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# Development and Mathematical Analysis of Double Gravity Well Exhibit

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## *Abstract*

*Double Gravity Well is a highly interesting exhibit, demonstrating the movement of a ball of mass 'm' simultaneously under the attractive forces of two centers of gravitation. When attempts to simulate the exact space-time curvature of such a force field by trial and error method failed, we resorted to a mathematical modeling using 3-D analytical geometry and plotted the exact surface. We then devised a practical method of fabrication calculating the cross sections of the surface at regular intervals. The solution of the equation of motion of the ball was worked out by solving the Euler-Lagrange Equation in elliptical coordinates. The solution shows unpredictable trajectories of the ball in the space-time curvature which is highly sensitive to initial conditions. We also extended the method in exploring fields with three or more gravity wells. The exhibit demonstrates some important phenomena in classical mechanics, classical electrodynamics, molecular physics and planetary physics and in some other fields.*


Double Gravity Well exhibit demonstrates the movement of a ball (Planet) of mass *m* simultaneously under the attractive forces of two centers of gravitation (Suns). In physics, the problem is known as Euler-Jacobi problem or two-center Kepler problem. The mathematics involved in this particular exhibit is known as Euler's restricted three-body method which is used to solve for the motion of a particle under the gravitational field of two other point masses that are fixed in space. This problem is significant as an exactly solvable special case of the three-body problem, and an approximate solution for particles moving in the gravitational fields of prolate and oblate spheroids. This problem is named after Leonhard Euler, who discussed it in 1760. In this article, we apply Euler's method in designing a double gravity well as well as solving the equation of motion.

## *Modeling the exhibit*

For a single gravity well exhibit, the force acting on a rolling ball is $F(r) = -\frac{\mu}{r^2}$. The gravitational potential $V(r)$ is proportional to $\frac{1}{r}$. When we plot $V(r) \sim r$ curve, where $r^2 = x^2 + y^2$, we get a rectangular hyperboloid, which is our familiar space-time curvature of a single gravity well (fig. 1). The trajectories of motion on such a surface can be easily calculated to be elliptical in nature from classical mechanics.

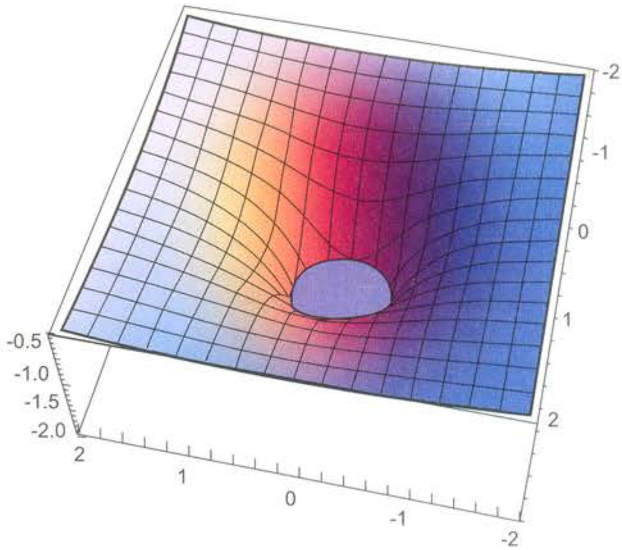


Fig. 1. A Single Gravity Well.

As the single gravity well modeling was successful, we applied the method of modeling to a double gravity well. It can be understood that the equation of the surface must have singularities at two points on the surface, say at $x=\pm a$, $y=0$. Thus we assume two centers of force acting on the ball to be fixed in space and let these centers be located along the x-axis at $\pm a$. The ball is likewise assumed to be confined to a fixed plane containing the two centers of force. The potential energy of the ball in the field of these centers is given by

$$V(x,y) = -\frac{\mu_1}{\sqrt{(x-a)^2 + y^2}} - \frac{\mu_2}{\sqrt{(x+a)^2 + y^2}} \quad \text{.................. (1)}$$

where $\mu_1$ *and* $\mu_2$ are constants. We have assumed that the two gravitational fields are identical. Therefore, $\mu_1 = \mu_2 = \mu$ *(say)*. To plot this surface, we take $V(x,y)$ along negative z-axis, and we get

$$z(x,y) = -\frac{1}{\sqrt{(x-1)^2 + y^2}} - \frac{1}{\sqrt{(x+1)^2 + y^2}} \quad \text{.................. (2)}$$

We have taken a=1 and for scaling, we have taken $\mu = 1$. The computer generated plot is a double gravity well (fig. 2).

