangle, scalene in the sense that its three edges are crystallographically dissimilar, there being twenty-four edges of each different kind. If O represent any edge formed by two adjacent faces of which the symbols differ in a sign, D an edge in which the highest index retains the same position in the symbols for the two adjacent faces, and H an edge in which the lowest index remains unchanged in its position in the two symbols; the normal-angles of the three kinds of edges are given by the equations

$$\cos O = \frac{h^2 + k^2 - l^2}{h^2 + k^2 + l^2}, \quad \cos D = \frac{h^2 + 2kl}{h^2 + k^2 + l^2}, \quad \cos H = \frac{2hk + l^2}{h^2 + k^2 + l^2};$$

and the following are the values for the edges of some of the more frequently occurring of these forms :

$\{321\}$; $O = 106° 36' 6''$, $D = 21° 47' 12''$, $H = 21° 47' 12''$;
$\{421\}$; $O = 25° 12' 31''$, $D = 17° 45' 10''$, $H = 35° 57' 2''$;
$\{531\}$; $O = 19° 27' 47''$, $D = 27° 39' 38''$, $H = 27° 39' 38''$
$\{543\}$; $O = 50° 12' 29''$, $D = 11° 28' 42''$, $H = 11° 28' 42''$

The distances of a pole of the hexakis-octahedron from the nearest, the next, and the most remote of the three least distant poles of the cube are given severally by the equations

$$\cos PA_1 = \frac{h}{\sqrt{S}}, \quad \cos PA_2 = \frac{k}{\sqrt{S}}, \quad \cos PA_3 = \frac{l}{\sqrt{S}}.$$

453. Special varieties of the forty-eight scalenohedron have been already considered, Article **175.** In one series of these the poles lie in zones with those of the rhomb-dodecahedron, the indices being determined by the equation $h = k+l$: the twenty-four edges D of such a form are parallel in groups of six to the four trigonal axes O of the system. In the other series the poles lie on zone-circles bisecting the angles between the deutero-systematic zone-circles Σ, and the forms correspond to those which in a trigonal system have been designated by the symbol $\{min\}$, where $2i = m+n$; and in a form of this kind normal-angles of the edges H and D have the same values though the edges are not of equal length nor otherwise similar.

Forms of the first series would be $\{321\}$, $\{532\}$, $\{431\}$, &c., and of the second $\{321\}$, $\{531\}$, $\{432\}$, &c.

454. It has been seen that, from the nature of the symmetry of the Cubic system, it is only the general scalenohedron that is capable of undergoing a hemi-symmetrical partition of its faces in accordance with both of the laws designated by the symbols $\sigma\{hkl\}$ and $\pi\{hkl\}$. The former of these symbols is that of the hexakis-tetrahedron, the latter is that of the trapezoid dodecahedron or diplohedron.

The *hexakis-tetrahedron* $\sigma\{hkl\}$, Article **181,** has three sets of edges, the normal-angles of which are determined by the equations

$$\cos D = \frac{h^2+2kl}{S}, \quad \cos H = \frac{2hk+l^2}{S}, \quad \cos \Delta = \frac{h^2-2kl}{S},$$

where the angles D and H are those belonging to the respective forms of the hexakis-octahedron.

The following are the normal-angles for the edges Δ of various hexakis-tetrahedra, Δ being the edge in which the signs

of two of the indices of the faces forming the edge are different:

$$\sigma\{321\};\quad \Delta = 158°\,12'\,47'';$$
$$\sigma\{421\};\quad \Delta = 55°\,9'\,0'';$$
$$\sigma\{531\};\quad \Delta = 57°\,7'\,18'';$$
$$\sigma\{543\};\quad \Delta = 88°\,51'\,14''.$$

The diplohedron, $\pi\{hkl\}$: Article **184.** The edges which form the four sides of the trapezoidal faces of this figure are of three kinds, representing three different normal-angles. The edges G which meet in a trigonal quoin do not lie in the Σ-planes, and the normal-angle of the faces meeting in each such edge G is given by the equation

$$\cos G = \frac{hk + kl + lh}{S}.$$

The two other edges O and Ω of each face meet in a tetragonal quoin, and lie in proto-systematic planes perpendicular to each other. They are formed by faces the symbols of which in each case differ in one of their signs; and their normal-angles may be computed from the equations

$$\cos\Omega = \frac{h^2 + k^2 - l^2}{S},\quad \cos O = \frac{h^2 - k^2 + l^2}{S};$$

where Ω is the edge representing the longer and O that representing the shorter side of the trapezoid.

The following are the normal-angles for G, Ω, and O for various diplohedra:

$$\pi\{321\};\quad G = 31°\,0'\,10'',\quad \Omega = 106°\,36'\,6'',\quad O = 64°\,37'\,23'';$$
$$\pi\{421\};\quad G = 48°\,11'\,23'',\quad \Omega = 25°\,12'\,31'',\quad O = 51°\,45'\,12'';$$
$$\pi\{531\};\quad G = 48°\,55'\,4'',\quad \Omega = 19°\,27'\,47'',\quad O = 60°\,56'\,27'';$$
$$\pi\{543\};\quad G = 19°\,56'\,54'',\quad \Omega = 50°\,12'\,29'',\quad O = 68°\,53'\,59''.$$

The pentagonal icositetrahedon, $a\{hkl\}$: Article **177.** Of the forms in the Cubic system the general scalenohedron is the only one which can undergo a hemi-symmetrical suppression of its faces in accordance with the law denoted by the symbol a; the resulting semiform being the pentagonal icositetrahedron $a\{hkl\}$.

The semiform $a\{hkl\}$ has three sets of edges G, V, W (see

Figs. 83 and 84, Art. **177**), the normal-angles of which are determined by the equations

$$\cos G = \frac{hk+kl+lh}{h^2+k^2+l^2}, \quad \cos V = \frac{h^2}{h^2+k^2+l^2}, \quad \cos W = \frac{2hk-l^2}{h^2+k^2+l^2}.$$

Thus for $a\{785\}$;

$$G = 18^\circ\, 19'\, 39'', \quad V = 62^\circ\, 22'\, 10'', \quad W = 50^\circ\, 55'\, 4'';$$

for $a\{986\}$;

$$G = 15^\circ\, 59'\, 12'', \quad V = 63^\circ\, 24'\, 57'', \quad W = 53^\circ\, 22'\, 2''.$$

Four edges V meet in each tetragonal quoin; three edges G in each trigonal quoin; and each edge W would be truncated by a face of the form $\{110\}$.

The Hexagonal System.

455. As referred to three axes lying in the proto-systematic planes S symmetrically with regard to the zone-axis of these

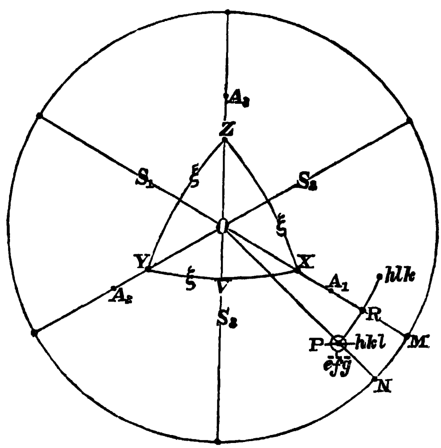


Fig. 367.

planes, a crystal of the Hexagonal system has for its parametral plane a face of the pinakoid form $\{111\}$ which is parallel to the equatorial systematic plane C: this last plane being also the plane of the zone-circle $[111]$.

The only variable element of the system is the axial angle ξ.

If X, Y, Z (Fig. 367) be the axial points, A_1, A_2, A_3 the poles of the axial planes YZ, ZX, and XY severally; then A_1 is 100, A_2 is 010, A_3 is 001, and the arcs A_1Y, A_1Z, A_2X, A_2Z, A_3Y, A_3X are quadrants; and the triangles $A_1A_2A_3$ and XYZ are polar, and by the symmetry of the construction A_1 and X, A_2 and Y, A_3 and Z lie on the same side of (111), on quadrants of the great circles S which meet in that pole.

Further, $$OX = OY = OZ,$$
and $$OA_1 = OA_2 = OA_3,$$
$$A_1A_2 = A_2A_3 = A_3A_1,$$
$$XOY = YOZ = ZOX = 120°.$$

Since the axes X and Y are symmetrically situate in respect to a great circle S_3, the arc XY is bisected, say in V, by that great circle; and in the triangle XOV,
$$\sin VX = \sin OX \sin VOX,$$
or $$\sin \tfrac{1}{2}\xi = \sin OX \sin 60°,$$
$$\sin OX = \sin OY = \sin OZ = \frac{2}{\sqrt{3}} \sin \tfrac{1}{2}\xi. \quad \text{(i)}$$

456. Let P be the pole hkl of a discalenohedron (Fig. 367). Then the pole symmetrical with it in regard to S_1 is hlk, and the edge of the faces hkl and hlk would, Article **133**, be truncated by a face R with the symbol $(2h, k+l, k+l)$, which is of the form $h'k'k'$. The faces hkl, $\bar{e}\bar{f}\bar{g}$ which are symmetrical to the equatorial plane C and lie in the zone $[111, hkl]$ are truncated by a face N for which the symbol is obtained by the addition of the symbols $(3h, 3k, 3l)$ and $\bar{e}\bar{f}\bar{g}$; that is to say, the symbol is
$$(2h-k-l,\ 2k-l-h,\ 2l-h-k),$$
or (pqr), where $p+q+r = 0$, and the face thus belongs to the equatorial zone of the crystal.

457. Let R be the pole hkk of a rhombohedron (Fig. 367). Since the parameters are equal,
$$\frac{\cos RX}{h} = \frac{\cos RY}{k} = \frac{\cos RZ}{l} \text{ and } \cos RY = \cos RZ, \quad \text{(ii)}$$
and $\cos RX = \cos(OR - OX) = \cos OR \cos OX + \sin OR \sin OX$,
also $\cos RY = \cos OR \cos OY + \sin OR \sin OY \cos 120°$,
$$= \cos OR \cos OX - \tfrac{1}{2} \sin OR \sin OX;$$

whence by substitution in (ii),

$$\left(\frac{h}{2}+k\right)\sin OR \sin OX = (h-k)\cos OR \cos OX,$$

and $$\tan OR = 2\,\frac{h-k}{h+2k}\cot OX \;\ldots\ldots\ldots\ldots\ldots\ldots \text{(iii)}$$

Hence, for a pole A_1 of the form $\{100\}$ on the hemisphere which is positive relatively to the zone $[111]$,

$$\tan OA = 2\cot OX\,; \;\ldots\ldots\ldots\ldots\ldots\ldots \text{(iv)}$$

and if R is a pole of a form $\{hkk\}$ lying also on the positive side of $[111]$ and on the same great circle with A_1,

$$\tan OR = \frac{h-k}{h+2k}\tan OA\,; \;\ldots\ldots\ldots\ldots\ldots\ldots \text{(v)}$$

OR and OA having the same or opposite signs according as R and A are on the same or opposite sides of O; i.e. according as the rhombohedron $\{hkk\}$ is direct or inverse.

And since, from equation (i), sin OX is known when ξ is given, it is seen from equation (iv) that tan OA determines ξ: and OA may therefore serve as the *arc-element* of a crystal in this system.

458. Every pole of a form $\{hkk\}$ is at a quadrant's distance from two opposite poles of the form $\{01\bar{1}\}$ (Fig. 183); so that khk and kkh being poles of the form $\{hkk\}$ on the same side of $[111]$, the great circles $[khk, kkh]$ and $[111]$ intersect in the poles $01\bar{1}$ and $0\bar{1}1$, and from either of these poles the pole hkk is distant a quadrant; the arc-distance of khk and kkh from them is either $90^\circ+\frac{1}{2}\lambda$ or $90^\circ-\frac{1}{2}\lambda$, where λ is the arc joining any two of the poles hkk, khk, kkh.

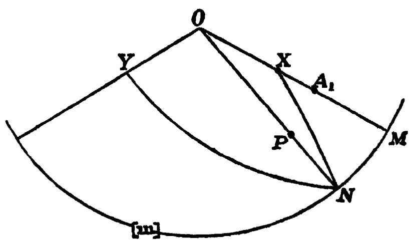


Fig. 368.

459. R being one of the above poles of the form $\{hkk\}$ and T a pole of the form truncating by its faces the edges of $\{hkk\}$, the symbol of the truncating form is $\{2k, h+k, h+k\}$, and we have evidently, equating values of tan OA given by (v),

$$\tan OR = -2\tan OT.$$

The zone-circles $[111]$ and $[111, 100]$ or $[S_1]$ intersect in M or $2\bar{1}\bar{1}$, a pole of the proto-hexagonal prism, and $[\Sigma_1]$ or $[2\bar{1}\bar{1}]$, the zone-circle perpendicular to $[S_1]$, intersects $[111]$ in $01\bar{1}$, a pole of the deutero-hexagonal prism.

For N, a pole (pqr) of the dihexagonal prism lying on the zone-circle $[111]$ (Fig. 368),

$$\frac{\cos NX}{p} = \frac{\cos NY}{q}.$$

From the triangles XON, YON, ON being a quadrant,

$$\cos NX = \sin OX \cos XON,$$

$$\cos NY = \sin OY \cos YON = \sin OX \cos(120° - A_1ON),$$

and since A_1ON is MN we have $\frac{q}{p} = \frac{1}{2}(\sqrt{3}\tan MN - 1)$;

but since $p + q + r = 0$,

$$\tan MN = \tan A_1 ON = \frac{1}{\sqrt{3}}\frac{q-r}{p}. \quad \text{.. .. ..} \quad \text{(vi } a\text{)}$$

Now the zone-circle $[111]$ is an isogonal zone with abortive symmetry (Article **99**), and every face in it has for its pole the pole of a possible zone, the indices in the symbols for the zone and for the face being identical: hence every face in the zone $[111]$ is perpendicular to another possible face of the zone.

If V or $p'q'r'$ be the face at a quadrant's distance from N,

$$\tan MV = \tan(MN + 90°) = -\cot MN,$$

$$\text{and} \quad \frac{1}{\sqrt{3}}\frac{q'-r'}{p'} = \sqrt{3}\frac{p}{r-q}; \quad \text{.. .. ..} \quad \text{(vi } b\text{)}$$

$$\text{whence} \quad \frac{p'}{q'} = \frac{q-r}{q+2r} \text{ and } \frac{p'}{r'} = \frac{q-r}{-(2q+r)}. \quad \text{.. .. ..} \quad \text{(vii)}$$

If now N lie in the zone $[111, hkl]$, its symbol (pqr) is

$(2h-k-l,\ 2k-l-h,\ 2l-h-k)$, and $\frac{q-r}{p} = \frac{3(k-l)}{2h-k-l}$;

so that, P being $h\,k\,l$,

$$\tan A_1OP = \tan MN = \sqrt{3}\frac{k-l}{2h-k-l}. \quad \text{.. .. ..} \quad \text{(viii)}$$

If R be a pole $h'k'k'$ of the dirhombohedron the faces of which truncate the alternate edges hkl, hlk, &c. of the discalenohedron

$\{hkl\}$, the angle at R in the triangle POR is a right angle, and

$$\tan OR = \tan OP \cos ROP$$

$$= 2\,\frac{h'-k'}{h'+2k'} \cot OX \text{ by (v)};$$

and substituting values of hkl for $h'k'k'$ (Article **456**),

$$\tan OX \tan OP \cos MOP = \frac{2h-k-l}{h+k+l},$$

$$\text{or } \tan OP = \frac{1}{2}\,\frac{\tan OA_1}{\cos A_1OP}\,\frac{2h-k-l}{h+k+l}, \qquad \text{(ix)}$$

which may be written

$$\tan OA_1 = 2\,\frac{h+k+l}{2h-k-l}\,\frac{\cos A_1OP}{\cot OP}.$$

460. The position of a pole P or hkl is determined by the arc OP and the angle MOP, i.e. the arc MN, where N is

$$(2h-k-l,\ 2k-l-h,\ 2l-h-k).$$

And if the position of P is known, its symbol can be determined from equations (viii) and (ix). For since the terms of the expression $2h-k-l$ may be represented by any numbers, whole or fractional, in the ratios of $2h$, k, and l, we may take

$$2h-k-l = 1; \quad \ldots\ldots\ldots \quad \text{(x)}$$

then from (viii), $$k-l = \frac{\tan A_1OP}{\sqrt{3}};$$

and from (ix), $$2h+2k+2l = \frac{\tan OA_1}{\tan OP \cos A_1OP}:$$

hence $$3(k+l) = \frac{\tan OA_1}{\tan OP \cos A_1OP} - 1,$$

while $$3(k-l) = \sqrt{3} \tan A_1OP; \quad \ldots\ldots \quad \text{(xi)}$$

two equations from which and from equation (x) the symbol of P is readily obtained.

If the symbols were those of two poles of the discalenohedron belonging to the semiform $\{efg\}$ correlative with $\{hkl\}$, for instance, the poles gfe and gef symmetrical on S_1, expressions identical in form lead to similar results.

461. For a pole K lying on a deutero-systematic zone-circle Σ, and having a symbol $m i n$; since $A_1OK = \frac{\pi}{6}$ and, by Article **121**, $\frac{m+n}{n} = i$, we have, for OK where the symbol is given,

(xii)

and for the ratios of the indices of the symbols, where is known,

$$\frac{m}{n} = \frac{\tan OA + \sqrt{3}\tan OK}{\tan OA - \sqrt{3}\tan OK}, \quad \frac{i}{n} = \frac{\tan OA}{\tan OA - \sqrt{3}\tan OK}. \quad \text{(xiii)}$$

If $\tan\phi = \sqrt{3}\tan OK$, we have

$$\frac{m}{n} = \frac{\sin(OA+\phi)}{\sin(OA-\phi)}, \quad \ldots\ldots\ldots\ldots \quad \text{(xiv)}$$

a form adapted to logarithmic computation.

462. To determine the arc joining two poles P and P' not lying on a systematic zone-circle, where the symbols hkl and $h'k'l'$ of P and P', respectively, are given; the angles AOP and AOP', and therefore POP', are to be found from equation (viii), and the arcs OP and OP' being found from equation (ix), the third side of the triangle POP', that is to say, the arc PP', is directly found by spherical trigonometry.

Or we may proceed as follows:—

Let $[\mathrm{u\,v\,w}] = [hkl, h'k'l']$, and the zone-circle PP' or $[\mathrm{u\,v\,w}]$ meet the zone-circle $[111]$ in a pole Q or efg; then

$$e = \mathrm{v} - \mathrm{w},\ f = \mathrm{w} - \mathrm{u},\ g = \mathrm{u} - \mathrm{v};$$

and it may be shown that if $[OP]$ meet $[111]$ in N,

Similarly, $\tan QN' = -$

Then, PP' is known, for

$\cos QP = \cos QN \sin OP$, and $\cos QP' = \cos QN' \sin OP'$.

463. *To find the arc-element of a hexagonal crystal.*

Where the pole of any face of the crystal has been given in position with respect to the pole (111) and the zone-circle [111, 100], or $[OA_1]$, or the zone-circle passing through O and the nearest pole of the form {100}, the equations (ix) give the arc-element directly; provided that the symbol of the face is given, and that the face neither belongs to the zone [111] nor is the face (111) itself.

Where however we have only symbols and one measured arc of the crystal for data, the problem takes the form of having to

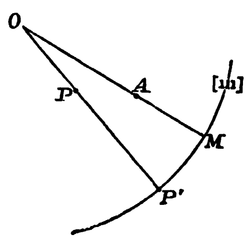


Fig. 369.

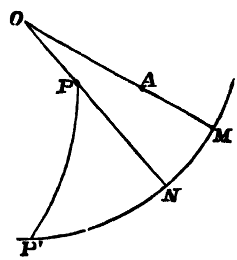


Fig. 370.

determine the arc-element from the measured arc-distance between two poles whereof the symbols only are given or can be determined. Here then we have the following cases: first, P' one of the poles may lie on the zone-circle [111], P the other lying either in or out of the zone containing O and P'. Or secondly, P' may lie on the zone-circle OP but not on [111]. Thirdly, both P and P' may be discalenohedral poles and their zone lie obliquely to the zone [111].

I. (*a*) Let P be the pole hkl, P' the pole pqr lying on the zone-circle [111] (Fig. 369). In this case (pqr) is

$$(2h-k-l,\ 2k-l-h,\ 2l-h-k).$$

From equation (viii) A_1OP can be determined, since

$$\tan A_1OP = \sqrt{3}\,\frac{k-l}{2h-k-l}.$$

Also PP' being given, OP is known, for $\tan OP = \cot PP'$; hence AO the arc-element can be found by equation (ix).

I. (*b*) In the case where P' is not in the zone OP, let N be the pole in which $[OP]$ and $[111]$ intersect (Fig. 370). Thus NM or AOP is known, from (viii); so is $P'M$ and therefore $P'N$; hence PN in the right-angled triangle PNP' can be computed. Thus OP also is known and P given in position; so that AO can be calculated from equation (ix).

If the symbol of P' had not been given but only that of $[\mathbf{uvw}]$, a zone traversing PP', then the symbol of P' is $\left\|\begin{matrix}\mathbf{u} & \mathbf{v} & \mathbf{w} \\ 1 & 1 & 1\end{matrix}\right|$

II. If neither pole is on the zone-circle $[111]$ but both are in a zone with (111), the arc PP' and the symbol of P being given; then, if N be the pole in which the zone-circles $[OP]$ and $[111]$

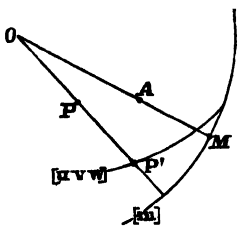


Fig. 371.

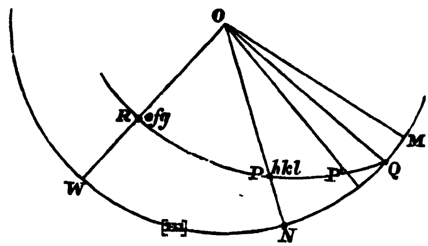


Fig. 372.

intersect, the symbol of N can be found, and it determines, as before, the angle AOP (Fig. 371). The four poles N, P', P, O lie in a zone, and of these the symbols of O or (111) and P or (hkl) are given. Also N lies in the zone $[111]$; and let the symbol $[\mathbf{uvw}]$ of a zone traversing P' be also given. Then, by Article **52**,

$$\sin(2PN-PP') = (2\Lambda-1)\sin PP',$$

where $$\Lambda = \frac{(h+k+l)(\mathbf{u}+\mathbf{v}+\mathbf{w})}{3(h\mathbf{u}+k\mathbf{v}+l\mathbf{w})}:$$

an equation by which NP, and therefore $\tan OP$, can be computed. So that P is given in position and the element can be found.

III. If P and P' (Fig. 372) are both independent poles heterozonal to O, let the zone-circle $[111]$ intersect $[PP']$ in Q and $[OP]$

in N. Then the symbols of Q and N are known. Let W be the symbol of a pole at a quadrant's distance from Q on the zone [111], of which therefore the symbol may be found by equations (vii), and let R be the pole in which $[OW]$ and $[QP]$ intersect. Then, of the four poles Q, P, P', R let the known symbols of P and R be hkl and efg; the symbols also of the zone [111] traversing Q, and of a zone [uvw] traversing P', which may be $[OP']$, are known, as are the arc QR which is a quadrant, and the arc PP' which has been found by measurement.

The arc QP can thus be found by the equation

$$\sin(2PQ - PP') = (2\Lambda - 1)\sin PP',$$

where $$\Lambda = \frac{h+k+l}{e+f+g} \cdot \frac{e\mathrm{u}+f\mathrm{v}+g\mathrm{w}}{h\mathrm{u}+k\mathrm{v}+l\mathrm{w}};$$

QM and NM, and therefore also QN, can be computed by equation (viii); and thus PN in the right-angled triangle PNQ can be found.

Hence OP and AOP are determined, and the arc-element and the axial angle of the crystal can be computed by equation (ix).

464. The least unsymmetrical mode of referring a hexagonal crystal to axes of which one is at right angles to the rest (Art. **118**) is that employed by Bravais, and adopted in recent times in the Zeitschrift für Krystallographie.

The axial system adopted in this method assumes for the four axial planes the systematic planes Σ and C, the axial system consisting of three lateral axes which are the zone-lines $[\Sigma_1 C]$ $[\Sigma_2 C]$ and $[\Sigma_3 C]$ and of a vertical axis which will be the zone-line $[\Sigma_1 \Sigma_3]$.

The X-axis and the Y-axis are the first and third of these, the fourth or Z-axis being the morphological axis perpendicular to them, while the redundant lateral axis may be designated by the letter U; the general symbol for a form being $\{h\ k\ i\ l\}$.

The directions of the three lateral axes are so taken that the poles designated in this treatise as $01\bar{1}$, $1\bar{1}0$ and $\bar{1}01$ lie on the positive side of the origin, upon the three axes respectively.

The constant relation of the index i, corresponding to the redundant axis U, to the indices on the axes X and Y may be easily established. If in Fig. 373 the face HKL intersect the axis U in a point I we have for the intercepts of this plane

$$\frac{a}{h}, \frac{a}{k}, \frac{a}{i}, \frac{c}{l}.$$

Since area HOI + area KOI = area HOK, we have

$$OH \,.\, OI \sin 60^\circ + OK \,.\, OI \sin 60^\circ = OH \,.\, OK \sin 120^\circ.$$

Hence $OH \,.\, OI + OK \,.\, OI = OH \,.\, OK$; or, if we regard lengths as negative when measured on the negative part of an axis,

$$OH \,.\, OI + OK \,.\, OI + OH \,.\, OK = 0,$$

and
$$\frac{1}{OH} + \frac{1}{OK} + \frac{1}{OI} = 0;$$

but
$$OH = \frac{a}{h}, \;\; OK = \frac{a}{k}, \;\; OI = \frac{a}{i},$$

$$\therefore \;\; h + k + i = 0.$$

In the symbol $(h\,k\,i\,l)$ of the face $HKIL$, therefore, the third index is equal to the algebraical sum of the first two with its sign changed.