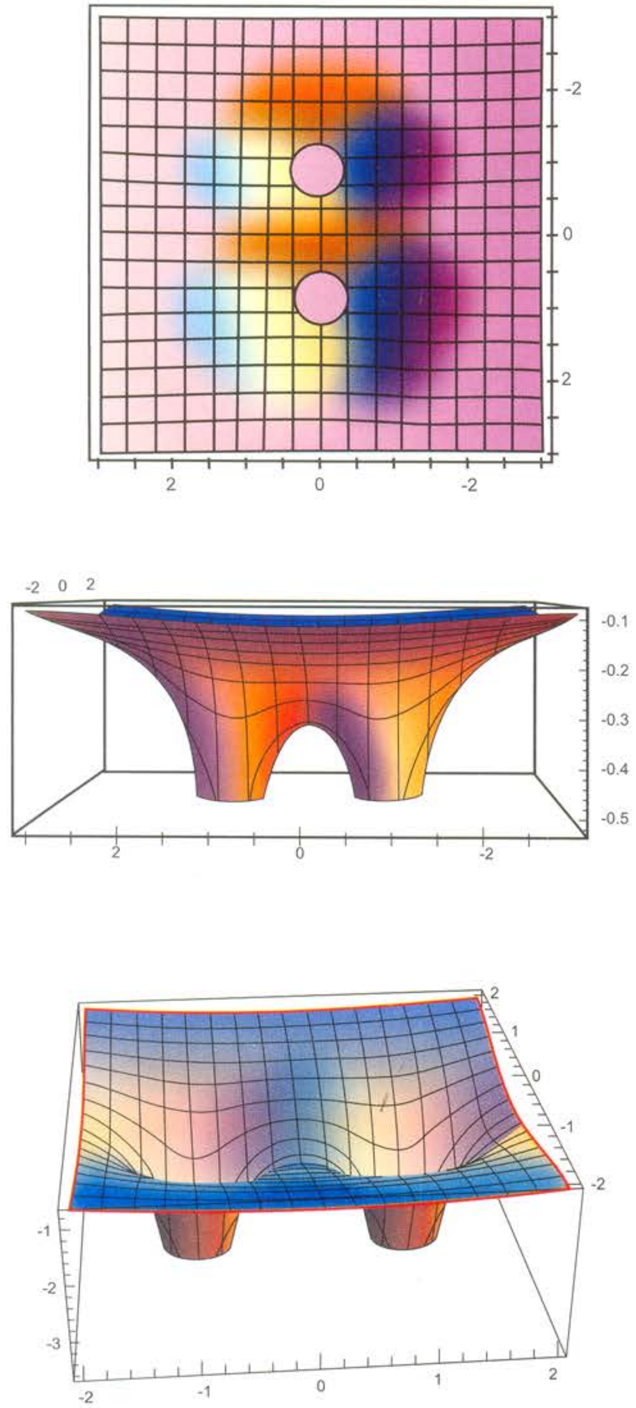

Fig. 2. Double Gravity Well (three different views).

However, if the strengths of the fields are different, the well takes an asymmetric form (fig. 3).

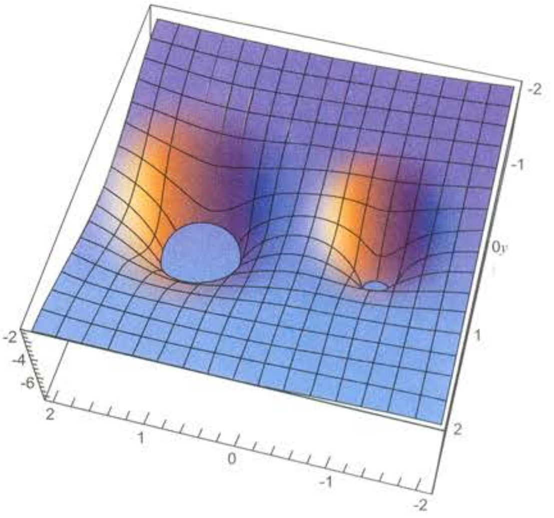

Fig. 3. Asymmetric Double Gravity Well.

The symmetric well has a cross section at its center as given below.

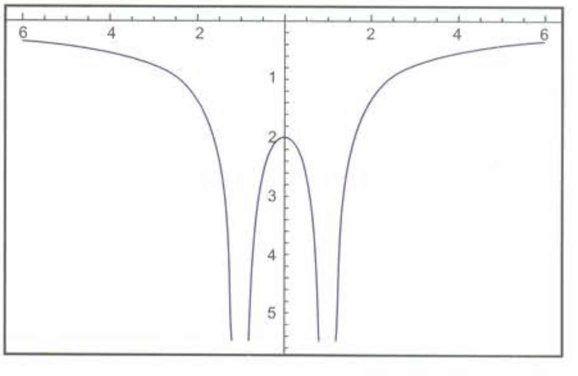

Fig. 4. Vertical Cross Section of a Double Gravity Well.