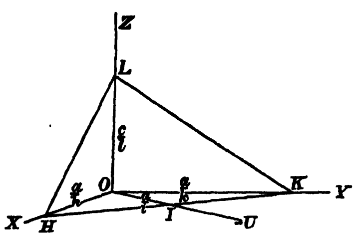


Fig. 373.

For the derivation of symbols for zones from those of faces, or the inverse process, the usual operations are resorted to; the determinant is taken of the symbols of any two faces but with one of the first three indices, and of course the same one throughout, omitted from the face-symbol.

The symbol of a face obtained from those of two zones has then to be completed by the introduction of the fourth index in accord with the relation $h + k + i = 0$.

Fig. 374 represents a projection of some of the forms of a hexagonal crystal referred to such an axial system, the four-indexed symbols representing the poles which in this treatise have the three-indexed symbols conjoined with them in the figure.

Had the *S*-planes, instead of the deutero-systematic planes, been selected for the axial planes, the symmetry of the distribution of

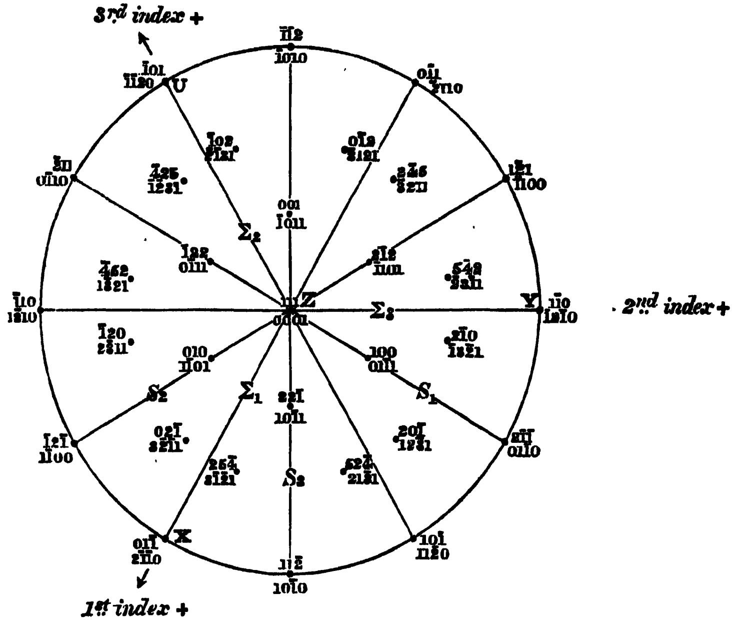


Fig. 374.

the scalenohedral forms, and of the indices in their symbols, would have presented greater symmetry and more geometrical elegance.

CHAPTER IX.

THE REPRESENTATION OF CRYSTALS.

465. The representation of a crystal by a model 'in the round' may best be effected by apparatus specially constructed for cutting sections of the wood, or other material employed, in directions inclined to each other at angles identical with those of the edges of the crystal; or they may be cut by the help of determinations deduced from the projection of the crystal on a plane surface, whereby the angles are given at which the edges meet in each quoin of the crystal.

But for use in written description recourse is had to representing the crystal by a drawing of its edges, which are projected on a plane, as on a screen, by lines emanating from the eye and traversing each point to be projected. Where the eye is near the crystal the lines are more or less divergent, and the figure on the screen obeys the ordinary rules of perspective. In proportion as the eye is removed to a distance, these rays approximate to parallelism; so that, by conceiving the eye to be placed at an infinite distance, a figure of the crystal may be projected by means of parallel rays.

Perspective figures of the former kind involve complex processes in the drawing from which those of the latter kind are free, while also the projection by the aid of parallel rays offers some important advantages.

466. *Parallel projection.* A projection of this kind offers the following advantages :—

I. Parallel lines on the crystal, the edges for instance belonging to a zone, are represented as parallel lines in the projection.

The converse however is not true, since lines parallel in the projection do not necessarily represent lines parallel on the crystal; for, if we consider a plane passing through any two rays, all lines lying in that plane will be seen to be represented on one and the same line in the projection, whatever may be the angles at which they are inclined to the two rays and to each other.

II. A second important property of the parallel projection consists in this—that a straight line on the crystal, if divided in any ratio, is represented in the projection by a line divided in the same ratio. Thus AB, BC, divisions of a line AC inclined at any angle to the plane of projection, are represented by ab, bc on ac, the projection of AC; and Aa, Bb, Cc being parallel to each other, we have

$$\frac{ab}{bc} = \frac{AB}{BC}.$$

It follows from this that, if the axes and parameters of a crystal are given in projection, intercepts on the actual axes will be represented on each projected axis by intercepts proportional to the original intercepts on that axis; and, from the indices of a plane, its projected intercepts on the axes can be taken.

It will also be seen that equal parallel lines on the crystal will be projected as equal parallel lines; and, further, a line parallel to a visual ray will be represented by a point, and a plane containing a visual ray will be projected as a line.

Furthermore, all lines lying in planes parallel to the plane of projection if equal will be projected as equal lines, and if unequal will preserve their ratios in the projection.

467. *Orthogonal projection: the plane of projection a systematic plane. Systematic projection.* For a simple representation of a crystal, it is best to take the plane of projection perpendicular to the direction of the visual rays, i.e. to employ an *orthogonal projection.* The crystal may now be placed in any position in the path of the visual rays with a view to its being orthogonally projected; and the most suitable position will of course depend on the purpose of the projection. Sometimes this purpose is sufficiently fulfilled by making a systematic plane the plane of projection; the faces of the zone perpendicular to this plane becoming lines which bound the figure, while the edges of that zone are represented by the points in which these lines meet. An orthogonal projection of this kind may be termed a *systematic projection.*

Such a projection presents great facilities to the draughtsman, and is employed, except for the Cubic system, almost exclusively in Professor Miller's classical treatise on Mineralogy. It constitutes in fact a sort of graphic counterpart to the representation of a crystal by the stereographic projection of its poles. If the stereographic and the systematic projections of a crystal be constructed with the same systematic plane for the plane of projection in each, it will be seen that for the faces of the zone perpendicular to that plane the poles will be distributed on the circle of projection in the one case, while they are represented by lines in the other case: and these lines will evidently be parallel to tangents drawn through the poles on the circle of projection, if the two projections are similar in orientation. The direction in which the edge of any two faces of the crystal would be projected can be determined by tracing the zone-circle containing the poles of the two faces and drawing a tangent to the projection-circle through the pole in which it and the projection-circle intersect.

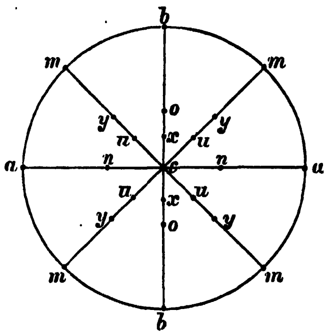


Fig. 375 (*a*).

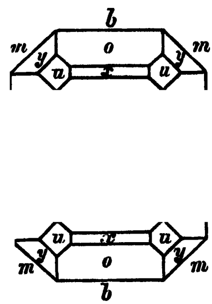


Fig. 375 (*b*).

All the lines that are to represent the edges of the front half of the crystal can thus be determined so far as their *directions* are concerned. Their actual position on the projection will depend on the relative magnitudes to be given to the several forms, and for this a small free-hand drawing of the crystal or a view of the crystal itself is desirable. Where faces of the same form are symmetrically repeated, care has to be taken that the area of corresponding faces is maintained constant by keeping corresponding symmetrical lines on the crystal everywhere of equal length in the projection. In practice it will be found that the parallelism of the edges of the faces belonging to each zone rapidly indicates the contour of the faces enclosed by the various edges; and in fact, when the stereographic projection of the crystal is once correctly figured, the systematic projection is produced by a simple process of evolution from it. In this way the systematic projection of a crystal of bournonite given in Fig. 375 (*b*) can be immediately obtained from the stereographic projection of the poles given in Fig. 375 (*a*).

468. *Orthogonal projection: the plane of projection not a systematic plane.* In order, however, to obtain a more complete view of a crystal than is given by the systematic projection, it is necessary to be able to represent the various forms more as they would be seen in a perspective drawing, as well in order to give an aspect of solidity to the figure as to indicate the relative importance of the different forms.

The systematic projection, for instance, indicates only by its boundary-lines, and therefore in fact obliterates as faces, the planes in the zone of which the edges are parallel to the line of sight; and these frequently include important pinakoid-, dome-, or prism-faces, to which the crystal may owe its most characteristic aspect.

The drawings are therefore generally made in such a way as that the plane of projection is not in parallelism with any systematic plane, nor, intentionally, with any face. The crystal is in fact placed in such a position that while its orientation gives to the axes nearly the positions in space usually assigned them in geometry, certain rotations through small angles, forward and from right to left respectively, bring into view, and as it were open out, faces that in the systematic projection are represented by lines only; while the idea of solidity is further imparted to the figure by the difference in dimensions that faces of the same form assume when thus moved into positions in which they are differently inclined to the line of sight.

469. *Cubic system.* Let Ox, Oy, Oz be three rectangular lines of reference fixed in space, Ox and Oy being horizontal and Oz vertical; let xOz be the plane of orthogonal projection, and the visual rays have directions parallel to yO. Taking as the simplest case a cubic crystal, we may suppose the initial position of the crystal to be such that the crystal-axes OX, OY, OZ are coincident with the fixed lines Ox, Oy, Oz, respectively: in the corresponding projection, the projections of OX, OZ will coincide with the lines Ox, Oz respectively, and the projection of OY will be the *point* O (Fig. 376).

Next suppose the crystal to be rotated, in the direction of the hands of a watch, through a small angle δ round the fixed line Oz.

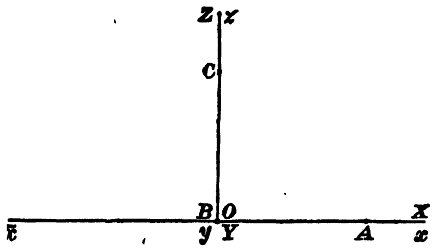


Fig. 376.

The Y-axis will now be represented by a *line* coincident in direction with the line $O\bar{x}$; and A', the projection of any point A on the axis OX, will move along Ox towards the origin while the rotation is proceeding (Fig. 377). Since the parameters are equal

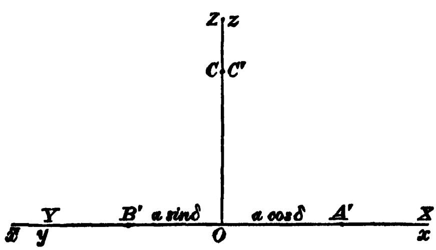


Fig. 377.

we may take in the initiatory position $OA = OB = OC = a$ for the parameters on the axes X, Y, and Z. During the rotation of the crystal, A describes a circle: the visual rays are in the plane of this circle and are perpendicular to the initial position of OA. If we describe a circle (see Fig. 378) through A and B with its centre at the origin, we may determine by a simple construction the values of OA' and OB', representing the projected lengths a on the X-axis and Y-axis, after the rotation of the axes through the angle δ.

Making $AOD = \delta$, we may suppose A to have moved on the circle to D; when seen by visual rays in the plane of the circle

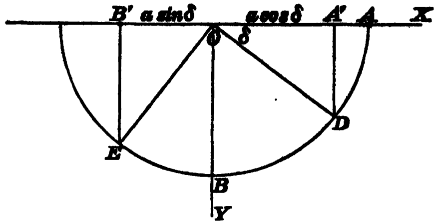


Fig. 378.

and perpendicular to the initial position of OA, A will appear as at A', where DA' is perpendicular to OA.

Similarly, after the rotation, B will have moved to E and be seen as at B': and $OA' = a \cos \delta$, $OB' = a \sin \delta$.

The projections OA'', OB', OC' of the lines OA, OB, OC on the plane xOz, after this single rotation of the crystal through an angle δ about the line Oz, are given in Fig. 377.

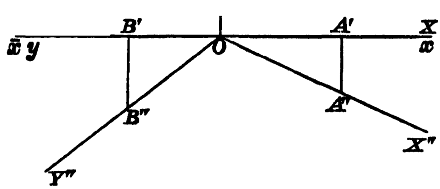


Fig. 379.

Let now the system further revolve round the line Ox through a small angle ϵ, the point C or C' moving forward, and its projection C'' downward along the axis Z. Then the projections of the axes X and Y on the plane xOz will become disengaged from their former coincidence with the line Ox, and will take new directions in OX'' and OY'': the points A' and B' will appear in the projection to move along lines perpendicular to Ox to the new positions A'' and B'', while C moves to C'' (Fig. 379).