The above method can be used to design multiple well potential surfaces as shown below:

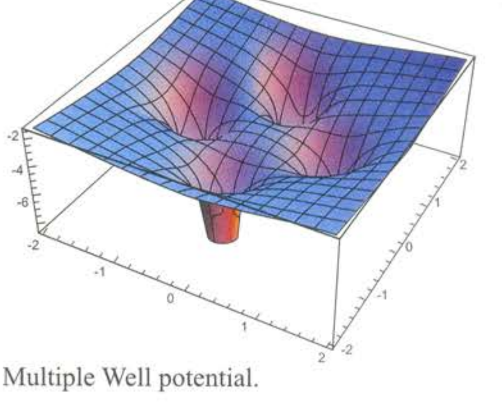

Fig. 5. Multiple Well potential.

## Equation of motion and its solution

The problem of two fixed centers conserves energy. The total energy $E$ is a constant of motion. In our problem (taking a symmetric double gravity well), the potential energy is given by

$$V(x,y) = -\frac{1}{\sqrt{(x-1)^2 + y^2}} - \frac{1}{\sqrt{(x+1)^2 + y^2}} \quad \text{..................(3)}$$

The kinetic energy can be simply written as

$$T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \text{..................(4)}$$

where z is again related to x and y by the equation (2). So, there are practically two degrees of freedom for the rolling ball.

However, as the ball moves in a curved path, it has a rotational kinetic energy too. Taking this into account and taking mass and all other constants equal to one, the Lagrangian takes the form as given below ($\omega$ is the angular velocity of the ball, which is not a constant):

$$L = \left[\tfrac{1}{2}\left[\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right]\right] + \frac{1}{\sqrt{(x-1)^2 + y^2}} + \frac{1}{\sqrt{(x+1)^2 + y^2}} + \tfrac{1}{2}\omega^2(x^2 + y^2)$$

$$\text{..................(5)}$$

Theoretically, the equation of motion can be solved by solving Lagrangian equation (6) for each generalized coordinate ($q$).

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 \text{ ............(6)}$$

We shall, however, solve the equations of motion by Euler's method in elliptic coordinate. The two centers of attraction can be considered as the foci of a set of ellipses. By treating Euler's problem as a Liouville dynamical system, the exact solution can be expressed in terms of elliptic integrals. Introducing elliptic coordinates,

$$x = a \cosh\zeta \cos\eta \text{ ..........(7a)}$$

$$y = a \sinh\zeta \sin\eta \text{ ..........(7b)}$$

and using the particular solutions for a Liouville dynamical system ($\mu_1 = \mu_2 = 1$), we get

$$u = \int \frac{d\xi}{\sqrt{E\cosh^2\xi + 2\cosh\xi - \gamma}} = \int \frac{d\eta}{\sqrt{-E\cosh^2\eta + \gamma}} \text{ ..........(8)}$$

Since these are elliptic integrals, the coordinates $\xi$ and $\eta$ can be expressed as elliptic functions of $u$.

## The Orbits

The solutions give a series of orbits as shown below [Ref. Select bibliography No. 8]. The interesting point is the 8-shaped paths where the ball forms cross loops around the wells:

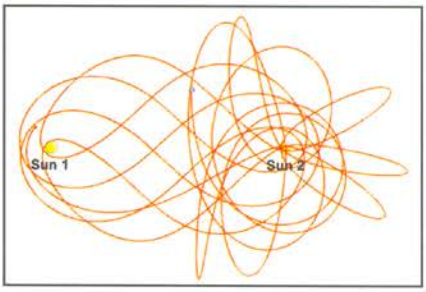


Fig. 6. Orbits around a double gravity well.

However, the practical orbits will be much different from this theoretical model because of three reasons:

1. Due to friction between the ball and the surface, the ball will continuously lose energy and move from an outer orbit to an inner orbit. In fact, the friction will act as a perturbation and we can see the precession of the orbit. The actual path will also depend upon the initial conditions.
2. Due to roughness of the ball and the surface at microscopic level, the ball may take any unpredictable turn near the central hump.
3. The effective potential of the moving ball is different from pure gravitational potential because the centrifugal force acting on the orbiting ball will contribute a potential of the value $m\omega^2(x^2 + y^2)/2$ which will oppose the gravitational potential.

For an asymmetric well, if the mass of Sun 1 is double that of Sun 2, then the ball will be drawn towards the bigger mass even if it is released near the smaller one (fig. 7).