The axial system has now taken a new aspect. The axes X and Y come to be represented by lines from the origin through A'' and B'', and the parameters are measured on these axes and on Oz by the ratios $$OA'' : OB'' : OC'';$$
which, it will be seen by the constructions in Figs. 378 and 380, may be determined by the expressions

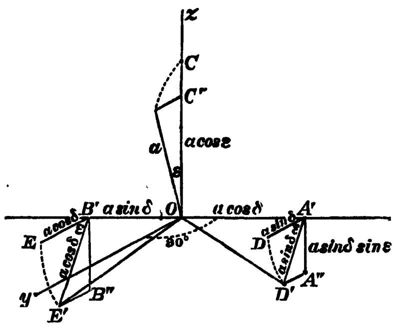


Fig. 380.

$$\left.\begin{array}{l} OA' = a\cos\delta \\ A'A'' = a\sin\delta\sin\epsilon \end{array}\right\}, \quad \left.\begin{array}{l} OB' = a\sin\delta \\ B'B'' = a\cos\delta\sin\epsilon \end{array}\right\}, \quad OC'' = a\cos\epsilon,$$

and $$OA''^2 = a^2(1-\sin^2\delta\cos^2\epsilon),$$

For the angles between OX'', OY'', and OZ'', the projections of the axes OX, OY, and OZ, we have (Fig. 379)

$$\cot Z''OX'' = \cot(90^\circ + xOX'') = -\tan xOX''$$
$$= -\frac{A'A''}{OA'} = -\tan\delta\sin\epsilon;$$

$$\cot Z''OY'' = \cot(90^\circ + \bar{x}OY'') = -\tan\bar{x}OY''$$
$$= -\frac{B'B''}{OB'} = -\cot\delta\sin\epsilon;$$

whence also $X''OY''$ can be computed.

470. The values of δ and ϵ that give to a crystal the most advantageous position for exhibiting its features have been differently taken by different crystallographers.

Haidinger, who has been very generally followed, took $\tan\delta = \frac{1}{3}$ and $\sin\epsilon = \frac{1}{8}$, corresponding to $\delta = 18^\circ\, 26'$ and $\epsilon = 7^\circ\, 11'$, nearly; the projected parametral ratios being

$$a : b : c = 0{\cdot}957 : 0{\cdot}3404 : 1,$$

and the axial angles as seen in projection

$$ZOX = 92^\circ\, 23' \text{ and } YOZ = 110^\circ\, 33\tfrac{1}{2}'.$$

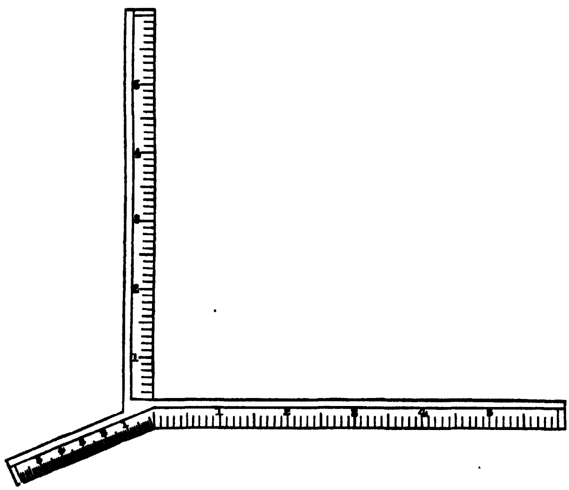


Fig. 381.

471. *Crystallographs.* When the projection of the axes has been determined for the Cubic system, that for any other system is readily deduced from it. And it is very convenient to have an instrument in the form of Fig. 381, made of brass or other metal of about $\frac{1}{8}$ in. thick, cut to the angles of the projected axes, and graduated along each of its three limbs with divisions subdivided to tenths and in the ratios of the projected parameters of a cubic crystal. Such an instrument or *crystallograph* serves for laying down on a piece of card, or a sheet of smooth paper strained on a frame, the projection of a cubic axial system, from which that of the axes and parameters for any other crystallographic system can be deduced in the following manner.

(*a*) *Orthosymmetric systems.* The Tetragonal and Orthorhombic systems are easily dealt with. In the latter, for instance, the orientation of the axes as projected by the crystallograph remains unchanged, but we have to find parameters on each of the three axes proportional to those of the crystal, but interpreted in accord with the parametral units of length on each limb of the crystallograph. In the Tetragonal system, only the parametral length on the *Z*-axis has to be so adjusted.

(*b*) *Hexagonal system.* For the purpose of transforming a projection for a cubic axial system into one for a hexagonal crystal, let the axes of the cubic system be *XYZ*, those of the hexagonal crystal *X'Y'Z'*.

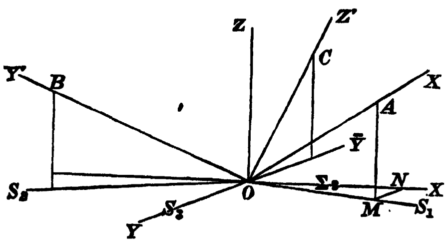


Fig. 382.

If now a projected cubic system of axes be drawn, and the axis *Z* be taken as the morphological axis of the hexagonal crystal, the axes *X* and *Y* being taken for the symmetry-axes $[\Sigma_3 C]$ and $[S_3 C]$ or $[1\bar{1}0]$ and $[11\bar{2}]$ respectively, the axis *Z'* of the hexagonal crystal will lie in the *YZ* plane between *Z* and $\bar{Y}$, and the axes *X'* and *Y'* will lie in the systematic planes S_1 and S_2 which are inclined at 30° to the plane Σ_3 and at 60° to S_3 (Fig. 382).

Let *ABC* be any three points on *X'Y'Z'* equidistant from *O* the origin of the system. Let a perpendicular from *A* meet the plane *XOY* in *M*, and draw through *M* a line parallel to the axis *Y* and cutting the axis *X* in *N*. Then *ON*, *NM*, *MA* are the rectangular coordinates of A, as referred to the rectangular axes OX, OY, OZ; and

$$OM = OA \cos AOM = OA \sin AOZ,$$
$$ON = OM \cos NOM = OA \cos 30^\circ \sin AOZ,$$
$$NM = OM \sin NOM = OA \sin 30^\circ \sin AOZ,$$
$$MA = OA \sin AOM = OA \cos AOZ:$$

whence, dividing each by $OA \sin AOZ$, the coordinates of A may be taken as

on axis X; $\cos 30^\circ$ or 0.866;
on axis Y; $\sin 30^\circ$ or 0.5;
on axis Z; $\cot AOZ$:

where $\cot AOZ = \frac{1}{2} \tan (100 \cdot 111)$, an angle which measures the characterising element of the particular crystal (Art. 457).

Similarly, after dividing by $OA \sin AOZ$ as before, the coordinates of B are found to be

on axis OX; $-0{\cdot}866$;
on axis OY; $0{\cdot}5$;
on axis OZ; $\cot AOZ$:

and the coordinates of C are

on axis OX; 0;
on axis OY; -1;
on axis OZ; $\cot AOZ$.

The positions of the projected axes of the hexagonal crystal can now be determined by means of the crystallograph; lengths corresponding to the above coordinates are measured along the projected cubic axes, the unit being in each case determined by the limb along which the measurement is made (Art. 466).

(*c*) *Oblique systems.* (1) For the graphic representation of a crystal in the Monosymmetric system by aid of the crystallograph, the directions of the Y-axis and of the Z-axis are identical in the projections of the crystal and of the cubic axial system, and that of the Z-axis is retained in a vertical position.

The elements of the monosymmetric crystal being $a:b:c$, η, it is necessary, as in the last Article, to find the rectangular coordinates of A, B and C relative to the cubic axes OX, OY, OZ, where

$OA = a$, $OB = b$, $OC = c$ and $AOC = \eta$. If AH be drawn parallel to OZ meeting OX in H (Fig. 383), it will be seen that the coordinates of A in space are

on axis X, $OH = OA \cos(\eta - 90^\circ)$;
on axis Y, 0;
on axis Z, $-HA = -OA \sin(\eta - 90^\circ)$.

The coordinates of B are

on axis X, 0; on axis Y, b; on axis Z, 0.

The coordinates of C are

on axis X, 0; on axis Y, 0; on axis Z, c.

By measuring along the limbs of the crystallograph lengths representing these coordinates, the projected axes of the monosymmetric crystal are determined as in the last Article.

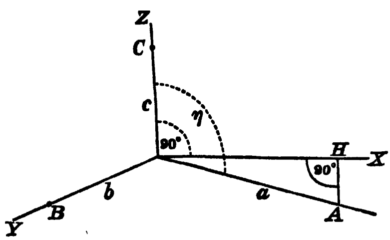


Fig. 383.

(2) The projection of the axes and parameters of an anorthic crystal can best be obtained by finding the values of the coordinates for a point at parametral distance from the origin for each axis.

The Z-axes for the cubic axial system and the anorthic crystal may be taken as vertical, and the X-axis of the crystal may be taken to lie in the plane ZX of the cubic system. Let the cubic axes as projected be $X' Y' Z'$.

The elements of the anorthic crystal being $a : b : c$, ξ, η, ζ, let OH, OK, OL be the parametral lengths of its axes. The axial angles of the crystal are ξ, η, ζ, and the angle of θ between the planes HOL, $KOL = 180^\circ - AB$, A and B being the poles of the faces 100 and 010; we have also, for the determination of the value of θ, the expression

$$\cot\frac{\theta}{2} = \sqrt{-\frac{\cos(S-\eta)\cos(S-\xi)}{\cos S \cos(S-\zeta)}},$$

where $S = \dfrac{\xi + \eta + \zeta}{2}$.

OH lies in the plane *XOZ*: drawing *HS* parallel to *ZO* and meeting the axis *OX* in *S*, we have (Fig. 384)

i. for the point *H* the coordinates in space

on the *X*-axis $OS = OH \cos SOH = a \sin \eta$,

" *Y*-axis $= 0$,

" *Z*-axis $SH = OH \sin SOH = a \cos \eta$;

ii. for the point *K*, draw the line *KR* perpendicular to the plane *XOY* and meeting it in *R*: draw *RT* perpendicular to *OX* and meeting it in *T*, then

$$OR = OK \cos ROK = OK \sin KOL = b \sin \xi.$$

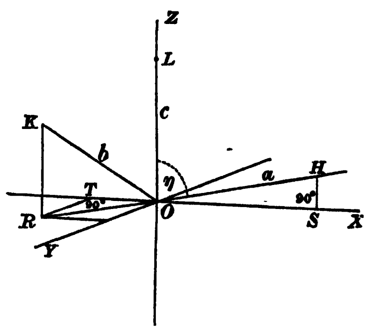


Fig. 384.

Further, *TO* and *RO* being both perpendicular to *OL*, the angle *TOR* is the angle between the planes *TOL* and *ROL*, that is, the supplement of the angle between the planes *HOL* and *KOL*. Hence $TOR = AB$.

We thus have for the coordinates of *K*,

on axis *X*, $OT = OR \cos ROT = -b \sin \xi \cos AB$,

" *Y*, $RT = OR \sin ROT = +b \sin \xi \sin AB$,

" *Z*, $RK = OK \cos KOL = b \cos \xi$;

and iii. for the coordinates of the point *L* we have, on the axes of *X* and *Y*, 0, on the axis of *Z*, *c*. Measuring these lengths along the limbs of the crystallograph, the projections of *OH*, *OK*, *OL* are immediately determined.

472. *Projections of twin-crystals.* In drawing a twin-crystal three preliminary steps have to be taken. We have to determine by the above methods the axial system for one of the individuals as seen in projection; next the twin-plane has to be represented; and, finally, the axial system for the second individual has to be determined.

We may first suppose the two crystals in apposition, and the twin-face to be the face of combination. We now need to find the position of the twin-axis relatively to the axial system of crystal No. I, or, which is the same thing, to find the point in which this axis meets the twin-face to which it is the normal.

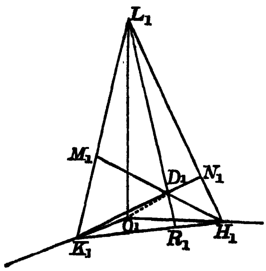


Fig. 385.

Let the twin-plane parallel to the face *hkl* intersect the actual axes in $H_1 K_1 L_1$; and let the twin-axis, the normal to the plane $H_1 K_1 L_1$, meet the plane itself in the point D_1 (Fig. 385). By the aid of the crystallograph it is possible, as already explained, to determine OH, OK, OL, the projections of $O_1 H_1$, $O_1 K_1$, $O_1 L_1$: it is now required to find OD, the projection of the normal $O_1 D_1$.

Join D_1 to the angles of the triangle $H_1 K_1 L_1$, and produce the lines to meet the sides of the triangle in $M_1 N_1 R_1$, respectively. By calculation or construction, determine the ratios

$$K_1 M_1 : M_1 L_1, \quad L_1 N_1 : N_1 H_1, \quad H_1 R_1 : R_1 K_1.$$

These ratios being unaltered by parallel projection (Art. **466**), the projected points M, N, R can now be found by dividing the projected lines KL, LH, HK in the same proportion. The intersection of the lines HM, KN, LR is the projection D of the extremity of the twin-axis.

473. When the axes are rectangular the construction of the above ratios is simple, for the lines $O_1 M_1$, $O_1 N_1$, $O_1 R_1$ are then perpendicular to the lines $K_1 L_1$, $L_1 H_1$, $H_1 K_1$ respectively, as may be thus proved: $K_1 L_1$ being the line of intersection of the planes $H_1 K_1 L_1$ and $O_1 K_1 L_1$ is perpendicular to the lines $O_1 D_1$, $O_1 H_1$, normal to those planes, and is thus at right angles to the plane $H_1 O_1 D_1$ and to every line in it; hence $K_1 L_1$ is perpendicular to $O_1 M_1$: similarly $L_1 H_1$ is perpendicular to $O_1 N_1$, and $H_1 K_1$ to

To construct the ratios we may proceed as follows:—construct three triangles $k_1 o_1 l_1$, $l_1 o_1 h_1$, $h_1 o_1 k_1$, equal to the triangles $K_1 O_1 L_1$, $L_1 O_1 H_1$, $H_1 O_1 K_1$, as illustrated in Figs. 386 *a*, *b*, *c*; draw the perpendiculars $o_1 m_1$, $o_1 n_1$, $o_1 r_1$; through $k_1 l_1 h_1$ in the respective triangles draw lines $k_1 l$, $l_1 h$, $h_1 k$ having lengths identical with those of the projected lines *KL*, *LH*, *HK*; join $l_1 l$, $h_1 h$, $k_1 k$

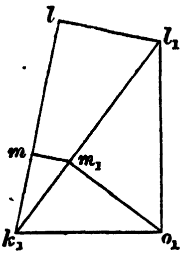


Fig. 386 (*a*).

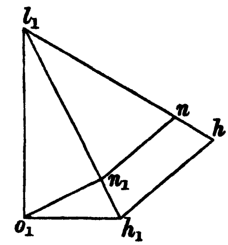


Fig. 386 (*b*).

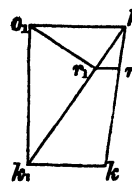


Fig. 386 (*c*).

and draw their parallels $m_1 m$, $n_1 n$, $r_1 r$. The positions of the points *M*, *N*, *R* on the sides of the triangle *HKL* are then determined by setting off lengths *KM*, *LN*, *HR* equal to the lengths $k_1 m$, $l_1 n$, $h_1 r$.

474. In order to represent the axial system of crystal No. II, we have recourse to the following construction:—

Project the axial system for crystal No. I (Fig. 387). Determine on the axes the parametral lengths *OA*, *OB*, *OC* for the face of the form {111} belonging to the octant, and also the intercepts *OH*, *OK*, *OL* of the twin-plane *hkl*.

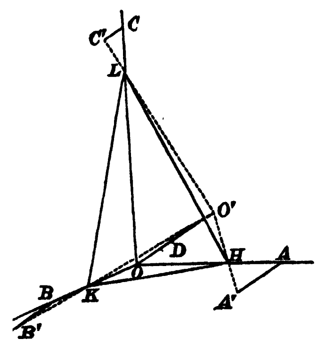


Fig. 387.

Find the normal *OD* of the plane *HKL*, meeting *HKL* in *D*, and continue *OD* to *O'* making *DO'* = *OD*. Then *O'* will represent the origin of the axial system of crystal No. II; and if *O'H*, *O'K*, *O'L* be drawn they will represent in projection the directions of the axes *OX'*, *OY'*, *OZ'* of the

It is in fact very desirable, and indeed, if the crystal presents several forms, necessary to have a free-hand drawing of it, constructed with sufficient care and accuracy to show the relative importance of the faces, and the consequent developement of the edges in which the various minor forms meet with each other and with the more important forms.

It will be necessary however in all cases to commence by laying down the axial system in projection by aid of the crystallograph. The axes should be drawn in considerable extension, the parameters and the intercepts for important faces being marked off at some considerable distance from the origin. If now the edges of two or three of the most prominent forms be drawn on this larger scale and the intersections of the faces of these forms with one another be determined, we have lines of appreciable length, to which the shorter lines nearer to the origin that are actually to represent these edges in the drawing can be made parallel with great accuracy. The edges formed by faces of less conspicuous forms, which are generally those with more complex symbols, and on this account often more numerous from the symmetrical repetition of the faces, can also be now drawn in over the lines representing the edges of the first drawn forms.

477. In order however to make any advance we have to be able to determine the direction of the edge for any two faces.

For this we may find the lines in which the two planes severally would intersect the axial planes. Let the two planes (hkl) and (pqr) intersect the axes in HKL and PQR (Fig. 388). HK and PQ lie in the plane OHK, and if produced will meet in a point U unless both are parallel to an axis. So also RQ and KL will meet in a point S, and HL and PR in a point T; and the points S, T, U are all points common to the planes hkl and pqr, and will lie in a straight line which is the edge of intersection of those two planes.

Another method, and in general the most satisfactory one, is really a particular case of the preceding. Since the direction of a plane depends, not on the absolute, but on the relative values of the intercepts, and the three intercepts of any plane on the axes may be multiplied by any arbitrary number, the intercepts of the two planes of which the edge is required should be so multiplied that the two intercepts on one of the axes—say the X-axis—are identical for both planes. The points corresponding to T, U, P and H then coincide, say at H' (Fig. 389), and a line joining this point to the

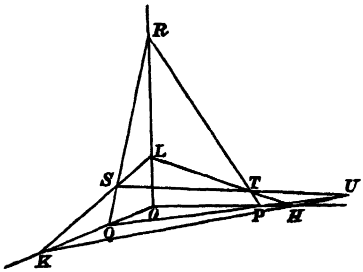


Fig. 388.

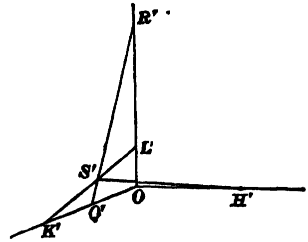


Fig. 389.

point of intersection of lines $Q'R'$ and $K'L'$, corresponding to QR and KL, will have the direction required. Thus—for the planes (362) (313)—the intercepts of the first plane on the axes will be in the proportion $\frac{a}{3} : \frac{b}{6} : \frac{c}{2}$ or $2a : b : 3c$; those of the second plane will have the ratios $\frac{a}{3} : \frac{b}{1} : \frac{c}{3}$, or $a : 3b : c$. If the second set be doubled, the intercepts of the second plane will be $2a$, $6b$, $2c$.

Hence, taking OH', OK', OL', OQ', OR', as $2a$, b, $3c$, $6b$, $2c$ respectively, draw $Q'R'$ and $K'L'$; the line $H'S'$, joining S' their point of intersection to H', has the direction required (Fig. 389).

And of course the direction of the edge may be likewise found by determining the coordinates of a point in the edge considered as an origin-edge. Thus [uvw] being the symbol of the edge, take off (Fig. 390),

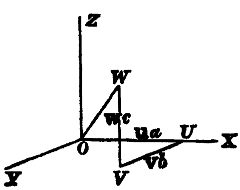


Fig. 390.

OU on axis $X = \mathbf{u}a$,

UV parallel to axis $Y = \mathbf{v}b$,

VW parallel to axis $Z = \mathbf{w}c$.

Then OW is the direction of the zone-axis in question (Art. 20).

478. Having thus the means of determining the direction of any edge to be represented, we have to introduce the edges parallel to such directions in their proper positions and with appropriate lengths. These depend on the relative importance to be given in the drawing to the different forms presented by the crystal. It is advisable in general to arrange that the crystal-figure shall be at a part of the paper not too near the origin; otherwise, as the drawing progresses, unnecessary confusion results through the frequent intersection of the growing figure by lines only drawn as aids to its construction. As in most cases the crystal is to be represented 'in equipoise,' care has to be taken that all the faces of each form are projected in equivalent magnitudes. For this purpose it is best to draw conspicuously the directions of the axes of symmetry in their due projection.

The edges which are not lines of symmetry but belong to conspicuous forms are then put in, and care is taken in doing so that the lengths of the lines representing corresponding edges are adjusted in accordance with the symmetry of the crystal as seen in projection.

The parallelism of all edges formed by the faces of a zone with each other simplifies very much the introduction of the lines representing such edges. It is generally advisable at first to draw the crystal with the omission of the smaller faces, and to introduce these subsequently; and to do this notwithstanding the necessity it involves of afterwards erasing lines already put in, when the faces of the less conspicuous forms are introduced. But the time lost is more than saved by the greater facility with which the proper positions and distances from the origin of the various faces and edges are maintained, while the projection of the crystal is growing under the hand.

479. *Drawing of merosymmetrical crystals.* In drawing a hemi- or tetarto-symmetrical crystal it is generally advisable to draw the crystal holosymmetrically and in equipoise, and then to deduce from this the hemisymmetrical projection by the omission of some and the extension of others of the edges of the crystal in accordance with the symmetry. Where the merosymmetrical forms occur only as replacements by small faces of certain of the quoins and edges of the crystal, these are best introduced after the construction of the holosymmetrical figure. Often however we have to note that some hemisymmetrical forms concur which are correlative and unequally developed on the crystal, and these will obviously have to be treated as separate forms, each with its distinct developement, and to be deduced from distinct imaginary holosymmetrical forms.

480. *Methods of verification.* For the verification of the exactitude with which the edges, and their intersections in quoins, have been projected, various methods may be had recourse to. Exact measurements of the lengths of corresponding edges should give those lengths precisely equal.

Further, inasmuch as where a crystal is centro-symmetrical any plane that divides it in half divides it, in the general case, into two antistrophic or enantiomorphous halves, it is evident that if half a holosymmetrical crystal be drawn accurately the other half can be added to it by the simple method of taking on tracing paper a tracing of the half-projection, rotating it in the plane of the drawing through 180°, and taking the quoins through by a pricking point on to the paper carrying the original half-projection. The lines joining these points should exactly correspond to the lines of the half-figure already drawn. This method, of little general use for the projection of the half of the front part of the crystal, is of great use in putting in the dotted lines that usually represent the edges of the other side: by thus repeating in reverse the lines and points of the projection on the front half, it is at once seen if they do not meet exactly, as they should do, when the two antistrophic semiprojections are thus brought into unison.

In the case of a twin-crystal, the equal developement of the two individuals is best secured by testing the position of corresponding points on the two crystals by drawing through them, or one of them, a parallel to the twin-axis, determining on it the point of intersection with the twin-plane, and noting that the corresponding quoins, or other corresponding points on the two edges to be compared, are exactly equidistant from the said point.

481. *Gnomonic projection.* In the stereographic projection the point of sight was taken at the surface of a sphere, and was the pole of a great circle in the plane of projection. In the *gnomonic projection* the centre of the sphere is the point of sight, and a tangent plane to the sphere is the plane of projection touching the sphere at a point termed the centre of projection. The planes of all great circles pass through the point of sight, and can only intersect the projection-plane in straight lines which will be of infinite length.

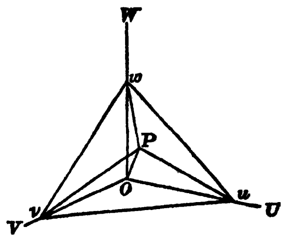


Fig. 391.

In Fig. 391 and subsequent Figures let O be the point of sight, P the centre of projection, and $OP = r$. OU, OV and OW are three normals to faces of the system, of which the mutual inclinations are

$$UV = \theta, \quad VW = \phi, \quad WU = \psi.$$

Then OP is the normal to the projection-plane, which may be any plane arbitrarily chosen but must intersect OU, OV, OW giving intercepts Ou, Ov, Ow: OP is inclined on those normals at known angles, viz.

$$POU = \alpha, \quad POV = \beta, \quad POW = \gamma.$$

hen

$$Ou = \frac{r}{\cos\alpha}, \quad Ov = \frac{r}{\cos\beta}, \quad Ow = \frac{r}{\cos\gamma},$$

$$vw = r\cos\alpha\sqrt{\cos^2\beta + \cos^2\gamma - 2\cos\beta\cos\gamma\cos\psi},$$

$$wu = r\cos\beta\sqrt{\cos^2\gamma + \cos^2\alpha - 2\cos\gamma\cos\alpha\cos\phi},$$

$$uv = r\cos\gamma\sqrt{\cos^2\alpha + \cos^2\beta - 2\cos\alpha\cos\beta\cos\theta},$$

$$Pu = r\tan\alpha, \quad Pv = r\tan\beta, \quad Pw = r\tan\gamma;$$

expressions by the aid of which the points $u\,v\,w$ and P can be projected on the surface of the drawing which is the plane of projection. The plane selected for the purpose, and which is determined in relation to the crystal by the angles α, β, γ, is arbitrary, but usually so chosen as in each different system to exhibit the zones belonging to at least one systematic triangle; the normals for reference being also selected so as to enable these zones to be laid down within convenient limits.