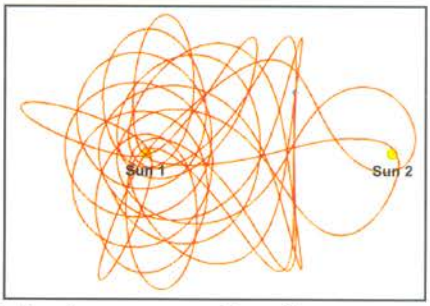


Fig. 7. Motion in an asymmetric well.

It has also been found that the motion is highly sensitive to initial condition and paths of two balls released from the same point diverge after some time (indicated with red and green lines):

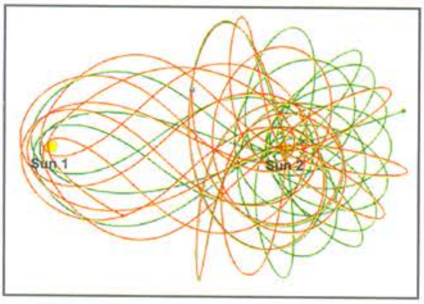


Fig. 8. Divergence of paths of two balls with same initial conditions.

One can easily see that if either center were absent, the ball would move on one of simple ellipses, as a solution to the Kepler's problem. Legendre and Bonnet have shown that according to the principle of superposition, the same ellipses are also the solutions for the Euler problem.

The ball's linear and angular momenta are not conserved in Euler's problem, since the two centers of force act like external forces upon the ball, which may yield a net force and torque on the ball. As a result, the ball will continuously change its path from an ideal superposition of two ellipses. However, the ball has a second conserved quantity that corresponds to the angular momentum or to the Laplace-Runge-Lenz vector as limiting cases.

The solution to the original Euler problem is an approximate solution for the motion of a particle in the gravitational field of a prolate spheroid. The corresponding approximate solution for a particle moving in the field of an oblate spheroid is obtained by making the positions of the two centers of force into imaginary numbers. The oblate spheroid solution is astronomically more important, since most planets, stars and galaxies are approximately oblate spheroids.

## *Fabrication*

To start fabrication, we must revisit the equation (2)

$$z(x,y) = -\frac{1}{\sqrt{(x-1)^2 + y^2}} - \frac{1}{\sqrt{(x+1)^2 + y^2}}$$

Putting different values of z at a fixed interval, we got a series of contours in $xy$ plane.

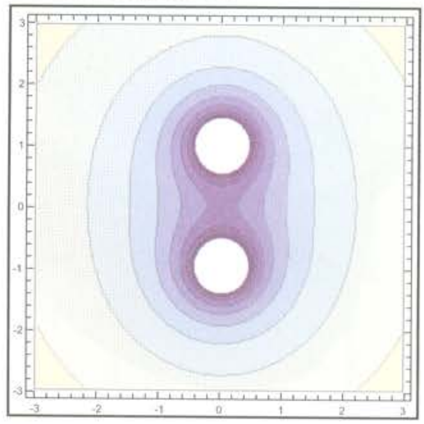

Fig. 9. Contour Plots.

These plots were used as templates to cut ply woods as shown below.

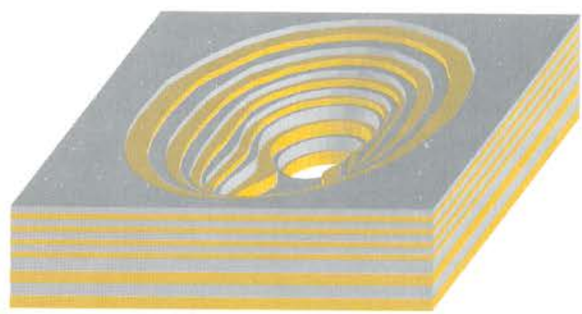

Fig. 10. The ply wood stack before surface finishing and molding.

Several slices have been taken from the contour plot of the double gravity well generated in computer. The plots were converted into Auto Cad drawing and the well was developed in 3D by laying those slices one above the other. These slices were cut in 6mm thick ply wood pieces and stacked in order to obtain the shape of the well just like generating topographical model of land forms. Since the contours are symmetrical, the foci were maintained by placing the contours at appropriate position with respect to four reference points (two on $x$ axis and two on $y$ axis). The contour of the well has been generated in discrete steps. The steps have been made smooth by applying epoxy resin putty in the groove to obtain a smooth surface. Once the shape of the well had been generated, it was checked by rolling a ball on it, which showed the motion similar to the simulation done by computer software. Then, a fiberglass mold has been taken. The final model of the well has been fabricated with fiberglass cast.

Fig. 11. The final exhibit (grid digitally superimposed).

## Further works

The technique developed by us can be applied to generate and investigate upon multiple gravity wells as well as any arbitrary trial potential which may yield interesting result.

## Acknowledgement

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