If a sphere of radius r be supposed described round O as centre and a great circle pass through two poles or other points A' and B' on the sphere, these points will lie on a straight line upon the plane of projection, which will be the projection of the great circle; and, if the points be poles, will be the projection of their zone-plane and zone-circle.

482. *To measure the angle between two lines of which the points of projection are given.*

P being the centre of projection, let AB be the line on which the two points A' and B' are projected in A and B (Fig. 392).

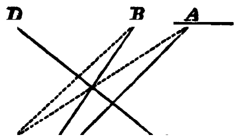


Fig. 392.

Then O lies on a line vertically over P, and AOB is the actual angle at O between the two normals OA' OB'. In order by a graphic method to measure this angle it will be requisite to lay down a triangle congruent with AOB on the plane of the drawing.

Through P draw a perpendicular to AB, produced if necessary, and meeting AB in D; also a line $P\omega = r$, perpendicular to PD: join $D\omega$, and on DP or its continuation take Ω such that

The triangle DOP will evidently coincide with the triangle $D\omega P$ if it be turned on its edge DP till O falls on ω, and

$$D\omega = DO = D\Omega.$$

Similarly the triangle DOA becomes coincident with $D\Omega A$ after turning round AD, and the angle $D\Omega A$ is the same angle as DOA which is measured in the projection by the line AD. Hence the angle between any lines represented by points on the projected line AB is the angle between two lines joining these points with a point Ω found by the above construction.

483. *Two zone-planes being projected in CA, CB, intersecting in C, to find the angle of their inclination; or, conversely, given one zone-plane in projection, to project a second zone-plane tautohedral with it at a given angle in a pole C given in projection.*

If OC be supposed drawn, and also PC, and a perpendicular to PC and therefore to the plane OPC, be drawn through P intersecting the two zone-projections in A and B, and continued to ω at a distance $= r$ from P, it will be seen that the triangle $P\omega C$ will be congruent with the triangle POC if the latter be turned round PC till it falls on the projection-plane.

If now K, outside the plane of the paper, be the point in which a plane AKB would be intersected by OC perpendicularly, the angle AKB would be the angle of the edge OC formed by the two zone-planes OAC and OBC.

Draw PD a perpendicular to $C\omega$ (Fig. 393); the line PK being perpendicular to OC would fall into congruence with PD when the triangle POC falls on the projection-plane, after rotation round PC.

In PC take $P\Omega = PD$, and draw $A\Omega$, $B\Omega$.

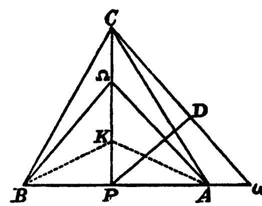


Fig. 393.

Then the triangle AKB when turned round AB till it falls on the projection-plane is congruent with $A\Omega B$, and the angle $A\Omega B$ is the actual angle between the zone-planes CA, CB. Hence the simple graphic construction by which PD, and thus $P\Omega$, is determined enables us to find the angle between any two zones intersecting in C, or to project a zone intersecting a zone given in projection at any required angle.

484. In the application of these artifices of construction to the projection of a crystal, that is to say to the projections of its zone-planes, it is clear that the area included in the projection must be so limited as not to embrace zones which by their angular distance from the centre of projection would be represented by lines too far removed from that centre. A distance representing 90° of angular measurement from P is infinite in length, and therefore an area the greatest diameter of which would represent an angle not larger than about 45° on each side of P, or 90°, is as large as can be conveniently employed.

This area of projection needs further to be so chosen as to exhibit as far as possible the essential features of the crystal in regard to symmetry: and for this a systematic triangle or a group of these naturally presents itself as offering the conditions required.

In the Ortho-symmetrical systems an octant bounded by three systematic planes, and in the Hexagonal system two adjacent systematic triangles included by two planes S and the plane C, are the usually projected portions of a crystal.

The normal OW would in the Ortho-symmetrical systems be the Z-axis [001], and in the Hexagonal system the zone-axis [111]; U and V would be normals lying in the zone-plane [001] or [111], in the two cases.

The intersections of two of the Σ- or two of the S-systematic planes with this zone-plane are conveniently taken for the two normals in the Cubic, Tetragonal and Hexagonal systems; and, in the Orthorhombic system, the axes X and Y, or two normals symmetrically situate in regard to X or Y. The projection-plane is in all such cases taken more or less symmetrically with regard to the normals OU, OV, OW, and therefore in some cases is not a crystal-face.

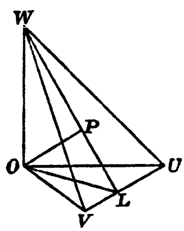


Fig. 394.

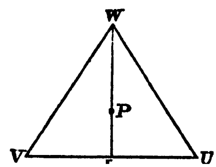


Fig. 395.

Thus in all the systems in which three perpendicular normals are allowed by the symmetry we may take $\phi=\psi=\frac{\pi}{2}$, and the plane uvw as equally inclined on the normals OU and OV. The centre of projection P will lie on WL, L being the middle point of UV.

If $\epsilon=\frac{1}{2}\,UOV=\frac{1}{2}\theta$,

then $\cos a=\cos\beta=\sin\gamma\cos\frac{1}{2}\theta=\sin\gamma\cos\epsilon$;

and in Fig. 394,

$$WL=\frac{r}{\sin\gamma\cos\gamma},\quad OL=\frac{r}{\sin\gamma},\quad VL=r\frac{\tan\epsilon}{\sin\gamma};$$

$$\tan UWL=\frac{LU}{LW}=\tan\epsilon\cos\gamma.$$

The triangle UVW can be constructed by the aid of this last ratio.

Also, $$\frac{PL}{WL}=\frac{r}{\tan\gamma}\,\frac{\sin\gamma\cos\gamma}{r}=\cos^2\gamma;$$

so that P can be found on WL,

and $$r=\frac{PW}{\tan\gamma}=\frac{PU}{\tan\alpha}, \text{ or } \frac{PU}{PW}=\frac{PV}{PW}=\frac{\tan\alpha}{\tan\gamma}.$$

In the Cubic, Tetragonal and Orthorhombic systems the angle ϵ is taken as 45°.

Generally γ is taken $=45^\circ$ and $\cos\gamma=\frac{1}{\sqrt{2}}$,

$$\therefore \quad \cos\alpha=\tfrac{1}{2} \text{ and } \alpha=60^\circ.$$

Then, $UL=\frac{1}{\sqrt{2}}WL$, $PL=\frac{1}{2}WL$, $PU=\sqrt{3}PW$.

If however we take $\alpha=\beta=\gamma$ and $\epsilon=45^\circ$,

then, Fig. 395, $$PU=PV=PW;$$

$$\cos\alpha=\sin\alpha\cos 45^\circ, \quad \tan\alpha=\sqrt{2},$$

$$\alpha=54^\circ\, 44\tfrac{1}{2}';$$

$$UL=\frac{1}{\sqrt{3}}WL; \quad PL=\tfrac{1}{3}WL.$$

In the Hexagonal system $\epsilon=30^\circ$.

Here $$\cos\gamma \text{ is taken } =\frac{1}{\sqrt{3}}.$$

Then $$\cos\alpha=\frac{1}{\sqrt{2}} \text{ and } \alpha=45^\circ.$$

$$UL=\tfrac{1}{3}WL=PL; \; PU=\sqrt{2}PW.$$

485. One of the advantages of the gnomonic projection lies in its application to the drawing of crystals in orthogonal projection. The lines in a gnomonic projection are the projections of zone-planes, and therefore the edges of the faces belonging to any zone are perpendicular to the line in which the zone-plane is projected; so that the projections of all the edges of a crystal can be drawn in the directions they would have were the crystal projected orthogonally on the plane taken for the gnomonic projection-plane.

486. To obtain a gnomonic projection corresponding to the axes used in Art. **470**, we proceed as follows. Let the rectangular axes fixed in space and passing through O the centre of the sphere be OX, OZ, respectively horizontal and vertical, parallel to the plane of the paper, and OY perpendicular to the plane of the paper (Fig. 397). Let the paper remain fixed, and the cubic axes OA, OB, OC, which initially coincide with OX, OY, OZ, move under it by first a rotation δ about OZ and then a rotation ϵ about OX. The first rotation brings the pole (010) to B_1, where $PB_1 = r\tan\delta$ along the horizontal line; the pole (001) remains at an infinite distance along the vertical line PZ, and (100) travels from an infinite distance towards P along the horizontal line. The second rotation brings (010) to B, found by taking $PQ = r\tan\epsilon$, and $QB = r\tan\delta$, as may be seen from the construction in Fig. 396, where $OQ = \frac{r}{\cos\epsilon}$, $QOB = POB_1 = \delta$,

$$QB = OQ\tan\delta = \frac{r\tan\delta}{\cos\epsilon}.$$

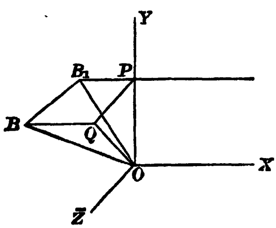


Fig. 396.

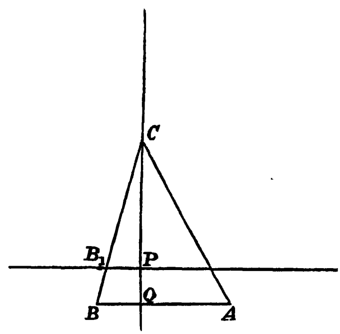


Fig. 397.

The line BQ is then the zone-line [010, 100]; BD the zone-line [010, 001] may be drawn as a line inclined to BQ at an angle representing 90° by the construction given in Art. **483**. The poles (100) and (010) may then be set off along these lines at distances BA, BC, representing 90° by the construction given in Art. **482**.

The above projection may be more conveniently drawn directly from the crystallograph by drawing the lines CA, CB perpendicular to the B and A arms of the latter respectively and setting off a distance $CP = r\cot\epsilon$ along a line parallel to the C arm (Fig. 397). Then through Q, where $PQ = r\tan\epsilon$, draw BA perpendicular to CQ; ABC are then the extremities of the rectangular axes, or the poles (100) (010) (001) in the Cubic, Tetragonal, or Orthorhombic systems. The complete projection for a crystal belonging to either of these systems is constructed by laying down the poles of (110) (011) (101) upon the sides of the axial triangle at distances corresponding to their inclinations to ABC.

The rhombohedral or hexagonal projection may be derived from the triangle by taking ABC as the poles $(10\bar{1})$ $(11\bar{2})$ (111) respectively, and laying down other poles upon these zones by their angular distances from ABC. The projection of a monosymmetric crystal is obtained by taking C_1 (001) at a distance from B representing the angle (100 : 001) along the line BC, and then laying down other poles in the axial zones as before.

The projection of an anorthic crystal is obtained by taking B_1 (010) at a distance from A, representing the angle (100 : 010) along the line AB and drawing the zone-lines B_1 C_1 and AC_1 such that their inclinations to AB represent the angles η and ξ respectively; and finally laying down other poles in the axial zones as before.

487. *Drawing implements.* It remains to notice the implements and materials requisite for carrying on the processes described for the projection of a crystal.

Good smooth drawing paper or card is the best material on which to carry out the drawing. If paper is used it should be mounted on a drawing board, and be fastened thereto at the corners by a little gum. It should be of sufficient size for carrying out the projection of the axial system of the crystal with axial lengths equivalent to from 5 to 10 inches according to the complexity of the crystal to be drawn.

It is generally well, where many faces have to be represented, to draw the crystal-projection at some little distance away from the axial projection obtained by aid of the crystallograph, as already pointed out, and even sometimes to project more than one such subsidiary axial system, all of course in absolutely parallel orientation with one another and with the corresponding lines in the crystal.

This, and indeed the ruling of all ordinary straight lines on the drawing, is best effected by help of two similar flat pieces of vulcanite or thin steel, each in the form of a triangle with angles of 90°, 60° and 30°, and having a hypotenuse 7 or 8 inches long;

for many figures a length of 2 or 3 inches is however sufficient. The sides of the triangles should be as straight as they can be made: the two implements when used together, the hypotenuse of one sliding in contact with that of the other, form an excellent parallel ruler.

Pencils of various degrees of hardness and blackness are needed, and they must be used with a very fine point. In use, fainter lines are employed for the purely structural lines, that form as it were the scaffolding of the projection, and these can be rubbed out when the final lines representing the complete drawing have been introduced. A hard pencil, as dark as is consistent with hardness and the retention of a fine point, is needed for the last process, unless a ruling pen with Indian ink is substituted for the pencil. The pen or pencil should lean away from the face of the ruler so that the point may be in close contact with the edge formed by the ruler and the paper. The dotted or broken continuity of line that is employed to indicate the edges of the back-half of the crystal needs an even and careful use of the pen or pencil point to give it uniformity, and the delicacy requisite to prevent its obtrusion on the eye.

The simplest way of introducing this representation of the back-half of the crystal is that of the use of tracing paper already described. The projected front-half is traced through and on to the tracing paper, and this traced figure is then turned round in its plane through 180° till corresponding points on it and on the original projection are brought to meet; the quoins are then pricked through on to the paper below: finally, these are to be joined by dotted or interrupted lines. This method of course only applies to diplohedral figures. In others the back-half of the crystal has to be drawn by the same methods as the front-half. Where the method of tracing can be applied it is of great advantage as a test of the accuracy of the original drawing, since all the parts of the one figure ought to join to and exactly meet with the corresponding parts of the other.

It is necessary to use tracing paper for this and for transferring a drawn figure to a plate on which it is to be engraved, since the thickness of drawing paper is sufficient to displace points pricked through it from their true position, owing to the difficulty of passing a needle-point quite perpendicularly in every case.

For transference we may also use a coloured under-surface to the tracing paper, on which a little rouge for instance has been rubbed; after laying the tracing on the sheet of paper or plate of metal or stone to which it is to be transferred, the lines are gone over with a hard point and a ruler, and are thus impressed on the surface below.

Where the figure has to be reversed, as for direct engraving or lithographing, the tracing has, of course, to be turned over and the figure reversed on the surface that is to be engraved or lithographed.

DESCRIPTION OF THE PLATES.

THE Plates I to VIII represent in stereographic projection the poles of a general independent form under all the varieties of mero-symmetry required by the symmetrical conditions of the six crystallographic systems. The total number of holo-symmetrical and mero-symmetrical types is thirty-two; they are numbered consecutively in the table given below. Some of the hemimorphous forms, as being readily deducible from those in the Plates, are not represented in the projections, and the two types belonging to the Anorthic system are too simple to require representation; the omitted types are marked with an asterisk in the table. The plan on which the projections are arranged has been described in Article **149**, p. 168. The merohedra corresponding to those forms of which the poles lie in, and the faces of which are perpendicular to, systematic planes of which the symmetry is in abeyance, in most cases reveal their mero-symmetrical nature in the physical characters of their faces.

In the Plates the projections of proto-systematic planes are indicated by darker lines than those of deutero-systematic planes. A trito-systematic plane is taken as the plane of projection.

Subjoined is a list of the thirty-two types of symmetrical forms, and one or two examples are given in each case of substances whose crystals present that particular type of symmetry, where such are known to exist.

PLATE I.

CUBIC SYSTEM.

Holo-systematic forms.

1. Holo-symmetric (p. 201) . $\{hkl\}$ fluor.
2. Haplohedral (p. 207) . $\alpha\{hkl\}$ cuprite.
3. . . . . . . (p. 212) . $\sigma\{hkl\}$ blende; fahlore.

Hemi-systematic forms.

4. Diplohedral (p. 215) . $\pi\{hkl\}$ pyrites; stannic iodide.
5. Haplohedral (p. 217) . $\sigma\pi\{hkl\}$ sodium chlorate; sodium bromate; lead nitrate.

PLATES II and III.

TETRAGONAL SYSTEM.

Holo-systematic forms.

6. Holo-symmetric (p. 247) . {hkl} cassiterite.
7. Haplohedral (p. 254) . α {hkl} { strychnine sulphate; ethylene-diamine sulphate.
8. . . . . . (p. 255) . s {hkl} / (p. 257) . σ{hkl} } chalcopyrite; edingtonite; urea.
9. . . . . . (hemimorphous) (p. 259) . ρ {hkl} iodo-succinimide.

Hemi-systematic forms.

10. Diplohedral (p. 259) . ϕ {hkl} scheelite, stoltzite.
11. Haplohedral (p. 262) . $s\sigma$ {hkl} (no example known).
12. . . . . . (hemimorphous) (p. 263) . $\rho\alpha$ {hkl} wulfenite.

PLATES IV–VII.

HEXAGONAL SYSTEM.

Holo-systematic forms.

13. Holo-symmetric (p. 276) . {hkl, efg} beryl.
14. Haplohedral (p. 287) . α {hkl, efg} { barium-potassium stibiotartrate and nitrate.
15. . . . . (p. 289) . x {hkl}, x {efg}, or ξ {hkl, efg} { no example known.

*16. . . . . . (hemimorphous) (p. 292) . ρ {hkl, efg} { greenockite; silver iodide.

Hemi-systematic forms.

17. Diplohedral (p. 293) . π {hkl} / . . . . (p. 298) . ψ {hkl} } calcite.
18. . . . . (p. 299) . ϕ {hkl, efg} apatite.
19. Haplohedral (p. 302) . $\alpha\pi$ {hkl} quartz; calcium dithionate.
20. . . . . (p. 304) . $x\phi$ {hkl} (no example known).

*21. . . . . . (p. 306) . $\rho\pi$ {hkl} or $\rho\psi$ {hkl} tourmaline.

*22. . . . . . (hemimorphous) (p. 307) . $\rho\phi$ {hkl, efg} nepheline.

Tetarto-systematic forms.

23 Diplohedral (p. 307) . $\pi\phi$ {hkl} or $\psi\phi$ {hkl, efg} dioptase, phenakite.

*24. Haplohedral (hemimorphous) (p. 308) . $\rho\pi\phi$ {hkl} sodium meta-periodate.

PLATE VIII.

Ortho-rhombic System.

Holo-systematic forms.

25. Holo-symmetric (p. 333) . {hkl} barytes.

26. Haplohedral (p. 336) . . *a* {hkl} { magnesium sulphate; potassium dextro-tartrate.

27. . . . . . . (hemimorphous) (p. 337) . *ρ* {hkl}, *x* {hkl} or *ξ* {hkl} hemimorphite, struvite.

Mono-symmetric System.

Holo-systematic forms.

28. Holo-symmetric (p. 354) . {hkl} augite.

29. Haplohedral (p. 358) . . *s* {hkl} scolecite.

30. . . . . . . (hemimorphous) (p. 359) . *a* {hkl} lactose.

Anorthic System.

*31. The Holohedron (p. 372) . {hkl} axinite.

*32. The Hemihedron (p. 373) . *a* {hkl} calcium thiosulphate.

The holo-symmetric form of the Hexagonal system, not represented on Plates IV–VII, is given in the adjoining figure on a scale somewhat larger than that of the projections in the Plates.

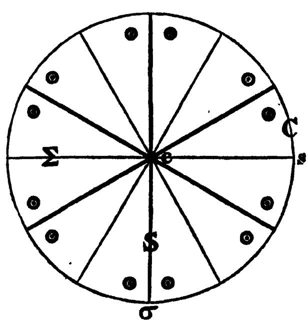
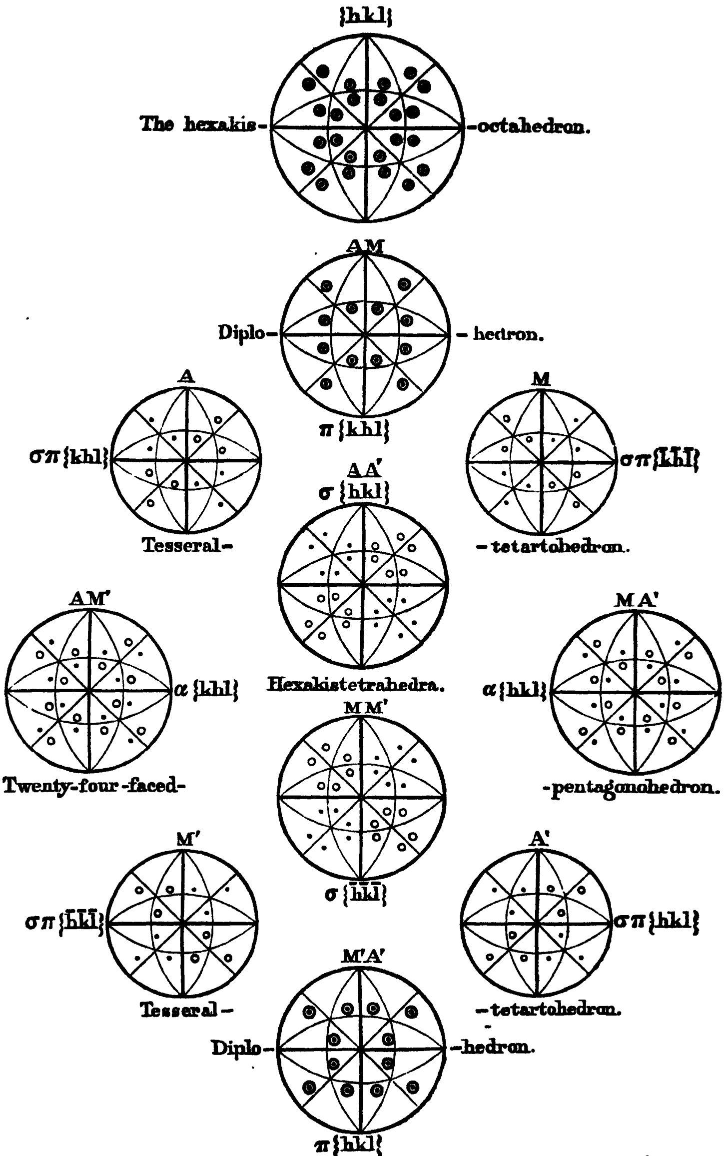
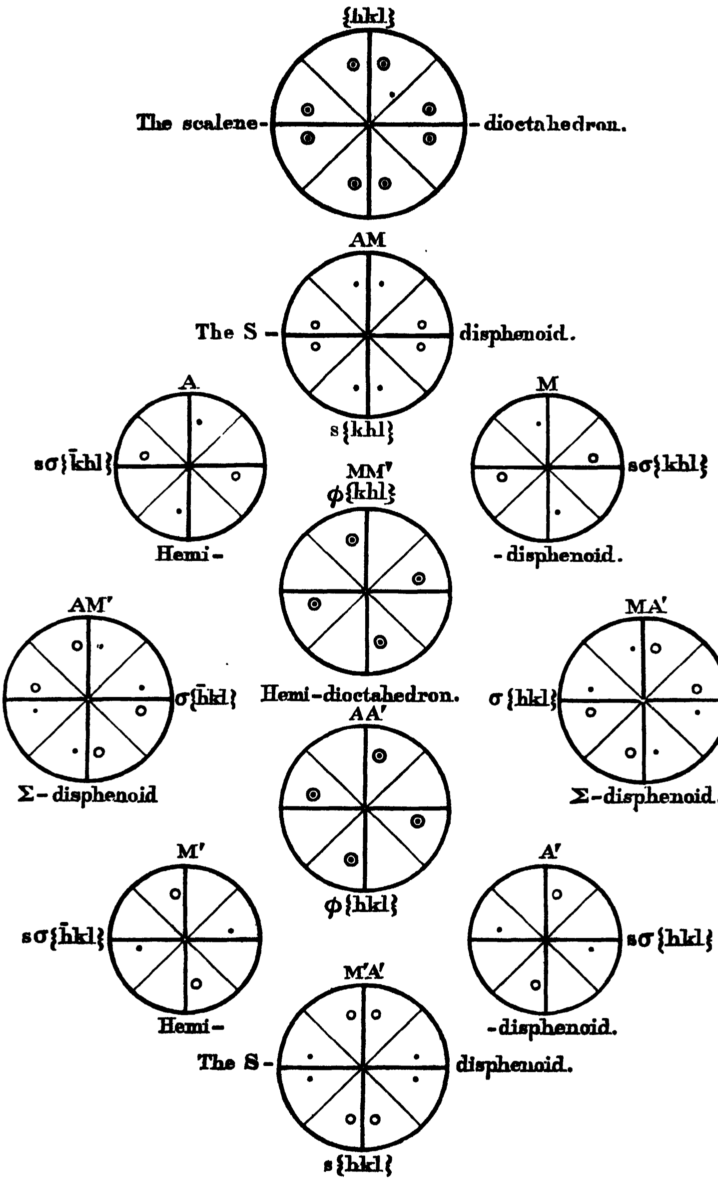


Fig. 398.


{hkl}
The hexakis- -octahedron.
AM
Diplo- - hedron.
π {khl}
A
σπ {khl}
Tesseral-
M
σπ {khl}
-tetartohedron.
AA'
σ {hkl}
Hexakistetrahedra.
AM'
α {khl}
Twenty-four-faced-
MA'
α {hkl}
-pentagonohedron.
MM'
σ {hkl}
M'
σπ {hkl}
Tesseral-
A'
σπ {hkl}
-tetartohedron.
M'A'
Diplo- -hedron.
π {hkl}


{hkl}
The scalene-
-dioctahedron.
AM
The S -
disphenoid.
s{khl}
A
sσ{k̄hl}
Hemi-
M
sσ{khl}
-disphenoid.
MM'
φ{khl}
Hemi-dioctahedron.
AM'
σ{h̄kl}
Σ-disphenoid
MA'
σ{hkl}
Σ-disphenoid.
AA'
φ{hkl}
M'
sσ{h̄kl}
Hemi-
A'
sσ{hkl}
-disphenoid.
M'A'
The S-
disphenoid.
s{hkl}

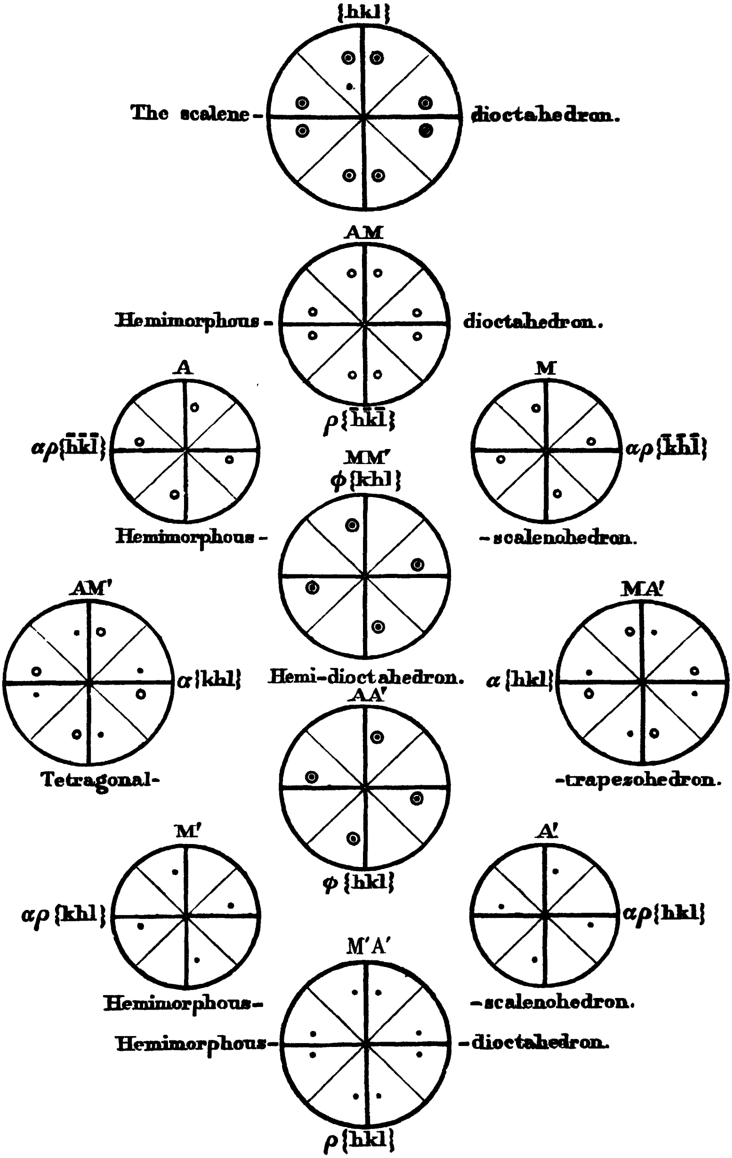

{hkl}
The scalene-
dioctahedron.
AM
Hemimorphous-
dioctahedron.
ρ{h̄k̄l̄}
A
αρ{h̄k̄l̄}
M
αρ{k̄h̄l̄}
Hemimorphous-
-scalenohedron.
MM'
φ{khl}
AM'
α{khl}
Hemi-dioctahedron.
α{hkl}
MA'
Tetragonal-
-trapezohedron.
AA'
M'
αρ{khl}
A'
αρ{hkl}
φ{hkl}
M'A'
Hemimorphous-
-scalenohedron.
Hemimorphous-
-dioctahedron.
ρ{hkl}

PLATE IV.—Hexagonal System.

For the Discalenohedron {hkl, efg} see Fig. 398, p. 504.

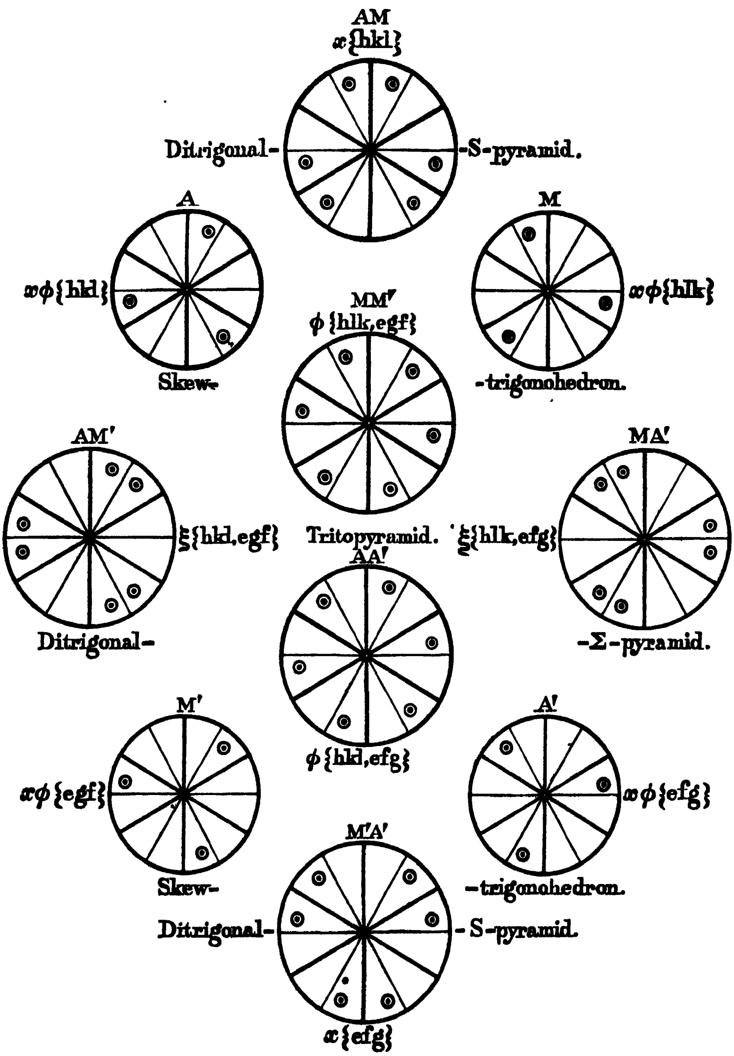


PLATE V.—Hexagonal System.

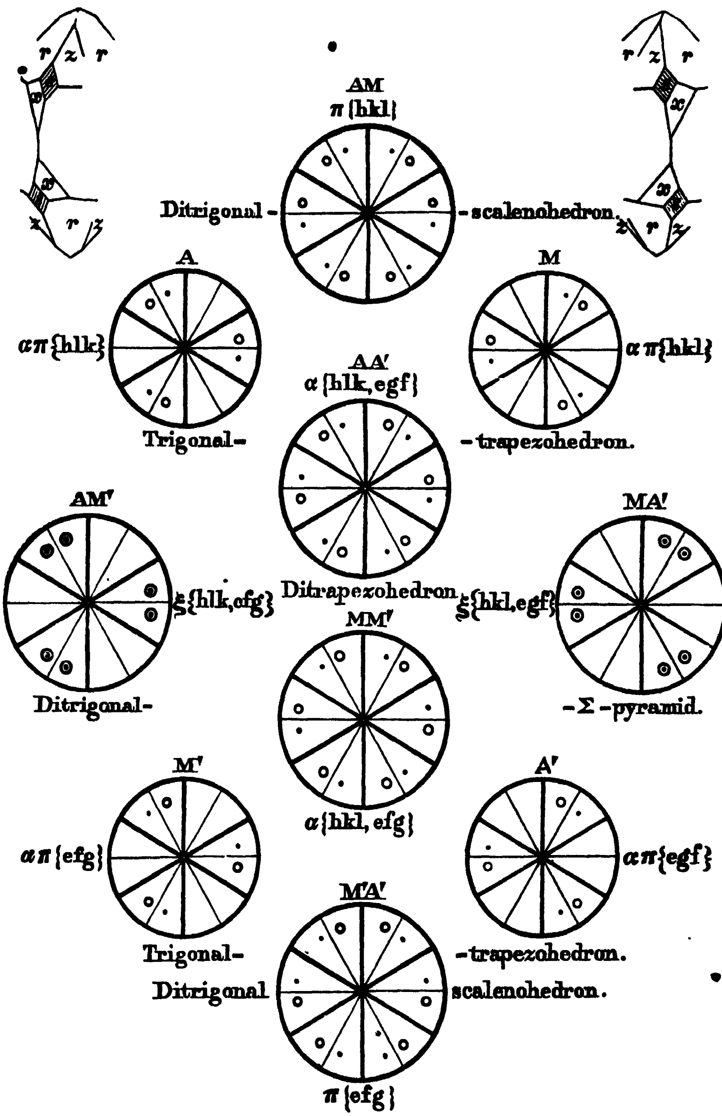


PLATE VII.—Hexagonal System.

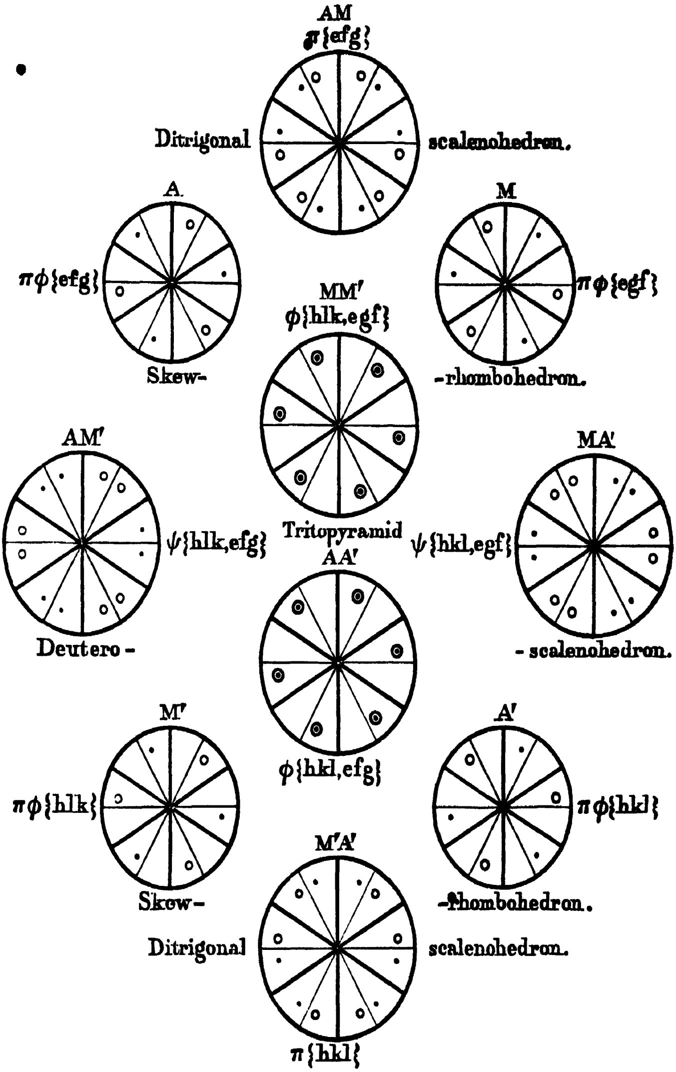
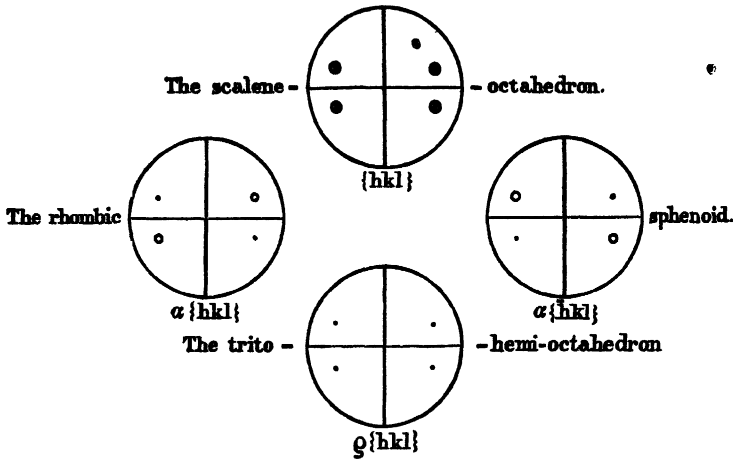
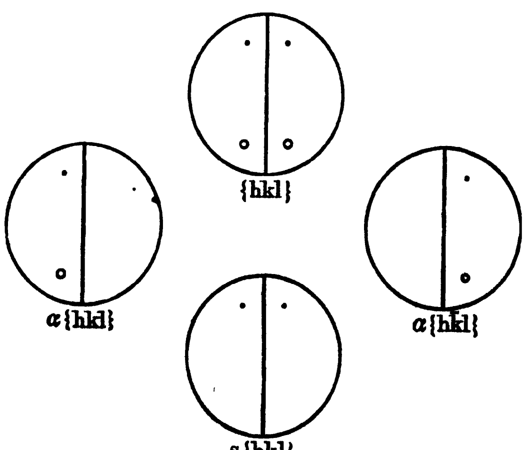


PLATE VIII.—Ortho-rhombic System.



Mono-symmetric System.



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